Abstract. We discuss a version of Newton’s quadrilateral theorem, where
the circle is replaced by a hyperbola. We present two geometric proofs of the
theorem in that case. One of them originates from Newton himself.

1. Introduction

Newton’s famous theorem states that if a circle can be inscribed in a
convex quadrilateral $ABCD$, then the center of the circle lies on the line
joining the midpoints of the diagonals $AC$ and $BD$ of the quadrilateral.

One of the proofs of this theorem (probably the most famous one) uses
the properties of signed areas (we will present this proof later in the text).
By looking at it, we easily come to the conclusion that this theorem can
be formulated and proved in the same way for any quadrilateral $ABCD$,
convex, concave, and even crossed: as long as we assume that there is
a circle tangent to the lines $AB$, $BC$, $CD$, $DA$, its center lies on the line
connecting the midpoints of the segments $AC$ and $BD$.

Moreover, since any affine transformation preserves midpoints, New-
ton’s theorem can be immediately generalized to the case of an ellipse.
The question arises: will Newton’s theorem remain true if we assume that
the lines $AB$, $BC$, $CD$, $DA$ are tangent to a certain hyperbola? Specifically, does the center of the hyperbola then lie on the line connecting the midpoints of $AC$ and $BD$?

Surprisingly, it turns out this is true, despite the fact that a hyperbola cannot be transformed affinely into a circle. Also, the use of a projective transformation, which maps a hyperbola onto a circle, will not bring this configuration to Newton’s theorem with a circle, since the midpoints of $AC$ and $BD$ will not be preserved under such a transformation.

Newton’s theorem with a circle is well-known and can be found in many sources (see for example [1] or Google), but its above version with a hyperbola is (almost) nowhere to be found. Internet search engines only point to one recent paper [3]. It contains a calculational proof of Newton’s theorem for an ellipse or a hyperbola (see Conjecture 1.1 and Theorem 4.3 in [3]) and it refers to only one source [2] — a video on a YouTube channel — where the theorem for an ellipse or a hyperbola was formulated, but not proved. To the best knowledge of R. Kaldybayev — the author of [3], his paper [3] is the first to prove Newton’s theorem for an ellipse and a hyperbola.

In search of a source and information about the version of Newton’s theorem with a hyperbola, I initiated a discussion [5] on this topic in the Facebook group Romantics of Geometry, which brings together over 18,000 geometry enthusiasts from around the world. In response, based on Wikipedia, Fedor Bakharev pointed out that Newton proved his theorem for any conic, but no reference to Newton’s proof was given on Wikipedia. The breakthrough information was provided by Vladimir Dubrovsky that Newton published his proof in his famous work Principia [4]. It turns out that Newton’s book [4] is available on the Internet and I have verified that it indeed contains a proof for an ellipse and a hyperbola ([4] Cor. 3, page 146)!
In this paper, I will present two geometric proofs of Newton's theorem for an ellipse or a hyperbola. One is my own, while the second is based on Newton's original idea. Thanks to the use of theorems that are now widely known, I was able to simplify and write Newton's proof in a very concise way.

1. Signed Areas

The theory presented in this chapter is classical and well-known. I include it here for the convenience of the reader.

To any triangle in the plane we may assign a positive or a negative orientation, depending on the order we list its vertices in. Namely, we say that triangle $ABC$ is positively oriented, if the order $A \rightarrow B \rightarrow C \rightarrow A$ determines a counterclockwise orientation in the plane (Fig. 4); otherwise we say that triangle $ABC$ is negatively oriented (Fig. 5).

If triangle $ABC$ is positively oriented, so are triangles $BCA$ and $CAB$, while triangles $BAC$, $ACB$, and $CBA$ are then negatively oriented.

To each oriented triangle $ABC$ we assign a signed area, which is simply the area of $ABC$, if $ABC$ is positively oriented, or minus the area of $ABC$, if triangle $ABC$ is negatively oriented. We will denote the signed area of oriented triangle $ABC$ by $S(ABC)$. If points $A$, $B$, $C$ are collinear, then we set $S(ABC) = 0$. Therefore, for any triangle $ABC$ we have $S(ABC) = S(BCA) = S(CAB)$, while $S(ABC) = -S(BAC)$.

Sometimes the signed area is more convenient to use than the traditional area. For example, if $ABC$ is any triangle and $X$ is any point in the plane, then


Note that an analogous formula for the traditional area is true only if $X$ lies inside triangle $ABC$. An immediate induction yields now the following

**Theorem 1.1.**

Let $A_1, A_2, \ldots, A_n$ be arbitrary points. Then the sum

$$ \sum_{k=1}^{n} S(A_k A_{k+1} X) $$

does not depend on the choice of point $X$. (We assume $A_{n+1} = A_1$.)
This theorem allows to define a signed area for any oriented closed broken line with consecutive vertices $A_1, A_2, \ldots, A_n$. Namely, we set

$$S(A_1A_2 \ldots A_n) = \sum_{k=1}^{n} S(A_kA_{k+1}X),$$

where $X$ is an arbitrary point. Note that the broken line may be self-intersecting.

In the proof of the next theorem we shall use another useful property of the signed area, which fails for the traditional area. Namely, let $A \neq B$ be points and let $a$ be a real number. Then the set of all points $X$ in the plane, such that $S(ABX) = a$ is a line parallel to line $AB$.

**Theorem 1.2.**

Let $A, B, C, D$ be points such that $ABCD$ is not a parallelogram and let $a$ be a real number. Then the set of all points $X$ in the plane, such that

$$S(ABX) + S(CDX) = a$$

is a line (Fig. 6).

**Proof**

Observe that point $X$ satisfies $S(ABX) + S(CDX) = a$ if and only if it satisfies $S(BCX) + S(DAX) = a'$, where $a' = S(ABCD) - a$. Therefore, since $ABCD$ is not a parallelogram, we may without loss of generality assume that lines $AB$ and $CD$ are not parallel and intersect at $P$.

Let $K$ and $L$ be the points determined by $PK = AB$ and $LP = CD$ (Fig. 7).
Then we obtain

\[ S(ABX) + S(CDX) = S(PKX) + S(LPX) = S(KLP) - S(KLX). \]

Therefore, \( S(ABX) + S(CDX) = a \) if and only if \( S(KLX) = b \), where \( b = S(KLP) - a \). This means point \( X \) lies in a line parallel to line \( KL \).

2. **Proof of Newton’s Theorem for a circle**

We are ready to prove Newton’s Theorem for a circle. The proof we are going to present is classical and well-known.

For simplicity we assume that \( ABCD \) is convex and has an excircle (Fig. 8). The other circular configurations, including the most classical one shown at Figure 1, can be proved analogously.

Let \( J \) be the excenter of \( ABCD \) and let \( K, L \) be the midpoints of diagonals \( AC, BD \), respectively. By Theorem 1.2 it suffices to show that the sum \( S(ABX) + S(CDX) \) attains the same value for \( X = J, K, \) and \( L \).

Choose \( X = L \) first. Then we have

\[ S(ABL) + S(CDL) = \frac{1}{2} S(ABD) + \frac{1}{2} S(CDB) = \frac{1}{2} S(ABCD). \]

Similarly, we show that \( S(ABK) + S(CDK) = \frac{1}{2} S(ABCD) \).

Let \( X = J \). Since \( ABCD \) has an excircle, we have \( DA + AB = BC + CD \), or \( AB - CD = BC - DA \). Therefore, we obtain

\[ S(ABJ) + S(CDJ) = \frac{1}{2} AB \cdot r - \frac{1}{2} CD \cdot r = \frac{1}{2} BC \cdot r - \frac{1}{2} DA \cdot r = S(BCJ) + S(DAJ). \]

But \( S(ABJ) + S(CDJ) + S(BCJ) + S(DAJ) = S(ABCD) \), which implies that

\[ S(ABJ) + S(CDJ) = \frac{1}{2} S(ABCD). \]

This completes the proof.
3. Proof of Newton’s Theorem for a hyperbola (and an ellipse)

The signed area is also used in my proof of Newton’s Theorem for a hyperbola and an ellipse. We start with another application of Theorem 1.2.

Theorem 3.1.

Let $A, B, C, D$ be points such that lines $DA$ and $BC$ are not parallel and intersect at $P$ (Fig. 9). Let $Q$ be any point. Then the midpoints of segments $AC, BD,$ and $PQ$ are collinear if and only if $S(ABQ) = S(DCQ)$.

Proof

Denote by $K, L,$ and $O$ the midpoints of segments $AC, BD,$ and $PQ,$ respectively (Fig. 10). Moreover, let $X, Y$ be symmetric points to $P$ with respect to points $K, L,$ respectively. Then points $K, L, O$ are collinear if and only if point $Q$ lies on line $XY$.

Since $AXCP$ is a parallelogram, we get $S(ABX) = S(ACX) = S(DCX)$, or
$S(ABX) + S(CDX) = 0$. Similarly, we obtain $S(ABY) + S(CDY) = 0$. Therefore, applying Theorem 1.2 for $a = 0$, we infer that point $Q$ lies on line $XY$ if and only if $S(ABQ) + S(CDQ) = 0$, i.e. $S(ABQ) = S(DCQ)$. This completes the proof.

**Theorem 3.2.**

Two tangent lines $a$, $b$ to a hyperbola (or to an ellipse) with center $O$ meet at point $P$. Let $Q$ be a symmetric point to $P$ with respect to $O$ (Fig. 11). A variable tangent line to the hyperbola (or to the ellipse) intersects lines $a$ and $b$ at $X$ and $Y$, respectively. Then the signed area of triangle $QXY$ is constant.

![Fig. 11](image1)

**Proof**

Let $a'$ and $b'$ be symmetric lines to $a$ and $b$, respectively, with respect to $O$ (Fig. 12). Then $a'$ and $b'$ are tangent to the hyperbola (or to the ellipse). Let $C$ be the touch point lying on $a'$. Set $K = a' \cap b$ and $L = a \cap b'$.

![Fig. 12](image2)

Brianchon’s Theorem applied for hexagon $KCQLXY$ implies that lines
KL, CX, and QY are concurrent at some point T. Moreover, since XL || CK and LQ || KY, triangles XLQ and CKY are homothetic (or translated, if T is an infinity point). This yields QX || YC. Therefore, \( S(QXY) = S(QXC) = S(QPC) \), which is constant.

**Theorem 3.3.** (Newton’s Theorem for a hyperbola and an ellipse)

Let \( A, B, C, D \) be points such that lines \( AB, BC, CD, DA \) are tangent to a hyperbola (or to an ellipse) with center \( O \) (Fig. 13). Then the line passing through the midpoints of \( AC \) and \( BD \) passes through \( O \).

![Fig. 13](image13.png)

**Proof**

Set \( P = AB \cap CD \). Moreover, let \( Q \) be the point symmetric to \( P \) with respect to \( O \) (Fig. 14). By Theorem 3.2 we infer that \( S(QBC) = S(QAD) \), which by Theorem 3.1 implies that \( O \) and the midpoints of \( AC \) and \( BD \) are collinear. This completes the proof.

![Fig. 14](image14.png)
4. Newton's Proof of Newton's Theorem

Newton's original proof of Theorem 3.3 (see [4], p. 146) is based on projective properties of a conic. Using today's knowledge and terminology it can be presented in a concise way as follows.

We assume a conic tangent to lines $AB, BC, CD, DA$ is an ellipse or a hyperbola, with center $O$.

![Diagram](image)

Fig. 15

Label the tangents $AB$ and $CD$ by $a$ and $b$, respectively. Let $a', b'$ be lines symmetric to $a, b$, respectively, with respect to $O$. Then $a'$ and $b'$ are also tangent to the conic. Set $P = a \cap b', Q = a' \cap b$. Lines $a, b, a', b'$ bound a parallelogram, so $O$ is the midpoint of $PQ$.

A projection from line $a$ to $b$ through the conic preserves the double ratio, so

$$ (ABP \propto) = (DC \propto Q) = (CDQ \propto), $$

implying

$$ \frac{AP}{PB} = \frac{CQ}{QD}. $$

Thus degenerated triangles $ABP$ and $CDQ$ are similar, so the midpoints of $AC, BD$, and $PQ$ are vertices of a triangle similar to $ABP$ and $CDQ$. In particular, they are collinear, which completes the proof.

Remarks

Of course Newton didn’t use the double ratio and the similar triangles property in his arguments. He derived the special cases of these properties directly and stated them as lemmas.
References


