A Triangle Ratio Theorem and its Generalization to Higher Dimensions

Hari Rajesh

Ridgewood High School, Ridgewood, New Jersey

Abstract: In this paper, I discuss an interesting theorem involving a triangle and lines through a point parallel to the sides of the triangle, including both a geometric and a coordinate-based proof of the theorem. Later, I discuss and prove a generalization in higher dimensions. I conclude with a conjecture about a further generalization.

1 Introduction

I first noticed this 4 years ago while I was still in middle school. I made a little document to try and prove it, but I quickly forgot about it. I decided to look through my old stuff one day and reopened the document. I began to look into it and have since proven the theorem and generalized it.

2 A Statement of the Theorem

Take any triangle $\triangle ABC$ and an internal point P. Construct three line segments parallel to the sides of the triangle through P. This will create a diagram similar to the following:

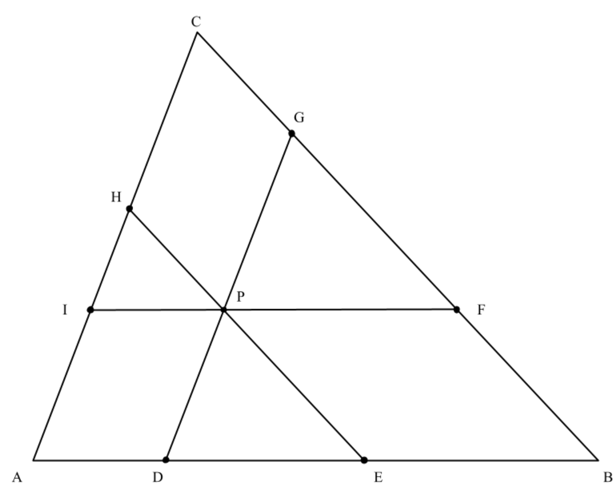

Figure 1: Triangle ABC with point P

The sum of the ratios between a segment and the corresponding parallel side of the triangle is 2:

$$\frac{\overline{IF}}{\overline{AB}} + \frac{\overline{EH}}{\overline{BC}} + \frac{\overline{GD}}{\overline{CA}} = 2 \tag{1}$$

Given that this involves ratios in a triangle, I will call it the Triangle Ratio Theorem.

3 Proof

First, notice that $\triangle DBG \sim \triangle ABC$. This implies that $\frac{\overline{DB}}{\overline{AB}} = \frac{\overline{GD}}{\overline{CA}}$. Then, notice that $\overline{IP} = \overline{AD}$ because they are parallel sides in the parallelogram $ADPI$. Thus, $\frac{\overline{IP}}{\overline{AB}} = \frac{\overline{AD}}{\overline{AB}}$. Next, we shall use the fact that $\frac{\overline{AD}}{\overline{AB}} + \frac{\overline{DB}}{\overline{AB}} = 1$. We can then use the previously found equalities to get that $\frac{\overline{IP}}{\overline{AB}} + \frac{\overline{GD}}{\overline{CA}} = 1$.

We may now repeat the same reasoning to get a second equation. $\triangle AEH \sim \triangle ABC$, so $\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{EH}}{\overline{BC}}$. Being parallel sides of parallelogram $EBFP$, $\overline{PF} = \overline{EB}$, so $\frac{\overline{PF}}{\overline{AB}} = \frac{\overline{EB}}{\overline{AB}}$. Since $\frac{\overline{AE}}{\overline{AB}} + \frac{\overline{EB}}{\overline{AB}} = 1$, and given the previous equalities, $\frac{\overline{PF}}{\overline{AB}} + \frac{\overline{EH}}{\overline{BC}} = 1$.

Adding the two equations gives us $\frac{\overline{IP}}{\overline{AB}} + \frac{\overline{PF}}{\overline{AB}} + \frac{\overline{EH}}{\overline{BC}} + \frac{\overline{GD}}{\overline{CA}} = 2$, and since $\frac{\overline{IP}}{\overline{AB}} + \frac{\overline{PF}}{\overline{AB}} = \frac{\overline{IF}}{\overline{AB}}$, we get the desired equation:

$$\frac{\overline{IF}}{\overline{AB}} + \frac{\overline{EH}}{\overline{BC}} + \frac{\overline{GD}}{\overline{CA}} = 2$$

3.1 Related Results

There are a few related results of the Triangle Ratio Theorem apart from the main one. The following three equations are all true:

$$\frac{\overline{DE}}{\overline{AB}} + \frac{\overline{FG}}{\overline{BC}} + \frac{\overline{HI}}{\overline{CA}} = 1$$
$$\frac{\overline{EB}}{\overline{AB}} + \frac{\overline{GC}}{\overline{BC}} + \frac{\overline{IA}}{\overline{CA}} = 1$$
$$\frac{\overline{AD}}{\overline{AB}} + \frac{\overline{BF}}{\overline{BC}} + \frac{\overline{CH}}{\overline{CA}} = 1$$

Each of these three equations can be proven either directly from the Triangle Ratio Theorem or independently. I shall leave the proofs to the reader, except for the first one, which I shall prove in Section 6.2.5 when discussing generalizations to the theorem.

4 Generalization: What if P is Outside of the Triangle?

P does not have to lie in the triangle for this to work. However, we need to make a few modifications in order to allow this to happen. First, we must extend the sides of the triangle by considering the whole line because otherwise, the segments would not hit anything, so no distance calculation could be made. The other modification is to turn the segments into vectors. Why? Return to Figure 1. Notice that point I is to the left of point F. However, if P is above point C, this will flip, so point F will be to the left of point I. This means that $\frac{\overline{IF}}{\overline{AB}}$ needs to be subtracted from the sum. This can be done easily using vectors. This negative sign is quite important.

In Figure 2, the full lines are not shown for simplicity. Let's take vectors $\overrightarrow{IF}$ and $\overrightarrow{AB}$. If they are pointing in the same direction, we add $\frac{\overrightarrow{IF}}{\overrightarrow{AB}}$ but if they point in the opposite direction, we subtract $\frac{\overrightarrow{IF}}{\overrightarrow{AB}}$. This can be easily accomplished using the dot product.

Recall the following identity involving the dot product:

Given vectors $\mathbf{u}$ and $\mathbf{v}$,
$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Thus, if $\overrightarrow{IF}$ and $\overrightarrow{AB}$ are parallel, $\cos(\theta)$ will be 1 and if they are anti-parallel, $\cos(\theta)$ will be -1. This gets us the desired sign, so we can just multiply it by the ratio and add them all up.

Let $\mathbf{u_n}$ be one of the vectors through P and let $\mathbf{v_n}$ be the parallel/anti-parallel side on the triangle. Using the previous formula multiplied by the ratio, we get the following:

$$\frac{\mathbf{u_n}\cdot\mathbf{v_n}}{\|\mathbf{u_n}\|\|\mathbf{v_n}\|}\frac{\|\mathbf{u_n}\|}{\|\mathbf{v_n}\|} = \frac{\mathbf{u_n}\cdot\mathbf{v_n}}{\|\mathbf{u_n}\|^2}$$

We can then get the following equation which we will need to prove:

$$\sum \frac{\mathbf{u_n}\cdot\mathbf{v_n}}{\|\mathbf{u_n}\|^2} = 2$$

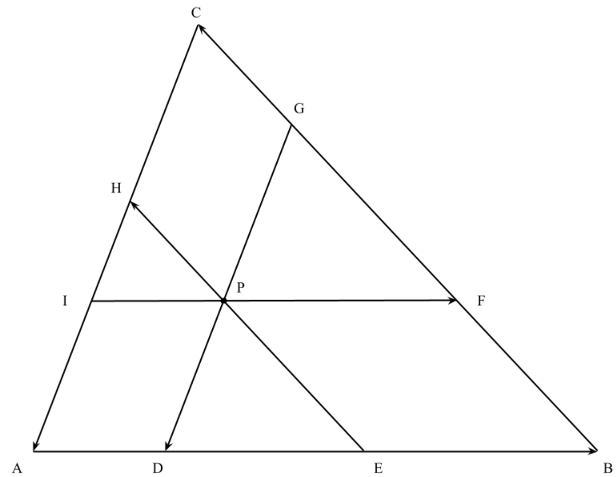

Figure 2: Triangle ABC with point P using vectors instead of segments

5 Proof of the Generalization

The first step in proving this generalization will be to find a way to remove vectors from the equation. This is not that difficult. Since $\mathbf{u_n}$ and $\mathbf{v_n}$ are either parallel or anti-parallel, they are scalar multiples of each other. If you define the scalar k_n as $\frac{\mathbf{u_n}}{\mathbf{v_n}}$, then we are simply looking for $\sum k_n$ and showing that it equals 2.

The second step is to define the triangle in terms of points on the coordinate plane. You could define it as (x_a, y_a), (x_b, y_b), and (x_c, y_c), but this is computationally inefficient and unnecessary. Instead, we can assume without loss of generality that A lies at $(0, 0)$ and B lies at $(1, 0)$. We can do this because any translation of the triangle will maintain the ratios and since we are dealing with length, scaling the triangle does not change the ratios. We thus get the following diagram:

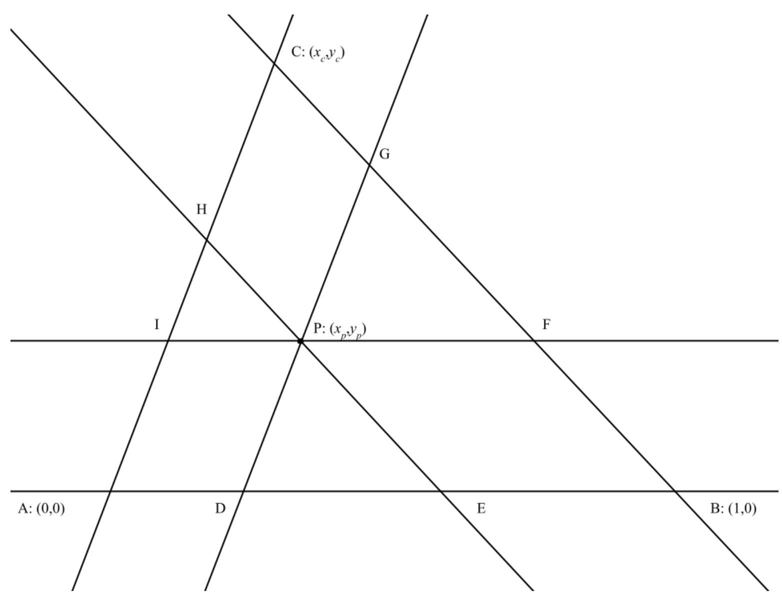

Figure 3: Triangle ABC with point P using lines

Note that while Figure 3 appears the same as any of the other triangles, P can lie anywhere on the plane. In order to find the ratio of $\overline{IF}$ to $\overline{AB}$, we can simply use the fact that $\frac{\overline{IF}}{\overline{AB}} = \frac{\overline{FC}}{\overline{BC}}$, following the logic from the original proof. In this case, however, we can simply use a ratio of the y-coordinates of P and C because $\overline{IF}$ is parallel to the x-axis. This ratio works out to $\frac{y_c - y_p}{y_c}$. You can easily verify that this gives us the desired sign change when

$y_p > y_c$, which is the main goal of this generalization.

To do the other ratios, we will need to find equations for the lines through $\overline{EH}$ and $\overline{GD}$. This will allow us to find the intersection point with the line through $\overline{AB}$. From there, we can use the following two facts to find the ratios: $\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{EH}}{\overline{BC}}$ and $\frac{\overline{DB}}{\overline{AB}} = \frac{\overline{GD}}{\overline{CA}}$.

For the line through $\overline{EH}$, we'll use the slope of the parallel $\overline{BC}$, which is $\frac{y_c}{x_c-1}$. Using that and the point-slope form through P yields the following equation: $y = \frac{y_c}{x_c-1}(x - x_p) + y_p$. Doing the same for the line through $\overline{GD}$ yields the following equation: $y = \frac{y_c}{x_c}(x - x_p) + y_p$.

Now, we can solve for the x-value of the intersection between these lines and the line through $\overline{AB}$, which will be the x-values of points D and E, by simply setting $y = 0$. Doing this yields the following two results: $x_d = x_p - \frac{y_p x_c}{y_c}$ and $x_e = x_p - \frac{y_p(x_c-1)}{y_c}$. To find the ratios, we can first calculate $\overline{AE}$, $\overline{DB}$, and $\overline{AB}$. By definition, $\overline{AB} = 1$, which simplies the computations. $\overline{AE}$ simply equals $x_e - x_a = x_e$, so $\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{EH}}{\overline{BC}} = x_e = x_p - \frac{y_p(x_c-1)}{y_c}$. $\overline{DB}$ is $x_b - x_d = 1 - x_d$, so $\frac{\overline{DB}}{\overline{AB}} = \frac{\overline{GD}}{\overline{CA}} = 1 - x_d = 1 - x_p + \frac{y_p x_c}{y_c}$. You can verify that this also results in the necessary sign changes when P is beyond any of the corners of the triangle.

Now we can calculate $\frac{\overline{IF}}{\overline{AB}} + \frac{\overline{EH}}{\overline{BC}} + \frac{\overline{GD}}{\overline{CA}}$. First, we can expand and get a common denominator on all of the terms: $\frac{\overline{EH}}{\overline{BC}} = \frac{y_c x_p - y_p x_c + y_p}{y_c}$ and $\frac{\overline{GD}}{\overline{CA}} = \frac{y_c - y_c x_p + y_p x_c}{y_c}$. Now we can simply add.

$$\begin{aligned}
\frac{\overline{IF}}{\overline{AB}} + \frac{\overline{EH}}{\overline{BC}} + \frac{\overline{GD}}{\overline{CA}} &= \frac{y_c - y_p}{y_c} + \frac{y_c x_p - y_p x_c + y_p}{y_c} + \frac{y_c - y_c x_p + y_p x_c}{y_c} \\
&= \frac{y_c - \cancel{y_p + y_p} + \cancel{y_p x_c - y_p x_c} + \cancel{y_c x_p - y_c x_p} + y_c}{y_c} \\
&= \frac{2y_c}{y_c} = 2
\end{aligned}$$

Thus, we get the desired result. One easily identifiable error is when $y_c = 0$, but this makes sense as this would create a triangle that is a straight line, so any parallel lines that aren't collinear to the "triangle" would never intersect. There is a second issue that must be dealt with, however, when x_c is 0 or 1; this is because some of the slopes put x_c or $x_c - 1$ in the denominator. Notice, however, that neither scaling nor rotating the triangle affect the ratios. Thus, we can simply rotate/scale the triangle as necessary to create a similar triangle that has a different side on the x-axis and $x_c \neq 0$ or 1, as can be seen in Figure 4.

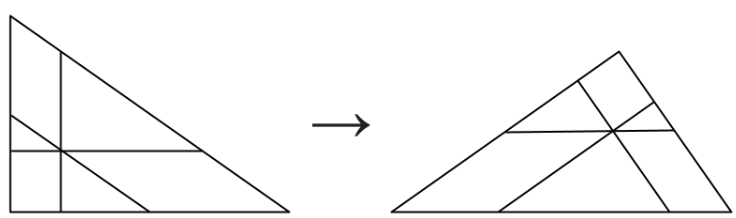

Figure 4: Transformation of a right triangle

6 Generalizing to Higher Dimensions

The seemingly obvious next step forward is to generalize the result to a tetrahedron. Take a tetrahedron $ABCD$ and an internal point P, as is shown in Figure 5.

Now, construct four planes through P, each parallel to a face of the tetrahedron. You might assume we can take the ratio of the areas, but if we do this, we would need to take the root of the ratio. Instead, we can just find the scaling factor. If you do this and add it up, you will find that the sum is 3. We will prove this later.

The next question is whether this applies to the higher dimensional cases. The fact that the 2- and 3-dimensional cases both work in a similar manner suggests that it will work. In fact, it also works in the 1-dimensional case, though trivially.

Figure 6 clearly shows that $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$. If P falls outside of the segment, then one can simply subtract as necessary. In any case, the reasoning behind this should require no proof. In the 0-dimensional case, where there is only a point, it is difficult to properly determine a notion of scale, so it would be difficult to conclude whether the theorem applies or not. However, this leads me to a more generalized theorem for any dimension ≥ 1.

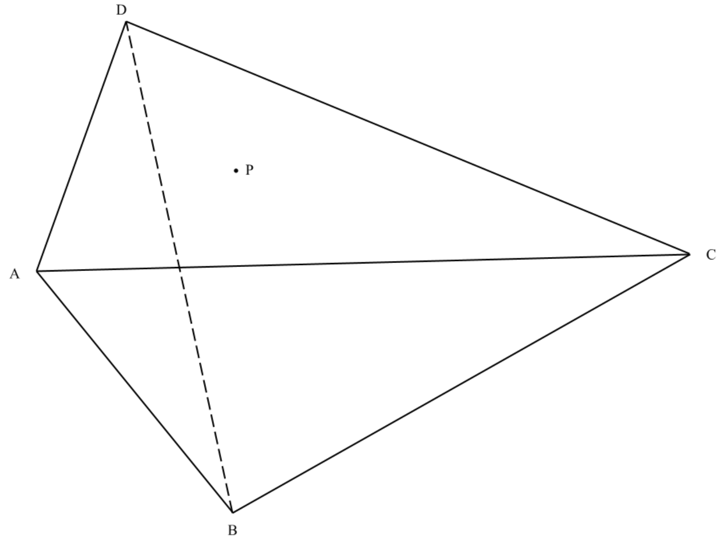

Figure 5: Tetrahedron $ABCD$ with point P

Figure 6: Segment $\overline{AB}$ with point P

6.1 The Simplex Ratio Theorem

In an n-dimensional space, take any n-simplex whose vertices are non-cohyperplanar, and an interior point P. Let u_i be a hyperplane through P parallel to v_i, an facet of the simplex. If the ratio of u_i and v_i is taken to be the scaling factor between them, then the following is true:

$$\sum_i \frac{u_i}{v_i} = n \tag{2}$$

I will now briefly explain what the theorem is saying in a bit more detail. First, a simplex is simply a generalization of a triangle to higher dimensions. Here is a little chart:

n-simplex	Common Name
0-simplex	Point
1-simplex	Line
2-simplex	Triangle
3-simplex	Tetrahedron
4-simplex	5-cell

A hyperplane is simply a generalization of a plane to higher dimensions. In 4-dimensions, for instance, a hyperplane is a 3-dimensional object. The Triangle Ratio Theorem fails when the triangle is just a line because the parallel lines through P never intersect the "triangle." Thus, the contraint that the vertices of the simplex are non-cohyperplanar must be made.

Facet refers to the generalization of the sides of a triangle. Triangles have sides, tetrahedrons have faces, 5-cells have cells, and so on, which are all considered facets of the n-simplex.

The scaling factor means the ratio between the hyperplane and the parallel facet while the fact that it is signed refers to the fact that it takes on a negative sign if the orientation of the hyperplane through P is opposite to that of the parallel facet.

The theorem states that the sum of those signed scaling factors should equal the number of dimensions. So far, I have shown it to be true in the 1- and 2-dimensional cases, but by the end of the paper, I will have proven it.

6.2 Lemmas

Before I prove the theorem, I must first provide a few lemmas. Most of these are necessary to proving the theorem while the first one is an important observation.

6.2.1 Lemma 1: When P is at a Vertex

If P is at a vertex, then the proof of the theorem is easy because each hyperplane is identical in size to their parallel facet (so the ratio for each one is 1), except for the facet opposite the vertex, whose ratio is 0. Since there are $n+1$ facets in an n-simplex, the ratio would add up to $n+1-1=n$. Thus, we have proven it for every n-simplex when P is at a vertex.

6.2.2 Lemma 2: When P is on a Facet, an (n-1)-case is Formed

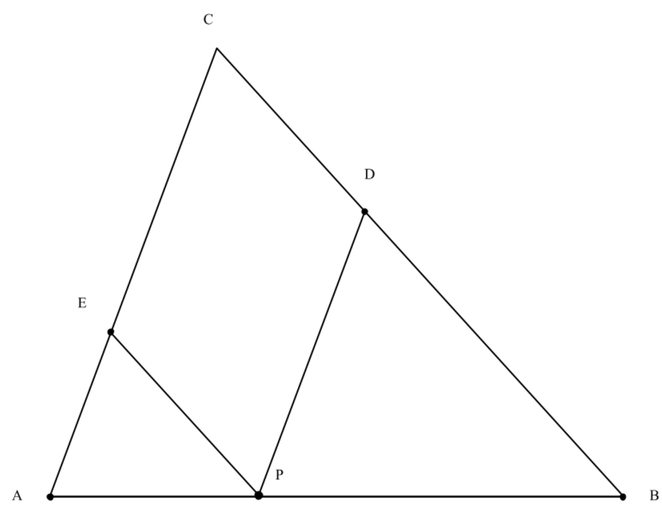

Figure 7: Triangle ABC with point P on $\overline{AB}$

Notice in Figure 7 that the bottom side is a 1-dimensional case, where $\frac{\overline{AP}}{\overline{AB}}+\frac{\overline{PB}}{\overline{AB}}=\frac{\overline{AB}}{\overline{AB}}=1$. This also holds for the 3-dimensional case, as can be seen in Figure 8. Notice that face ABD with point P forms the 2-dimensional case with lines parallel to $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

A proof of this is quite simple. For an n-simplex, where there are $n+1$ facets, there are $n+1$ hyperplanes through P. However, if P lies on a facet, then only n of those hyperplanes are non-cohyperplanar with the simplex. These hyperplanes all go through P and are parallel to the facets of the hyperplane through P. This is equivalent to having an $(n-1)$-simplex and hyperplanes through P parallel to the facets of that simplex, which is the $(n-1)$-case.

6.2.3 Lemma 3: A Hyperplane Through P Parallel to a Facet Forms an (n-1)-case

This is actually easiest to notice in the 3-dimensional case. Refer to Figure 9.

Notice that the intersection of plane KLM and the other planes forms a shadow that is identical to a 2-dimensional case. This is also true of the 2-dimensional case, where a line through P forms a 1-dimensional case. This naturally follows from Lemma 2. If you were to only consider the part of tetrahedron above plane KLM, you would end up with P being on the face of the tetrahedron $DKLM$, which we know forms the $(n-1)$-case from Lemma 2.

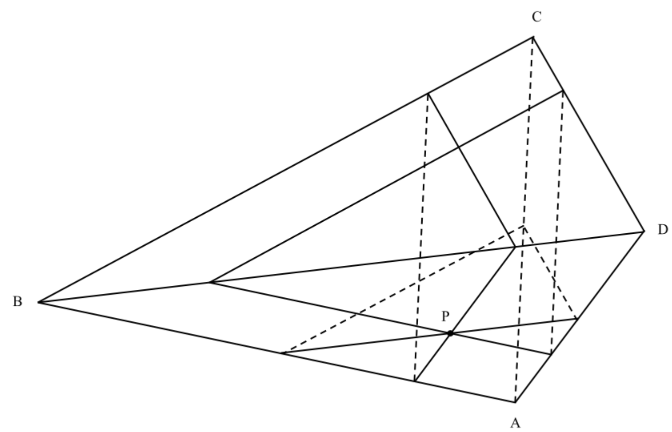

Figure 8: Tetrahedron $ABCD$ with point P on facet ABD

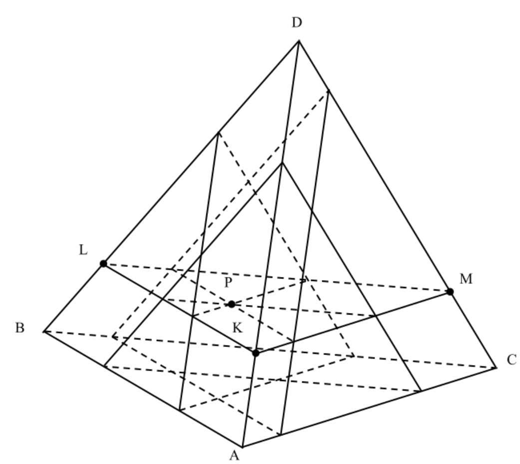

Figure 9: Tetrahedron ABCD with Point P and Plane KLM

6.2.4 Lemma 4: The Desired Sum is the Number of Facets Minus the Ratio of the Central Sections

In Figure 10, the central sections are segments $\overline{DE}$, $\overline{FG}$, and $\overline{HI}$. In higher dimensions, it works similarly. In Figure 11, the central sections are the outer faces of the small tetrahedra shown.

To prove the lemma, let's look at Figure 10. Notice that $\triangle ABC \sim \triangle IFC \sim \triangle DEP$. Additionally, notice the height of $\triangle DEP$ + the height of $\triangle IFC$ = the height of $\triangle ABC$. Thus, a linear quantity, such as the side lengths, of $\triangle DEP$ + the corresponding quantity of $\triangle IFC$ yields the corresponding quantity of $\triangle ABC$, by similar triangles. We can easily rearrange this and apply it to get the following: $\overline{IF} + \overline{DE} = \overline{AB}$. This is true of the other sides and this method can generalize to higher dimensions.

For example, in 3-dimensions, refer to Figures 9 and 11. In Figure 11, we shall use the bottom tetrahedron as an example. Notice that it, tetrahedron $KLMD$ in Figure 9, and tetrahedron $ABCD$ are all similar, and that the height of the bottom tetrahedron + the height of $KLMD$ yields the height of $ABCD$. Thus, as they are similar by a linear factor, all linear quantities such as side lengths of the bottom tetrahedron + tetrahedron $KLMD$ yields that of $ABCD$. Thus, the sum of the ratios of the central sections + the sum of the ratios of the hyperplanes, which is what we want, = the sum of the ratios of the facets themselves. The ratio of a facet to itself is simply 1,

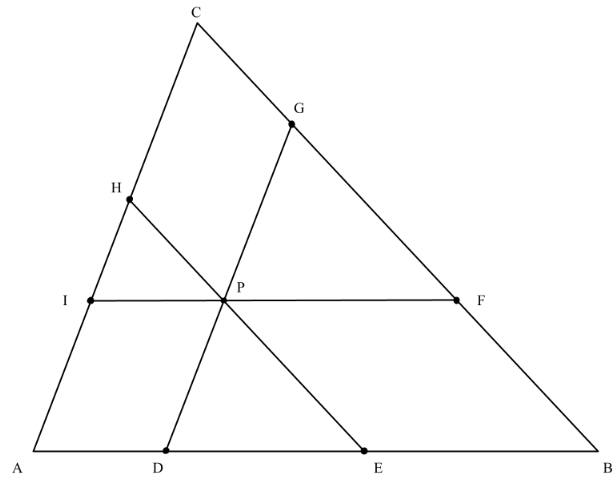

Figure 10: Triangle ABC with point P

so the sum of these ratios is the number of facets. Rearranging gives us that the desired sum of the ratios of the hyperplanes to their respective facets = the number of facets − the sum of the ratios of the central sections.

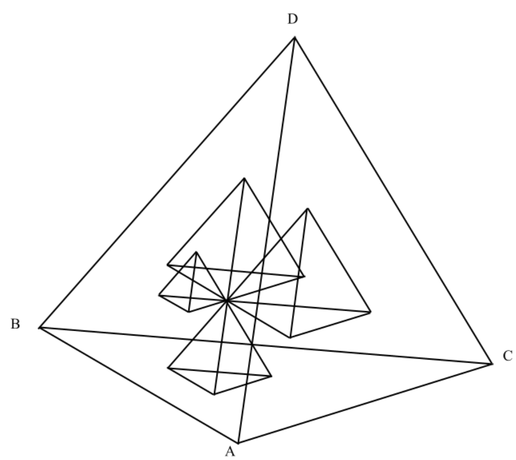

Figure 11: Tetrahedron $ABCD$ and Point P with Smaller Tetrahedra

6.2.5 Lemma 5: The Sum of the Ratios of the Central Sections to their Respective Sides is 1

In the 2-dimensional case, it follows naturally from the original theorem, as I mentioned in Section 3.1, when discussing related results of the Triangle Ratio Theorem. However, it will be important to start by proving this first, so here is a proof that doesn't use the Triangle Ratio Theorem as a prior step. More importantly, it also generalizes to higher dimensions extremely easily.

Return to Figure 10. First, notice the following:

$$\frac{\overline{IH}}{\overline{IC}} = \frac{\overline{IP}}{\overline{IF}}$$
$$\frac{\overline{GF}}{\overline{CF}} = \frac{\overline{PF}}{\overline{IF}}$$

This is due to similar triangles. We can then do the following:

$$\frac{\overline{IH}}{\overline{IC}} + \frac{\overline{GF}}{\overline{CF}} = \frac{\overline{IP}}{\overline{IF}} + \frac{\overline{PF}}{\overline{IF}} = 1$$

Next, notice that $\frac{\overline{IF}}{\overline{AB}} = \frac{\overline{CF}}{\overline{BC}} = \frac{\overline{IC}}{\overline{AC}}$, again by similar triangles. Thus, we can do the following:

$$\begin{aligned}\frac{\overline{IF}}{\overline{AB}} &= \left(\frac{\overline{IH}}{\overline{IC}} + \frac{\overline{GF}}{\overline{CF}}\right)\left(\frac{\overline{IF}}{\overline{AB}}\right) \\ &= \frac{\overline{IH}}{\overline{IC}}\frac{\overline{IF}}{\overline{AB}} + \frac{\overline{GF}}{\overline{CF}}\frac{\overline{IF}}{\overline{AB}} \\ &= \frac{\overline{IH}}{\overline{IC}}\frac{\overline{IC}}{\overline{AC}} + \frac{\overline{GF}}{\overline{CF}}\frac{\overline{CF}}{\overline{BC}} \\ &= \frac{\overline{IH}}{\overline{AC}} + \frac{\overline{GF}}{\overline{BC}}\end{aligned}$$

Now if we add in $\frac{\overline{DE}}{\overline{AB}}$, we get the following:

$$\frac{\overline{IH}}{\overline{AC}} + \frac{\overline{GF}}{\overline{BC}} + \frac{\overline{DE}}{\overline{AB}} = \frac{\overline{IF}}{\overline{AB}} + \frac{\overline{DE}}{\overline{AB}}$$

We already know $\frac{\overline{IF}}{\overline{AB}} + \frac{\overline{DE}}{\overline{AB}} = 1$ from Lemma 4, so we get our desired result:

$$\frac{\overline{IH}}{\overline{AC}} + \frac{\overline{GF}}{\overline{BC}} + \frac{\overline{DE}}{\overline{AB}} = 1 \tag{3}$$

The general process is similar in 3-dimensions. Return to Figure 11. You can prove that the sum of the scaling factor of the tetrahedrons is equal to 1 using similar logic. It revolves around the earlier fact that the plane through P forms the 2-dimensional case. Return to Figure 8 to see this more clearly. The sum of the central sections in plane KLM is 1. We can multiply both sides by, for instance, $\frac{\overline{KL}}{\overline{AB}}$. Then, adding in the last tetrahedra and using Lemma 4, we can get that the sum of the ratios of the inner sections is 1.

Thus, the n-dimensional case relies on the $(n-1)$-dimensional case. If we have an n-simplex and we look at just one of the hyperplanes through P, we get the $(n-1)$-case. Assuming that the $(n-1)$-case is true, then we can multiply by the scaling factor between the hyperplane and the parallel facet and add the remaining smaller n-simplex. By Lemma 4, this gives us 1. This sets us up for a proof by induction, which I shall now complete.

6.3 The Proof of the Simplex Ratio Theorem

First, take an n-simplex and a point P and construct hyperplanes through P parallel to the facets of the n-simplex. Assume that the $(n-1)$-case holds.

By Lemma 5, if the $(n-1)$-case holds, then the sum of the ratio of the center sections to their respective facets is 1 in the n-case. By Lemma 4, the desired ratio is the number of facets minus the sum of the ratios of the center sections. In an n-simplex, there are $n+1$ facets, so the desired ratio is $n+1-1=n$.

Given that the n-case is true if the $(n-1)$-case is true, and since it is true for n=2, then it is true for every n-simplex by induction.

7 Points Outside of the Simplex

The purely geometric proof applies to all internal points of the simplex. For external points, however, things are not as straightforward. You could simply define the scaling factor to be signed, but the geometric proof doesn't definitively prove that it works, though I am confident it is provable. A proof analogous to the coordinate-based one shown earlier is likely the best candidate, though it would be far more complex in higher dimensions, unfortunately.

Another thing I must mention is that while you can certainly just define the scaling factor to be signed, when dealing with computations or demonstrations, such as through Geogebra, there must be a way to calculate the sign. Using vectors still works in higher dimensions, though there are a few methods of using them. One of these is to use a unit vector normal to the facet and another unit vector normal to the hyperplane, then use the dot product of those vectors. If orientation is preserved by these vectors, then this will yield the correct sign. Another option is to simply have one vector lie on edge of the simplex while another one lies on the hyperplane parallel or anti-parallel to the first one, depending on the required sign. Then, the dot product of these vectors, divided by the product of the magnitudes, yields the sign required, which is analogous to the 2-dimensional case.

Again, this has not been proven, but given the fact that the 2-dimensional case has been proven and the n-dimensional case has been proven for internal points, I believe it can be done, so I will call it the General Simplex Ratio Conjecture.