Junior problems

J619. In triangle ABC,

$$AB^4 + BC^4 + CA^4 = 2AB^2 \cdot BC^2 + AB^2 \cdot CA^2 + 2BC^2 \cdot CA^2.$$

Find all possible values of $\angle A$.

Proposed by Adrian Andreescu, Dallas, USA

Solution by Polyahedra, Polk State College, USA
Let $a = BC$, $b = CA$, $c = AB$, $s = (a+b+c)/2$, and Δ be the area of ABC. We have

$$4b^2c^2 \sin^2 A = 16\Delta^2 = 16s(s-a)(s-b)(s-c) = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = b^2c^2,$$

thus $\sin A = 1/2$, so $A = 30°$ or $150°$.

Also solved by Ivan Hadinata, Jember, Indonesia; Sundaresh, Shivamogga, India; Maqsadbek Egamberdiyev, Bagat, Khorezm, Uzbekistan; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Matthew Too, Brockport, NY, USA; Sherzod Saidov, Urgench, Khorezm, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Tigran Gevorgyan, Quantum College, Yerevan, Armenia; Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Adam John Frederickson, Utah Valley University, UT, USA; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Clark College, WA, USA; Titu Zvonaru, Comănești, România.

J620. Let ABC be a right triangle and let M be the midpoint of the hypotenuse BC. It is known that $AM^2 = AB \cdot AC$. Find the measure of angle ACB.

Proposed by Vasile Lupulescu, Târgu Jiu, România

Solution by Polyahedra, Polk State College, USA

Let Δ be the area of ABC. Since $BC = 2AM$, we have

$$AM^2 \sin\angle AMB = \frac{1}{2} BC \cdot AM \sin\angle AMB = \Delta = \frac{1}{2} AB \cdot AC,$$

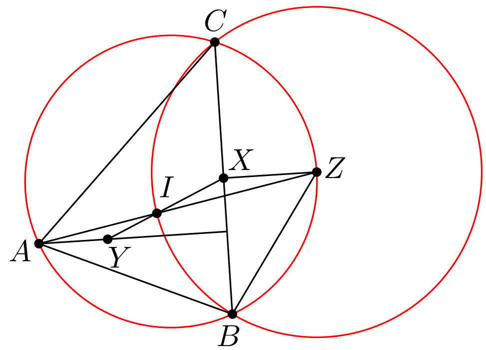
thus $\angle AMB = 30^\circ$ or 150°, so $\angle ACB = 15^\circ$ or 75°.

Also solved by Ivan Hadinata, Jember, Indonesia; Sundaresh, Shivamogga, India; Daniel Văcaru, Pitești, Romania; Tigran Gevorgyan, Quantum College, Yerevan, Armenia; Adam John Frederickson, Utah Valley University, UT, USA; Maqsadbek Egamberdiyev, Bagat, Khorezm, Uzbekistan; Batakogias Panagiotis, High School of Velestino, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Eckard Specht, Otto-von-Guericke University Magdeburg, Germany; Matthew Too, Brockport, NY, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Sherzod Saidov, Urgench, Khorezm, Uzbekistan; Soham Dutta, India; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Daniel Pascuas, Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Clark College, WA, USA; Soham Bhadra, India; Titu Zvonaru, Comănești, România.

J621. Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Let X be the midpoint of segment BC. Line XI intersects the altitude from A in Y. Prove that $AY = r$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Polyahedra, Polk State College, USA



Suppose that AI intersects the circumcircle of ABC at Z. It is well known that $ZI = ZB$. Therefore, $AY/XZ = AI/IZ = AI/BZ$, so $AY = AI \cdot XZ/BZ = AI \sin(A/2) = r$.

Also solved by Ivan Hadinata, Jember, Indonesia; Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Maqsadbek Egamberdiyev, Bagat, Khorezm, Uzbekistan; Corneliu Mănescu-Avram, Ploieşti, Romania; Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Sherzod Saidov, Urgench, Khorezm, Uzbekistan; Theo Koupelis, Clark College, WA, USA; Titu Zvonaru, Comănești, România.

J622. Let a, b, c be positive real numbers such that $a, b, c \in \left[\frac{1}{2}, 1\right]$. Prove that

$$\frac{a}{\sqrt{b}+\sqrt{c}} + \frac{b}{\sqrt{c}+\sqrt{a}} + \frac{c}{\sqrt{a}+\sqrt{b}} < 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Marius Stănean, Zalău, România

Since $a \le 1 \Rightarrow a \le \sqrt{a}$ and similarly for b, c, it suffices to prove that

$$\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}} + \frac{\sqrt{b}}{\sqrt{c}+\sqrt{a}} + \frac{\sqrt{c}}{\sqrt{a}+\sqrt{b}} < 2,$$

which is equivalent to

$$a\sqrt{a} + b\sqrt{b} + c\sqrt{c} < a(\sqrt{b}+\sqrt{c}) + b(\sqrt{c}+\sqrt{a}) + c(\sqrt{a}+\sqrt{b}) + \sqrt{abc}$$

that is true because

$$\sqrt{a} \le 1 < \sqrt{2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \le \sqrt{b} + \sqrt{c},$$

and similarly $\sqrt{b} < \sqrt{c} + \sqrt{a}$ respectively $\sqrt{c} < \sqrt{a} + \sqrt{b}$.

Also solved by Perfetti Paolo, Università degli studi di Tor Vergata Roma, Roma, Italy; Henry Ricardo, Westchester Area Math Circle; Polyahedra, Polk State College, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Soham Bhadra, India; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Clark College, WA, USA; Titu Zvonaru, Comănești, România.

J623. Let m_a, m_b, m_c be the medians in a triangle ABC. Prove that

$$\frac{m_a^4}{m_b+m_c-m_a}+\frac{m_b^4}{m_c+m_a-m_b}+\frac{m_c^4}{m_a+m_b-m_c}\geq m_a^3+m_b^3+m_c^3.$$

Proposed by Mihaly Bencze, Braşov and Neculai Stanciu, Buzău, România

Solution by Maqsadbek Egamberdiyev, Bagat, Khorezm, Uzbekistan
Without loss of generality, we can assume that $m_a \leq m_b \leq m_c$. Then $m_a^3 \leq m_b^3 \leq m_c^3$ and

$$\frac{m_a}{m_b+m_c-m_a}\leq\frac{m_b}{m_c+m_a-m_b}\leq\frac{m_c}{m_a+m_b-m_c}.$$

Applying *Chebyshev inequality*, we get

$$(m_a^3+m_b^3+m_c^3)\cdot\left(\frac{m_a}{m_b+m_c-m_a}+\frac{m_b}{m_c+m_a-m_b}+\frac{m_c}{m_a+m_b-m_c}\right)\leq$$
$$3\left(\frac{m_a^4}{m_b+m_c-m_a}+\frac{m_b^4}{m_c+m_a-m_b}+\frac{m_c^4}{m_a+m_b-m_c}\right).$$

So, it suffices to prove that

$$\frac{m_a}{m_b+m_c-m_a}+\frac{m_b}{m_c+m_a-m_b}+\frac{m_c}{m_a+m_b-m_c}\geq 3.$$

However, this is equivalent to

$$\frac{m_a^2}{m_a(m_b+m_c-m_a)}+\frac{m_b^2}{m_b(m_c+m_a-m_b)}+\frac{m_c^2}{m_c(m_a+m_b-m_c)}\geq 3.$$

From *T2's lemma*, we have

$$\frac{m_a^2}{m_a(m_b+m_c-m_a)}+\frac{m_b^2}{m_b(m_c+m_a-m_b)}+\frac{m_c^2}{m_c(m_a+m_b-m_c)}\geq\frac{(m_a+m_b+m_c)^2}{m_am_b+m_bm_c+m_am_c}\geq 3.$$

We are done!

Also solved by Polyahedra, Polk State College, USA; Sundaresh, Shivamogga, India; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Clark College, WA, USA; Titu Zvonaru, Comănești, România.

J624. In triangle ABC let M, N, P be the midpoints of BC, CA, AB, respectively, and let D, E, F be the feet of the altitudes on sides BC, CA, AB, respectively. Prove that

$$\frac{DM + EN + FP}{2} \geq \max\{m_a, m_b, m_c\} - \min\{m_a, m_b, m_c\}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Theo Koupelis, Clark College, WA, USA

Let a, b, c be the side lengths BC, AC, AB, respectively. We have

$$DM = \left|\frac{a}{2} - c \cdot \cos \angle B\right| = \left|\frac{a}{2} - c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right| = \frac{|b^2 - c^2|}{2a}.$$

Similarly we get $EN = |a^2 - c^2|/(2b)$, and $FP = |a^2 - b^2|/(2c)$. Without loss of generality let $a \geq b \geq c$, and thus $m_c \geq m_b \geq m_a$. The desired inequality is now equivalent to

$$\frac{b^2 - c^2}{4a} + \frac{a^2 - b^2}{4c} - \frac{a^2 - c^2}{4b} \geq m_c - m_a = \frac{4m_c^2 - 4m_a^2}{4(m_c + m_a)} = \frac{3(a^2 - c^2)}{4(m_c + m_a)},$$

or, after simplifying,

$$\frac{1}{4abc} \cdot (a + b)(a - c)(b + c)(a + c - b) \geq \frac{3(a - c)(a + c)}{4(m_c + m_a)}.$$

If $a = c$, then the triangle is equilateral and the inequality holds as an equality. Let $a > c$. Then we need to show that

$$m_c + m_a \geq \frac{3(a + c)abc}{(a + b)(b + c)(a + c - b)}.$$

But $\frac{2}{3}(m_c + m_a) > b$, and therefore it is sufficient to show that

$$\frac{3b}{2} \geq \frac{3(a + c)abc}{(a + b)(b + c)(a + c - b)} \Longleftrightarrow (a - b)(b - c)(a + b + c) \geq 0,$$

which is obvious.

Also solved by Polyahedra, Polk State College, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Senior problems

S619. Let $a, b, c \in [0, 1]$, no two of which are zero. Prove that

$$\frac{ab+1}{a+b} + \frac{bc+1}{b+c} + \frac{ca+1}{c+a} \geq \frac{ab+bc+ca+3}{a+b+c} + 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Pascuas, Barcelona, Spain

If $A \geq B > 0$ then $f_{A,B}(x) = \frac{A+x}{B+x} = 1 + \frac{A-B}{B+x}$ is a decreasing function on the interval $[0, \infty)$, so

$$\frac{A}{B} \geq \frac{A+C}{B+C} \qquad (A \geq B > 0,\, C \geq 0).$$

Now we may apply this inequality to $A = ab+1$, $B = a+b$ and $C = c$, since $C \geq 0$, $B > 0$, and $A \geq B$ (because $A - B = (1-a)(1-b) \geq 0$). We obtain that

$$\frac{ab+1}{a+b} \geq \frac{ab+1+c}{a+b+c}.$$

Similarly, we get that

$$\frac{bc+1}{b+c} \geq \frac{bc+1+a}{a+b+c} \quad \text{and} \quad \frac{ca+1}{c+a} \geq \frac{ca+1+b}{a+b+c}.$$

Finally, the inequality to be proved is obtained by summing the above three inequalities.

Also solved by Soham Bhadra, Patha Bhavan, India; Sundaresh, Shivamogga, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, România.

S620. Let a, b, c, d be positive real numbers. Prove that

$$(abc + abd + acd + bcd)^2 \geq 4abcd(ab + bc + cd + da).$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Matthew Too, Brockport, NY, USA

Since

$$\begin{aligned}(abc + abd + acd + bcd)^2 &= (abc)^2 + (abd)^2 + (acd)^2 + (bcd)^2 + 2abcd(ab + ac + ad + bc + bd + cd)\\ &= (abc + acd)^2 + (abd + bcd)^2 + 2abcd(ab + bc + cd + da),\end{aligned}$$

then it suffices to show that

$$(abc + acd)^2 + (abd + bcd)^2 \geq 2abcd(ab + bc + cd + da).$$

This inequality is true according to the QM-GM inequality since

$$(abc + acd)^2 + (abd + bcd)^2 \geq 2(abc + acd)(abd + bcd) = 2abcd(ab + bc + cd + da)$$

as required.

Also solved by Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Daniel Pascuas, Barcelona, Spain; Soham Bhadra, Patha Bhavan, India; Marian Ursărescu, Roman-Vodă National College, Roman, Romania; Maqsadbek Egamberdiyev, Bagat, Khorezm, Uzbekistan; Corneliu Mănescu-Avram, Ploieşti, Romania; Batakogias Panagiotis, High School of Velestino, Greece; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Christos Apostolidis, Alexandroupolis, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ivan Hadinata, Jember, Indonesia; Le Trong Khoi, Hanoi, Amsterdam High School for the Gifted, Hanoi, Vietnam; Sherzod Saidov, Urgench, Khorezm, Uzbekistan; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Toyesh Prakash Sharma, Agra College, Agra, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, România.

S621. Find all positive integers n for which there are positive integers a, b and a non-degenerate triangle with side lengths $n, 3^a, 5^b$.

Proposed by Josef Tkadlec, Czech Republic

Solution by the author

Clearly $n \leq 2$ is impossible, and $3 \leq n \leq 51$ are good e.g. due to 3^3 and 5^2. Take $x \geq 9$ a power of three and y the smallest power of 5 larger than x. Then one can show that all n with $x \leq n < 3x$ are good by a casework depending on whether $x < y < 2x$, $2x < y < 4x$, $4x < y < 5x$ (3 cases). Therefore, $n \geq 3$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S622. Let ABC be a triangle inscribed in a circle Γ of center O. The tangents at A and C to Γ intersect each other in P. The line BP intersect Γ in Q and let S be the midpoint of BQ. Prove that $\angle ACQ = \angle BCS$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Sherzod Saidov, Urgench, Khorezm, Uzbekistan

A well-known fact says that BP be the symmedian-line of the triangle ABC, and moreover, $ABCQ$ is a harmonic quadrilateral, i.e. $AB \cdot CQ = AQ \cdot BC$ and A, B, C, Q lie on a circle.

This gives us that CA is a symmedian-line of BCQ, and it is isogonal to the median CS. Hence, the lines CS and CA are isogonal with respect to the angle BCQ, therefore, $\angle ACQ = \angle BCS$.

Also solved by Ivan Hadinata, Jember, Indonesia; Soham Dutta, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Batakogias Panagiotis, High School of Velestino, Greece; Christos Apostolidis, Alexandroupolis, Greece; Kamran Mehdiyev, Baku, Azerbaijan; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Sherzod Saidov, Urgench, Khorezm, Uzbekistan; NUM Problem Solving Group; Titu Zvonaru, Comănești, România.

S623. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ $(n \geq 2)$ be positive numbers satisfying

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \ldots \geq \frac{a_n}{b_n}.$$

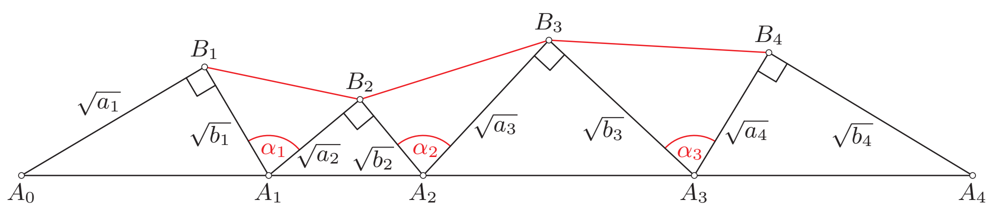
Prove that

$$\sqrt{a_1} + \sqrt{b_1 + a_2} + \sqrt{b_2 + a_3} + \ldots + \sqrt{b_{n-1} + a_n} + \sqrt{b_n} >$$
$$> \sqrt{a_1 + b_1} + \sqrt{a_2 + b_2} + \ldots + \sqrt{a_n + b_n}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by the author

Consider right-angled triangles with legs $\sqrt{a_i}$ and $\sqrt{b_i}$ as shown at the diagram. From the given inequalities it follows that $\alpha_i \leq 90°$.



Therefore,

$$B_iB_{i+1}^2 \leq b_i + a_{i+1} \quad \text{for } i = 1, 2, \ldots, n-1.$$

It follows that the left-hand side of the desired inequality is greater than or equal to the length of the broken line $A_0B_1B_2\ldots B_nA_n$. But this path in turn has the length greater than A_0A_n, which is the right-hand side of the desired inequality.

S624. Prove that the following inequality holds for all positive real numbers a, b, c:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt{\frac{b+c}{2a}}+\sqrt{\frac{c+a}{2b}}+\sqrt{\frac{a+b}{2c}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Ivan Hadinata, Jember, Indonesia

Let $x=\frac{a}{b}$, $y=\frac{b}{c}$, and $z=\frac{c}{a}$ where $x,y,z>0$ and $xyz=1$, so the wanted inequality is equivalent with

$$x+y+z \geq \sqrt{\frac{1}{2}(yz+z)}+\sqrt{\frac{1}{2}(xz+x)}+\sqrt{\frac{1}{2}(xy+y)} \tag{1}$$

By QM-AM inequality, note that

$$\sqrt{\frac{3}{2}\left(\sum_{cyc} xy+\sum_{cyc} x\right)} \geq \sqrt{\frac{1}{2}(yz+z)}+\sqrt{\frac{1}{2}(xz+x)}+\sqrt{\frac{1}{2}(xy+y)} \tag{2}$$

By AM-GM inequality, we have

$$\sum_{cyc} x \geq 3\sqrt[3]{xyz}=3 \quad\Longrightarrow\quad \begin{aligned}\left(\sum_{cyc} x\right)^2 &\geq \frac{1}{2}\left(\sum_{cyc} x\right)^2+\frac{3}{2}\sum_{cyc} x\\ &\geq \frac{3}{2}\sum_{cyc} xy+\frac{3}{2}\sum_{cyc} x \qquad (3)\end{aligned}$$

The combination of (2) and (3) yields inequality (1) which is proved.

Also solved by Batakogias Panagiotis, High School of Velestino, Greece; Adam John Frederickson, Utah Valley University, UT, USA; Christos Apostolidis, Alexandroupolis, Greece; Khakim Egamberganov, Uzbekistan; Le Trong Khoi, Hanoi, Amsterdam High School for the Gifted, Hanoi, Vietnam; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, România.

U619. Find all polynomials $P(x)$ with real coefficients such that

$$P(x)\left(P(x)-2P(y)\right)^2+\left(2P(x)-P(y)\right)^2P(y)=P(xP(x))+P(yP(y))$$

for all $x,y\in\mathbb{R}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Matthew Too, Brockport, NY, USA

The equation above is equivalent to

$$P(x)^3+P(y)^3=P(xP(x))+P(yP(y)).$$

Now, let $\deg(P(x))=n$. Since $\deg(P(x)^3+P(y)^3)=3n$ and $\deg(P(xP(x))+P(yP(y)))=n(n+1)$, then the only n for which the degrees are equivalent are $n=0$ and $n=2$.

For $n=2$, let $P(x)=ax^2+bx+c$ for $a,b,c\in\mathbb{R}$. Then substituting into equation (1) and equating coefficients, we get the system

$$\begin{cases} a^3=a^3\\ 3a^2b=2a^2b\\ 3a^2c+3ab^2=2a^2c+ab^2\\ 6abc+b^3=2abc+ab\\ 3ac^2+3b^2c=ac^2+b^2\\ 3bc^2=bc\\ c^3=c\end{cases}\implies\begin{cases} a^2b=0\\ a^2c+2ab^2=0\\ b^3+4abc-ab=0\\ 2ac^2+3b^2c-b^2=0\\ 3bc^2-bc=0\\ c(c-1)(c+1)=0\end{cases}$$

for which we consider the cases where $c=0,c=1$, and $c=-1$. If $c=0$, then $b=0$ and the system is satisfied for any a, so $(a,b,c)=(a,0,0)$ is a solution. For $c=\pm1$, both $b=0$ and $a=0$, so $(a,b,c)=(0,0,\pm1)$ are also solutions to the system.

For $n=0$, let $P(x)=d$ for $d\in\mathbb{R}$. Then equation (1) reduces down to $2d^3=2d$, or equivalently, $2d(d-1)(d+1)=0$. Thus, $d=0$ and $d=\pm1$ are solutions.

All together, this means that the polynomials satisfying the required equality are $P(x)=ax^2$, $P(x)=1$, $P(x)=-1$, and $P(x)=0$ for any $a\in\mathbb{R}$. A quick check shows that these polynomials do satisfy the required equality.

Also solved by Ivan Hadinata, Jember, Indonesia; Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Theo Koupelis, Clark College, WA, USA; Sebastian Fernandez, University of Costa Rica, Costa Rica.

U620. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{2\,k^2-2\,n\,k+n^2}.$$

Proposed by Vasile Lupulescu, Târgu Jiu, România

First solution by Henry Ricardo, Westchester Area Math Circle

We have

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} = n\cdot\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)^2}{\left(1-2\left(\frac{k}{n}\right)+2\left(\frac{k}{n}\right)^2\right)}\cdot\frac{1}{n},$$

which is n times a Riemann sum for the integrable function $f(x)=x^2/(1-2x+2x^2)$. Since the Riemann sum approaches $\int_0^1 x^2/(1-2x+2x^2)\,dx = 1/2$ as $n\to\infty$, we have $\lim_{n\to\infty} n\cdot(1/2)=\infty$.

Second solution by Henry Ricardo, Westchester Area Math Circle

We have

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} = \sum_{k=0}^{n}\frac{k^2}{k^2+(n-k)^2} = \sum_{k=0}^{n}\frac{(n-k)^2}{k^2+(n-k)^2},$$

from which it follows that

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} = \frac{1}{2}\sum_{k=0}^{n}\frac{k^2+(n-k)^2}{k^2+(n-k)^2} = \frac{n+1}{2} \to\infty$$

as $n\to\infty$.

Also solved by G. C. Greubel, Newport News, VA, USA; Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Soham Bhadra, India; Theo Koupelis, Clark College, WA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Ivan Hadinata, Jember, Indonesia; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, România.

U621. Let x, y, z be nonnegative real numbers such that $x + y + z = 2$. Find the minimum of

$$\sqrt{4+2x^2} + \sqrt{54-36\sqrt{2}+4y^2} + \sqrt{8+2z^2}.$$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Roma, Italy

Solution by Li Zhou, Polk State College, USA

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
\sqrt{(4+1)(4+2x^2)} &\geq 4+\sqrt{2}x,\\
\sqrt{5(54-36\sqrt{2}+4y^2)} = \sqrt{\left(\frac{9}{2}+\frac{1}{2}\right)\left(18(\sqrt{2}-1)^2+4y^2\right)} &\geq 9(\sqrt{2}-1)+\sqrt{2}y,\\
\sqrt{(4+1)(8+2z^2)} &\geq 4\sqrt{2}+\sqrt{2}z.
\end{aligned}$$

Equalities hold when $x = 1/\sqrt{2}$, $y = 1 - 1/\sqrt{2}$, and $z = 1$, respectively. Therefore, the minimum of the given expression is

$$\frac{4+9(\sqrt{2}-1)+4\sqrt{2}+(x+y+z)\sqrt{2}}{\sqrt{5}} = 3\sqrt{10}-\sqrt{5}.$$

Also solved by Theo Koupelis, Clark College, WA, USA; Ivan Hadinata, Jember, Indonesia; Arkady Alt, San Jose, CA, USA; Khakim Egamberganov, Uzbekistan; Matthew Too, Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U622. Prove that in any acute triangle ABC,

$$\left(\frac{4S}{3R}\right)^4 \geq \frac{3(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{a^2+b^2+c^2}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Let $a^2 = y+z$, $b^2 = z+x$, $c^2 = x+y$ where $x, y, z > 0$. The inequality becomes

$$\frac{256(xy+yz+zx)^4}{81(x+y)^2(y+z)^2(z+x)^2} \geq \frac{12xyz}{x+y+z},$$

$$64(x+y+z)(xy+yz+zx)^4 \geq 243xyz(x+y)^2(y+z)^2(z+x)^2,$$

because

$$16S^2 = 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4 = 2\sum_{cyc}(y+z)(z+x) - \sum_{cyc}(y+z)^2 = 4(xy+yz+zx)$$

and

$$R^4 = \frac{a^4b^4c^4}{(16S^2)^2} = \frac{(x+y)^2(y+z)^2(z+x)^2}{16(xy+yz+zx)^2}.$$

The inequality is homogeneous in x, y, z, so we can start by assuming $x+y+z = 3$. Hence we need to show that

$$64(xy+yz+zx)^4 \geq 81xyz(x+y)^2(y+z)^2(z+x)^2,$$

or

$$\frac{8(xy+yz+zx)^2}{9\sqrt{xyz}} \geq (x+y)(y+z)(z+x),$$

or

$$\frac{8(xy+yz+zx)^2}{9\sqrt{xyz}} + xyz \geq 3(xy+yz+zx).$$

Denote $p = x+y+z = 3$, $q = xy+yz+zx = 3(1-t^2)$, $t \in [0,1)$ and $r = xyz \leq (1-t)^2(1+2t) \leq 1$. The function $f(r) = \frac{8q^2}{9\sqrt{r}} + r - 3q$ is decreasing on $(0,1]$. Indeed, we have

$$f'(r) = 1 - \frac{4q^2}{9r\sqrt{r}} \leq 1 - \frac{4 \cdot 3pr}{9r\sqrt{r}} = 1 - \frac{4}{\sqrt{r}} < 0,$$

which means $f(r) \geq f\left((1-t)^2(1+2t)\right)$. So, it suffice to prove that

$$\frac{8(1-t^2)^2}{(1-t)\sqrt{1+2t}} + (1-t)^2(1+2t) \geq 9(1-t^2),$$

or

$$\frac{8(1+t)^2}{\sqrt{1+2t}} + (1-t)(1+2t) \geq 9(1+t),$$

or

$$\frac{4(1+t)^2}{\sqrt{1+2t}} \geq (2+t)^2,$$

or, after squaring both sides of the inequality and expanding, we get

$$t^2(2t^3+t^2-8t-8) \leq 0$$

which is clearly true.

Also solved by Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U623. Find all positive real numbers a for which the sequence

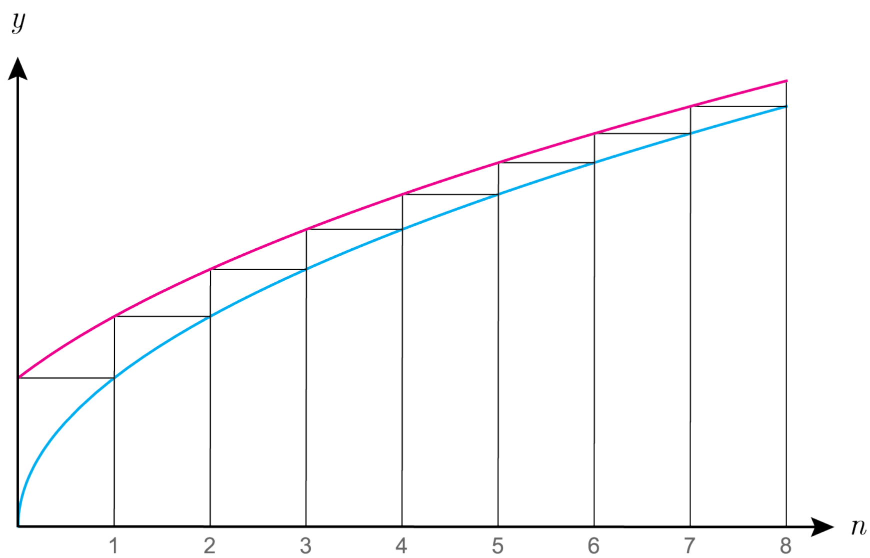
$$x_n = \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^a}$$

converges and find its limit in those cases.

Proposed by Mircea Becheanu, Canada

Solution by Adam John Frederickson, Utah Valley University, UT, USA
First, note that

$$x_n = \sum_{k=1}^{n} \frac{\sqrt{k}}{n^a} = \int_0^n \frac{\sqrt{\lceil t \rceil}}{n^a}\, dt.$$



The terms of the sequence x_n lie between the curves $y = \sqrt{t}/n^a$ (blue) and $y = \sqrt{t+1}/n^a$ (pink)

Then for any a (see the figure)

$$\frac{\sqrt{t}}{n^a} \le \frac{\sqrt{\lceil t \rceil}}{n^a} \le \frac{\sqrt{t+1}}{n^a} \text{ for all } t \ge 0$$
$$\int_0^n \frac{\sqrt{t}}{n^a}\, dt \le x_n \le \int_0^n \frac{\sqrt{t+1}}{n^a}\, dt$$
$$\frac{2}{3}\left(\frac{n^{3/2}}{n^a}\right) \le x_n \le \frac{2}{3}\left(\frac{(n+1)^{3/2} - 1}{n^a}\right).$$

If $a < 3/2$, then

$$x_n \ge \frac{2}{3}\left(n^{3/2-a}\right) \to \infty \text{ as } n \to \infty \qquad \Rightarrow \qquad \lim_{n\to\infty} x_n = \infty.$$

If $a = 3/2$, then

$$\frac{2}{3} \le x_n \le \frac{2}{3}\left(\frac{(n+1)^{3/2} - 1}{n^{3/2}}\right) \qquad \Rightarrow$$
$$\frac{2}{3} \le \lim_{n\to\infty} x_n \le \lim_{n\to\infty} \frac{2}{3}\left(\frac{(n+1)^{3/2} - 1}{n^{3/2}}\right) = \frac{2}{3} \qquad \Rightarrow$$
$$\lim_{n\to\infty} x_n = \frac{2}{3}.$$

If $a > 3/2$, then

$$0 < x_n \le \frac{2}{3}\left(\frac{(n+1)^{3/2}-1}{n^a}\right) \to 0 \text{ as } n \to \infty \qquad \Rightarrow \qquad \lim_{n\to\infty} x_n = 0.$$

In summary,

$$\lim_{n\to\infty} x_n = \begin{cases} \infty & \text{if} \quad a < 3/2, \\ 2/3 & \text{if} \quad a = 3/2, \\ 0 & \text{if} \quad a > 3/2. \end{cases}$$

Also solved by Arkady Alt, San Jose, CA, USA; Ivan Hadinata, Jember, Indonesia; Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Theo Koupelis, Clark College, WA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Matthew Too, Brockport, NY, USA; Henry Ricardo, Westchester Area Math Circle; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Soham Dutta, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U624. Let p be a prime number. For every positive integer n, denote by $\operatorname{rad}_p n$ the product of all prime divisors of n, except p. Let $f:\mathbb{N}\to\mathbb{N}$ be a multiplicative function for which there is a non-zero integer c such that

$$\operatorname{rad}_p n \mid f(n+1)-c$$

for all $n\in\mathbb{N}$. Prove that $f(n)=n^r$ for some non-negative integer r.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Fix positive integers a,b. Let $q>\max(a,b,|c|,|cf(ab)-f(a)f(b)|)$ and $q\neq p$. There are positive integers r,s,x,y such that $ax=1+rq, by=1+sq$ and x,y are prime numbers. Moreover, $\gcd(x,ab)=\gcd(y,abx)=1$.

Hence, letting $abxy=qT+1, T=r+s+qrs$. Then, $f(a)f(x)=f(ax)=f(1+qr)\equiv c \pmod q$. Further,

$$f(b)f(y)=f(by)=f(1+sq)\equiv c \pmod q.$$

Finally, $f(ab)f(x)f(y)=f(abxy)=f(qT+1)\equiv c \pmod q$. By the choice of q it follows that $f(x)f(y)$ is not divisible by q. Thus,

$$f(a)f(b)\equiv cf(ab) \pmod q.$$

Moreover, by the choice of q it follows that $cf(ab)=f(a)f(b)$. Plugging $a=b=1$ yielding $c=1$. Hence, f is completely multiplicative. That is, $f(ab)=f(a)f(b)$.

Let N be a positive integer and q be a prime number such that $q\mid f(N)$. We argue that $q\mid pN$. Indeed, assume on the contrary that there is a prime q dividing $f(N)$ such that $\gcd(q,Np)=1$. Then there are positive integers x,y such that $Nx=qy+1$. Since $q\neq p$ it follows that

$$0\equiv f(N)f(x)=f(Nx)=f(1+qy)\equiv 1 \pmod q,$$

a contradiction. Hence, $q\mid pN$. Thus for each prime r we can write $|f(r)|=r^{\alpha_r}p^{\beta_p}$. Thus, $|f(p)|=p^{\alpha}$. Now, let n,s be positive integers

$$f(np^{2s})=f(1+(np^{2s}-1))\equiv 1 \pmod{\operatorname{rad}_p(np^{2s}-1)}$$

On the other hand,

$$n^{\alpha}f(np^{2s})=n^{\alpha}f(n)p^{2s\alpha}=f(n)(mp^{2s})^{\alpha}\equiv f(n) \pmod{\operatorname{rad}_p(np^{2s}-1)}$$

That is, for all $s,\operatorname{rad}_p(np^{2s}-1)$ divides $f(n)-n^{\alpha}$. Since set of primes dividing $x_s=np^{2s}-1$ is infinite, we can choose a suitable s to ensure that $f(n)=n^{\alpha}$.

Also solved by Besfort Shala, University of Bristol, United Kingdom; Juan José Granier, Universidad de Chile, Chile.

O619. Let l be a nonnegative integer. Prove that there are infinitely many positive integers $k \geq l$, for each of which there exist infinitely many blocks of k consecutive positive integers such that every such block contains precisely l numbers that can be represented as the sum of two squares of integers.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by the authors

Let S be the set of nonnegative integers that can be expressed as sums of two squares. For positive integers s and t, let $f(s,t)$ be the number of those integers among the t consecutive integers $s,\ldots,s+t-1$ that are representable as the sum of two squares (that is, the number of those among $s,\ldots,s+t-1$ that belong to S). We rely in our proof on the following helping results.

Lemma 1: We have $f(s+1,t)-f(s,t) \in \{-1,0,1\}$ for all positive integers s and t.
Proof: Indeed, we have:

- $f(s+1,t)-f(s,t) = -1$ whenever $s \in S$ and $s+t \notin S$;
- $f(s+1,t)-f(s,t) = 0$ whenever either $s \in S$ and $s+t \in S$, or $s \notin S$ and $s+t \notin S$;
- $f(s+1,t)-f(s,t) = 1$ whenever $s \notin S$ and $s+t \in S$.

Lemma 2: Let l be a nonnegative integer. Then for any positive integer N there exist positive integers $s > N$ and t such that $f(s,t) > l$. Moreover, t only depends on l (it can be chosen the same for every s).

Proof: We choose $s = m^2 > N$ and $t \geq l^2+1$. Then the numbers $m^2, m^2+1^2, \ldots, m^2+(l-1)^2, m^2+l^2$ are, evidently, from S and, also, they are among the t consecutive integers $s,\ldots,s+t-1$, as $s+t-1 = m^2+t-1 \geq m^2+l^2$. Consequently, $f(s,t) \geq l+1 > l$. The second part is clear, as $t \geq l^2+1$ yields no dependence on such s.

Lemma 3: For any positive integers t and N, there exists a positive integer $s > N$ such that $f(s,t) = 0$.

Proof: This is well known. Let $p_1 = 3, p_2 = 7, p_3 = 11, \ldots$ be the sequence of primes congruent to 3 modulo 4. By the Chinese remainder theorem there exist infinitely many integers x satisfying the system of congruences

$$x \equiv p_i - i + 1 \bmod p_i^2, \quad \text{for } i = 1,\ldots,t.$$

All these integers are the terms of an arithmetic sequence $(x_0 + jp_1^2 \cdots p_t^2)_{j \in \mathbb{Z}}$, with common difference $p_1^2 \cdots p_t^2$, and the first term x_0 being one fixed solution of the system. Consequently, there are solutions simultaneously verifying all these congruences and that are as big as we want. Choose then one solution $s > N$ and note that, due to

$$s + i - 1 \equiv p_i \bmod p_i^2,$$

$s+i-1$ is divisible by p_i, but not by p_i^2, for each $1 \leq i \leq t$. As it is well known, this implies that every $s+i-1$ $(1 \leq i \leq t)$ does not belong to S (is not a sum of two squares), that is, we have $f(s,t) = 0$ for this choice of s (actually there are infinitely many such s).

And now we can solve our problem. Choose, as in lemma 2, $k \geq l^2 + 1$, which we fix in what follows (but note that there are infinitely many possible choices of k). As shown in the proof, every block of k consecutive integers of the form $n, \dots, n + k - 1$, with $n = m^2$ (any square) has the property that $f(n, k) > l$. Start then with $s_1 = 1$, for which $f(s_1, k) > l$, then choose (as lemma 3 assures that we can do) some $s_2 > s_1$ for which $f(s_2, k) = 0$. By lemma 1, $f(s, k)$ cannot pass from the value $f(s_1, k) > l$ to $f(s_2, k) = 0$ without taking the value l. In other words, we can find some n_1 between s_1 and s_2 (actually, n_1 can equal s_2, if $l = 0$) such that $f(n_1, k) = l$, meaning that the block $n_1, \dots, n_1 + k - 1$ contains precisely l elements of S, that is, precisely l sums of two squares.

From now on it is clear that we continue in the same manner, by taking turns on applying lemmas 2, 3, and 1. Thus, by lemma 2, we can find $s_3 > n_1$ such that $f(s_3, k) > l$, and then lemma 3 ensures that it exists $s_4 > s_3$ with $f(s_4, k) = 0$. By lemma 1 some n_2 between s_3 and s_4 (possibly equal to s_4) can be found such that $f(n_2, k) = l$. Since $n_2 > n_1$ we have a new block $n_2, \dots, n_2 + k - 1$ of k consecutive integers containing precisely l elements of S. Clearly this process goes on and on indefinitely, and our proof is now complete.

O620. Prove that for any positive integer n there is at most one triplet of positive integers $a \le b \le c$ such that $(a+b)(b+c)(c+a)(a+b+c+n)$ is a power of a prime.

Proposed by Josef Tkadlec, Czech Republic and Ján Mazák, Slovakia

Solution by Theo Koupelis, Clark College, WA, USA

If $(a+b)(b+c)(c+a)(a+b+c+n) = p^s$, where p is a prime and s a positive integer, then we must have

$$\begin{aligned} a+b &= p^k, \\ b+c &= p^\ell, \\ c+a &= p^m, \\ a+b+c+n &= p^r, \end{aligned}$$

where k, ℓ, m, r are positive integers with $k+\ell+m+r = s$ and $1 \le k \le m \le \ell < r$. Subtracting the first two equations we get $c - a = p^k(p^{\ell-k} - 1)$, and using the third equation we get $2c = p^k\left[p^{\ell-k} + p^{m-k} - 1\right]$. If p is an odd prime, then the quantity in brackets is an odd integer, and thus $2 \mid p$, which is a contradiction. Thus $p = 2$. Solving the first three equations we get

$$\begin{aligned} a &= 2^{k-1}\left(2^{m-k} - 2^{\ell-k} + 1\right), \\ b &= 2^{k-1}\left(2^{\ell-k} - 2^{m-k} + 1\right), \\ c &= 2^{k-1}\left(2^{\ell-k} + 2^{m-k} - 1\right). \end{aligned}$$

But $a > 0$ and thus we must have $2^{m-k} > 2^{\ell-k} - 1$, with $\ell \ge m$. Therefore, $m = \ell$ and thus $a = b = 2^{k-1}$, and $c = 2^{k-1}\left(2^{m-k+1} - 1\right) = 2^m - 2^{k-1}$. From the last equation we now get $n + 2^m + 2^{k-1} = 2^r$.

If there is another triplet of positive integers a', b', c' satisfying the given condition, we would have another set (k', ℓ', m', r'), with $\ell' = m'$, and $r' > m' \ge k' \ge 1$, that satisfies the equation $n + 2^{m'} + 2^{k'-1} = 2^{r'}$. Subtracting, we get

$$d := 2^r + 2^{m'} + 2^{k'-1} = 2^{r'} + 2^m + 2^{k-1},$$

where d is a natural number. But every natural number has a unique base 2 representation; therefore, $r = r', m = m', k = k'$. Thus, there is at most one triplet of positive integers $a \le b \le c$ that satisfies the given condition.

Also solved by Daniel Pascuas, Barcelona, Spain; Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, West Bengal, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O621. Let $a_1, \ldots, a_k, b_1, \ldots, b_k$ be integers, with $a_1, \ldots, a_k$ positive and mutually distinct, and let ϵ be a positive real number. Prove that there are infinitely many positive integers n such that $(a_1 n + b_1) \cdots (a_k n + b_k)$ divides $(\lfloor \epsilon n \rfloor)!$. (As usual, $\lfloor x \rfloor$ denotes the integer part of the real number x.)

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by the authors

Let Q be the product of all the primes dividing at least one of the differences $a_j - a_i$, $1 \le i < j \le k$, and let s be a positive integer (which fulfills some conditions that will be immediately explained). Note that $Q \ge 2$ (since there exists at least such a prime, because there exists at least a difference $a_j - a_i$ which is greater than 1) unless $k = 1$ or $k = 2$ and $a_2 = a_1 + 1$. In both of this cases we choose $Q = 2$, and note that (for $k = 2$ and $a_2 = a_1 + 1$) $c_1 = 1 + 2^s a_1$ and $c_2 = 1 + 2^s a_2 = 1 + 2^s(a_1 + 1)$ are relatively prime, because any common divisor of them must divide their difference 2^s, but both c_1 and c_2 are odd. In any other situation, the numbers $c_i = 1 + Q^s a_i$, $1 \le i \le k$ are also pairwise relatively prime. Indeed, if p was a prime common divisor of c_i and c_j $(1 \le i < j \le k)$, then it would divide $(a_j - a_i)Q^s = c_j - c_i$. As p is relatively prime to Q^s (it divides $1 + Q^s a_i$), it remains that $p \mid a_j - a_i$. But, if this was the case, then p would divide Q (by the very definition of Q) — which is not possible. Thus c_i and c_j are relatively prime for every $1 \le i < j \le k$.

In what concerns s, we choose it such that the inequalities

$$\epsilon c_i - a_i > 0 \Leftrightarrow a_i(\epsilon Q^s - 1) + \epsilon > 0$$

are verified for each $i = 1, \ldots, k$. This happens, for example, if s satisfies

$$Q^s > \frac{1}{\epsilon}$$

which is perfectly achievable, as long as $Q \ge 2$.

Now the system of congruences

$$x \equiv Q^s b_i \bmod c_i, \quad 1 \le i \le k$$

has solutions by the Chinese remainder theorem, and we can choose (infinitely many) such solutions as big as we want. Let n be such a solution, big enough for all the inequalities

$$n > \max\left\{-\frac{b_i}{a_i} \mid i = 1, \ldots, k\right\},$$

$$n > \max\left\{\frac{1 + c_i}{\epsilon} \mid i = 1, \ldots, k\right\},$$

and

$$n > \max\left\{\frac{b_i + c_i}{\epsilon c_i - a_i} \mid i = 1, \ldots, k\right\},$$

to hold. According to these inequalities, we have

$$0 < \frac{a_i n + b_i}{c_i} < \epsilon n - 1 \;\text{ and }\; c_i < \epsilon n - 1$$

hence

$$0 < \frac{a_i n + b_i}{c_i} < \lfloor \epsilon n \rfloor \;\text{ and }\; c_i < \lfloor \epsilon n \rfloor$$

for any $i = 1, \ldots, k$. Say that all these inequalities hold for n bigger than some positive integer N_1.

Observe that once (for the given numbers a_i, b_i) we fixed Q and s, and thus the numbers c_i, the values of n for which two of the numbers

$$c_1, \dots, c_k, \frac{a_1 n + b_1}{c_1}, \dots, \frac{a_k n + b_k}{c_k}$$

can be equal are only finitely many (possibly none; this is because all of $\frac{a_1}{c_1}, \dots, \frac{a_k}{c_k}$ are positive and distinct; actually, if, say, $a_1 < \cdots < a_k$, then $\frac{a_1}{c_1} < \cdots < \frac{a_k}{c_k}$). So, there exist some positive integer N_2 such that all the above numbers are pairwise distinct for every $n > N_2$.

Finally note that, n being a solution of the above system of congruences, we have

$$n \equiv Q^s b_i \bmod c_i \Leftrightarrow a_i n \equiv (a_i Q^s) b_i \equiv -b_i \bmod c_i, \ \ 1 \le i \le k,$$

so every $\frac{a_i n + b_i}{c_i}$ $(1 \le i \le k)$ is an integer (the equivalence is due to the fact that a_i and c_i are relatively prime for any i, and to the fact that $a_i Q^s \equiv -1 \bmod c_i$). Thus, for a solution n of the system of congruences, also satisfying $n > \max\{N_1, N_2\}$ (and there are infinitely many such solutions), all the numbers

$$c_1, \dots, c_k, \frac{a_1 n + b_1}{c_1}, \dots, \frac{a_k n + b_k}{c_k}$$

are distinct positive integers among $1, \dots, \lfloor \epsilon n \rfloor$; consequently, their product

$$c_1 \cdots c_k \frac{a_1 n + b_1}{c_1} \cdots \frac{a_k n + b_k}{c_k} = (a_1 n + b_1) \cdots (a_k n + b_k)$$

is a divisor of $(\lfloor \epsilon n \rfloor)!$, as desired, and the problem is solved.

O622. Determine all positive integers n for which the numbers $1, 2, \ldots, n$ can be written on a paper in such an order that for each $k = 1, 2, \ldots, n$ the sum of the first k numbers is a multiple of k.

Proposed by Josef Tkadlec, Czech Republic

Solution by Khakim Egamberganov, Uzbekistan

Assume that for some $n \geq 4$ there exists such permutation of the numbers $\{1, 2, \ldots, n\}$. The sum of all numbers $1, 2, \ldots, n$ should be divisible by n, i.e. n divides $\frac{n(n+1)}{2}$. So, $\frac{n+1}{2}$ is an integer, and n should be an odd number.

Let α be the last number on the paper, written in such order. Then, we have that

$$\frac{n(n+1)}{2} - \alpha = \frac{n+1}{2}(n-1) + \frac{n+1}{2} - \alpha$$

should be divisible by $n-1$. Since n is an odd number, we get that $n+1$ and $n-1$ are even numbers, and $n-1$ must divide the number $\frac{n+1}{2} - \alpha$. This implies $\alpha = \frac{n+1}{2}$.

Now, let β be the number written before α. Then, we know that $\frac{n(n+1)}{2} - \alpha - \beta$ should be divisible by $n-2$, that is $n-2$ must divide

$$\frac{(n-2)(n+1)}{2} + n + 1 - \alpha - \beta.$$

As $n+1$ is an even number, we get that $n-2$ divides $n+1-\alpha-\beta$. Since $\alpha = \frac{n+1}{2}$, it leads to $\frac{n+1}{2} - \beta$ should be divisible by $n-2$. We know that $|\frac{n+1}{2} - \beta| \leq \frac{n-1}{2} < n-2$ (since $n \geq 4$), and from which we get $\beta = \frac{n+1}{2}$. This contradicts the fact that $\frac{n+1}{2}$ has already been placed by α.

Thus, $n < 4$ and since n should be an odd positive integer, we find that $n = 1$ and $n = 3$ satisfy the given condition.

Also solved by Ivan Hadinata, Jember, Indonesia; Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Theo Koupelis, Clark College, WA, USA.

O623. Prove that there is a positive integer n and a list of bases $b_1, b_2, \ldots, b_{2022}$ such that n is a 2023-palindrome in each of the bases $b_1, b_2 \ldots, b_{2022}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Choose any positive integer m that has more than $2(2022)+1$ positive divisors. Let $1 = d_1 < d_2 < \cdots < d_{2022}$ be the smallest 2022 divisors of m. Letting $a_i = (d_i)^{2022}$ we can also ensure that $d_i \leq \lfloor \sqrt{m} \mid -1$. and $a_i \leq \lfloor \sqrt{m} \rfloor^{d-1} - 1$. Let

$$n = \binom{2022}{1011}^{2022} m^{2022^2}$$

Defining $b_i = \sqrt[2022]{\frac{n}{a_i}} - 1 = \binom{2022}{1011}\frac{m^{2022}}{d_i} - 1$.We claim the number

$$n = (\binom{2022}{2022}a_i, \binom{2022}{2021}a_i, \binom{2022}{2020}a_i, \ldots, \binom{2022}{1}a_i, \binom{2022}{0}a_i)b_i$$

is indeed the desired one.

Notice that it is a palindrome and has 2023 digits. Further,

$$n = \sum_{j=0}^{2022} \binom{2022}{j} a_i b_i^j = a_i(1+b_i)^{2022} = n$$

It remains to prove that all the digits, i.e., $\binom{2022}{2022}a_i, \binom{2022}{2021}a_i, \binom{2022}{2020}a_i, \ldots, \binom{2022}{1}a_i, \binom{2022}{0}a_i$ are smaller than b_i. In doing so, it suffices to show that the largest one, i.e., $\binom{2022}{1011}a_i$ is smaller than $b_i = \binom{2022}{1011}\frac{m^{2022}}{d_i} - 1$. Since $a_i \leq \lfloor \sqrt{m} \rfloor^{d-1} - 1$ the result follows.

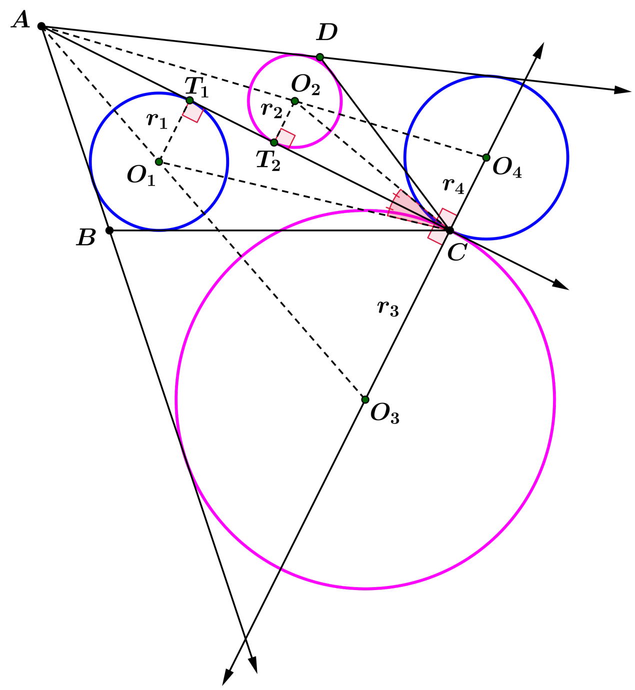
O624. Let $ABCD$ be a convex quadrilateral with $\angle BCA = \angle DCA$. Let r_1 and r_2 be the inradii of triangles ABC and ACD, respectively. Let r_3 be the radius of a circle that passes through C and is tangent to rays AC and AB. Similarly, let r_4 be the radius of a circle that passes through C and is tangent to rays AC and AD. Prove that

$$\frac{1}{r_1} + \frac{1}{r_4} = \frac{1}{r_2} + \frac{1}{r_3}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Kousik Sett, India

Let $O_1(r_1)$ and $O_2(r_2)$ be the incircles of the triangles ABC and ACD respectively, $O_3(r_3)$ be the circle that passes through C and is tangent to rays AC and AB, $O_4(r_4)$ be the circle that passes through C and is tangent to rays AC and AD. T_1 and T_2 be the orthogonal projections of O_1 and O_2, respectively, on AC. Join O_1, T_1; O_2, T_2; C, O_1; C, O_2.



Since $O_1(r_1)$ and $O_3(r_3)$ are tangent to rays AB and AC, therefore, A, O_1, and O_3 are collinear. Join A, O_3. Again, since $O_1(r_2)$ and $O_4(r_4)$ are tangent to rays AC and AD, therefore, A, O_2, and O_4 are collinear. Join A, O_4.

It is given that $\angle BCA = \angle DCA$. Therefore,

$$\angle T_1CO_1 = \angle ACO_1 = \frac{1}{2}\angle BCA = \frac{1}{2}\angle DCA = \angle ACO_2 = \angle T_2CO_2.$$

Also, $\angle O_1T_1C = \angle O_2T_2C = 90^\circ$. Hence $\triangle O_1T_1C \sim \triangle O_2T_2C$. Therefore,

$$\frac{CT_1}{CT_2} = \frac{O_1T_1}{O_2T_2} = \frac{r_1}{r_2} \implies \frac{AC - AT_1}{AC - AT_2} = \frac{r_1}{r_2}.$$

Since circles, $O_3(r_3)$ and $O_4(r_4)$ pass through C and tangent to AC, therefore, $O_3O_4 \perp AC$. Also, $O_1T_1 \perp AC$ and $O_2T_2 \perp AC$. Hence $\triangle AO_1T_1 \sim \triangle AO_3C$ and $\triangle AO_2T_2 \sim \triangle AO_4C$.
Hence

$$\frac{AT_1}{AC} = \frac{O_1T_1}{O_3C} = \frac{r_1}{r_3} \implies \frac{AC - AT_1}{AC} = \frac{r_3 - r_1}{r_3},$$

and

$$\frac{AT_2}{AC} = \frac{O_2T_2}{O_4C} = \frac{r_2}{r_4} \implies \frac{AC - AT_2}{AC} = \frac{r_4 - r_2}{r_4}.$$

Therefore, by the equations above we obtain

$$\frac{r_3 - r_1}{r_3 r_1} = \frac{r_4 - r_2}{r_4 r_2} \implies \frac{1}{r_1} + \frac{1}{r_4} = \frac{1}{r_2} + \frac{1}{r_3}.$$

Also solved by Khakim Egamberganov, Uzbekistan; Theo Koupelis, Clark College, WA, USA.