Junior Problems

J625. The rectangular box $ABCDA'B'C'D'$ has volume 2023 and total area 2550. Given that

$$\frac{1}{AB}+\frac{1}{AD}-\frac{1}{AA'}=\frac{4}{7},$$

find the dimensions of the box.

Proposed by Adrian Andreescu, Dallas, USA

J626. Let a, b, c, d be positive real numbers. Prove that

$$\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\geq 2\sqrt{a^2b^2+b^2c^2+c^2d^2+d^2a^2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J627. Let a, b, c be positive real numbers such that $a+b+c=6$. Prove that

$$\frac{ab}{\sqrt{a^2+3a+6}}+\frac{bc}{\sqrt{b^2+3b+6}}+\frac{ca}{\sqrt{c^2+3c+6}}\leq 3.$$

Proposed by Mihaela Berindeanu, Bucharest, România

J628. Find all positive integers n for which $(n+3)!-n!+7n^3+2023$ is the cube of a prime.

Proposed by Adrian Andreescu, Dallas, USA

J629. Let a, b, c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$bc\sqrt{2a+b+c}+ca\sqrt{2b+c+a}+ab\sqrt{2c+a+b}\geq \frac{2}{\sqrt{a+b+c}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J630. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{7}{a}+\frac{7}{b}+\frac{7}{c}+\frac{16}{a+b}+\frac{16}{b+c}+\frac{16}{c+a}+\frac{27}{a+b+c}\geq 54.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S625. Prove that there are infinitely many positive integers n such that n^2+5 has a proper divisor greater than $8n/5$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S626. Let a, b, c, d, e be nonnegative real numbers such that $ab+bc+cd+de+ea=1$. Prove that

$$3 < \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} + \frac{1}{e+1} \le 4.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

S627. Let a, b, c be positive real numbers. Prove that

$$2a\sqrt{9b^2+16c^2} + 2b\sqrt{9c^2+16a^2} + 2c\sqrt{9a^2+16b^2} +$$

$$+15abc\left(\frac{1}{2b+3c} + \frac{1}{2c+3a} + \frac{1}{2a+3b}\right) \ge 13(ab+bc+ca).$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

S628. Prove that there are infinitely many positive integers n such that precisely two of the numbers $n-2$, $n+2$, $5(n-2)$, $5(n+2)$ are perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S629. Let $ABCD$ be a rectangle with area $[ABCD]$ and O a point in its plane. Prove that

$$|OA \cdot OC - OB \cdot OD| \le [ABCD] \le OA \cdot OC + OB \cdot OD.$$

Proposed by Jozsef Tkadlec, Czech Republic

S630. Let $n \ge 2$ and $x_1, x_2, \ldots, x_n$ be real numbers, not all zero and adding up to zero. Moreover, for each positive real number t there are at most $1/t$ pairs (i, j) such that $|x_i - x_j| \ge t$. Prove that

$$x_1^2 + \cdots + x_n^2 < \frac{1}{n}\left(\max_{1\le i\le n} x_i - \min_{1\le i\le n} x_i\right).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Undergraduate Problems

U625. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi\sin\frac{\pi k}{2n}}{2n(\sin\frac{\pi k}{2n}+\cos\frac{\pi k}{2n})}.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

U626. Let $P(x)=99x^6+3x^5+x^4+2x^3+4x^2-1$. Prove that there is a prime q such that $P(q)=(q-1)!$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U627. Find the best constant c such that for any polynomial with real coefficients $f(x)$ satisfying

$$\iint_{[0,1]\times[0,1]}(f(x)-f(y))^2dxdy=1,$$

the function $g(x)=x(1-x)f'(x)$ is Lipschitz with constant bounded by $c(\deg f)^3$.

Proposed by Gabriel Dospinescu, E.N.S. Lyon, France

U628. Let S be the set of symmetric rational functions with real coefficients in two variables x,y. Find all positive integers a,b such that the set of rational functions with real coefficients in x^a+y^a, x^b+y^b coincides with S.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U629. Let $D=\{z\in\mathbb{C}:|z|<1\}$ and let $f:D\longrightarrow\mathbb{C}$ be a holomorphic function such that $|f(z)|\le 1$ and $f(0)=0$. Prove that for $z\neq 0$ the following inequality holds

$$\frac{|f(z)|}{|z|(1+|f'(0)|)}+\frac{|z||f'(0)|}{|z|+|f(z)|}\le 1.$$

Proposed by Alessandro Ventullo, Milan, Italy

U630. Evaluate

$$\int_{-\infty}^{+\infty}\frac{(\tanh y)(\sinh y)^2}{2+(\sinh(2y))^2}y^3\,dy$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Olympiad Problems

O625. Find all primes p such that

$$\frac{(p-2)!-1}{p^2}=\frac{2(p^4+3p^2-9)}{p-1}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O626. Prove that there are infinitely many triples (a,b,k) of positive integers such that

$$\frac{a+1}{b}+\frac{b+1}{a}=k$$

and find all possible values of k.

Proposed by Mircea Becheanu, Canada

O627. Prove that $k=3$ is the largest value of the positive constant k such that

$$\frac{1}{ab+k}+\frac{1}{bc+k}+\frac{1}{cd+k}+\frac{1}{da+k}\geq\frac{4}{1+k}$$

holds for all nonnegative real numbers a,b,c,d satisfying the condition $ab+ac+ad+bc+bd+cd=6$.

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

O628. Let x,y,z be positive real numbers such that $x^2+y^2+z^2+xyz=4$. Prove that

$$(x+y+z-2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{2}\right)\geq\frac{5}{2}.$$

Proposed by Marius Stănean, Zalău, România

O629. Let a,b,c,d be positive integers such that $4(a^2+b^2)=5(c^2+d^2)$ and $ad-bc$ divides c^2+d^2. Prove that

$$2(ac+bd)=L_{2n+1}|ad-bc|$$

for some nonnegative integer n, where L_k is the k^{th} Lucas number defined by $L_0=2$, $L_1=1$ and $L_{k+1}=L_k+L_{k-1}$, $k=1,2,3,\ldots$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O630. Prove that there are infinitely many positive integers m such that $m+1, 2m+1, 3m+1$ are all composite and divide 2^m-1.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran