Junior problems

J625. The rectangular box $ABCDA'B'C'D'$ has volume 2023 and total area 2550. Given that

$$\frac{1}{AB}+\frac{1}{AD}-\frac{1}{AA'}=\frac{4}{7},$$

find the dimensions of the box.

Proposed by Adrian Andreescu, Dallas, USA

Solution by Jodie Burdick, SUNY Brockport, NY, USA

Let $AB=x, AD=y, AA'=z$.

Thus the volume is xyz and the total area is $2(xy+xz+yz)$.

This means that $xyz=2023$ and $2(xy+xz+yz)=2550 \iff xy+xz+yz=1275$ and so

$$\frac{xy+xz+yz}{xyz}=\frac{1275}{2023} \iff \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1275}{2023} \iff \frac{1}{x}+\frac{1}{y}=\frac{1275}{2023}-\frac{1}{z}$$

We also know that

$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{4}{7} \iff \frac{1}{x}+\frac{1}{y}=\frac{4}{7}+\frac{1}{z}$$

Hence

$$\frac{4}{7}+\frac{1}{z}=\frac{1275}{2023}-\frac{1}{z} \iff \frac{2}{z}=\frac{1275}{2023}-\frac{4}{7} \iff \frac{2}{z}=\frac{1}{17} \iff z=34$$

$xyz=2023$ implies that

$$xy=\frac{2023}{34}=\frac{119}{2}$$

Moreover

$$\frac{1}{x}+\frac{1}{y}=\frac{4}{7}+\frac{1}{z} \iff \frac{x+y}{xy}=\frac{143}{238} \iff x+y=\frac{143}{238}xy=\frac{143}{238}\frac{119}{2}=\frac{143}{4}$$

Hence x and y are the solutions of the quadratic equation

$$t^2-\frac{143}{4}t+\frac{119}{2}=0 \iff 4t^2-143t+238=0$$

with solutions $t_1=34, t_2=7/4$.

Therefore the solutions of the problem are

$$x=34, \quad y=7/4, \quad z=34$$

and

$$x=7/4, \quad y=34, \quad z=34$$

Also solved by Soham Dutta, West Bengal, India; Jimi Nguyen, Brockport College, Brockport NY, USA; Matthew Too, Brockport, NY, USA; Vishwesh Ravi Shrimali, Jaipur, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Kay Yeon Lee, Choate Rosemary Hall, Wallingford, CT, USA; Austin Lee, The Lawrenceville School, NJ, USA; Soham Bhadra, India; Daniel Pascuas, Barcelona, Spain; Theo Koupelis, Clark College, WA, USA.

J626. Let a, b, c, d be positive real numbers. Prove that

$$\frac{bcd}{a} + \frac{acd}{b} + \frac{abd}{c} + \frac{abc}{d} \geq 2\sqrt{a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2}$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Daniel Pascuas, Barcelona, Spain

Since $\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d} = abcd(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2})$, the inequality can be rewritten as

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \geq 2\sqrt{\frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{c^2d^2} + \frac{1}{d^2a^2}},$$

or equivalently

$$x + y + z + t \geq 2\sqrt{xy + yz + zt + tx} \qquad (x, y, z, t > 0),$$

which directly follows from the AM-GM inequality since $xy + yz + zt + tx = (x + z)(y + t)$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Soham Dutta, West Bengal, India; Konstantinos Nakis, Athens, Greece; Henry Ricardo, Westchester Area Math Circle; Arkady Alt, San Jose, CA, USA; Jiang Lianjun,Quanzhou Middle School,Guilin,China; CFAL Problem Solving Group, Centre for Advanced Learning, Mangalore, India; Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia; Marian Ursărescu, Romania; Kay Yeon Lee, Choate Rosemary Hall, Wallingford, CT, USA; Austin Lee, The Lawrenceville School, NJ, USA; Soham Bhadra, India; Alexis Norcross, SUNY Brockport, NY, USA; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Sundaresh H R, Shivamogga, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Theo Koupelis, Clark College, WA, USA.

J627. Let a, b, c, d be positive real numbers. Prove that

$$\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\geq 2\sqrt{a^2b^2+b^2c^2+c^2d^2+d^2a^2}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let

$$w=\frac{bcd}{a},\quad x=\frac{cda}{b},\quad y=\frac{dab}{c},\quad \text{and}\quad z=\frac{abc}{d}.$$

Then

$$yz=a^2b^2,\quad zw=b^2c^2,\quad wx=c^2d^2,\quad \text{and}\quad xy=d^2a^2,$$

and the desired inequality becomes

$$w+x+y+z\geq 2\sqrt{yz+zw+wx+xy}.$$

Now, square both sides of this inequality and combine like terms to obtain

$$(w-x+y-z)^2\geq 0,$$

which is clearly true.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Arkady Alt, San Jose, CA, USA; Jiang Lianjun,Quanzhou Middle School,Guilin,China; Kay Yeon Lee, Choate Rosemary Hall, Wallingford, CT, USA; Soham Dutta, India; Alexander Lee; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Suvankar Saha, Indian Statistical Institute, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Batakogias Panagiotis, High School of Velestino, Greece; Henry Ricardo, Westchester Area Math Circle; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Theo Koupelis, Clark College, WA, USA.

J628. Find all positive integers n for which

$$(n+3)! - n! + 7n^3 + 2023$$

is a cube of a prime number.

Proposed by Adrian Andreescu, Dallas, USA

Solution by CFAL Problem Solving Group, Centre for Advanced Learning, Mangalore, India
Let

$$S(n) = (n+3)! - n! + 7n^3 + 2023.$$

Since 7 is a factor of 2023, if $n \geq 7$, then $S(n)$ is a multiple of 7. The only prime cube that is a multiple of 7 is 7^3. But

$$[(n+3)! - n!] + 7n^3 + 2023 > 343$$

for $n \geq 7$ since the first two terms (i.e. $[(n+3)! - n!], 7n^3$) are non-negative.

Thus we must have $n < 7$. If we try the numbers $n = 1, 2, 3, 4, 5, 6$, we see that only $n = 2$ works for which the prime is 13.

Also solved by Soham Dutta, West Bengal, India; Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia; Kay Yeon Lee, Choate Rosemary Hall, Wallingford, CT, USA; Austin Lee, The Lawrenceville School, NJ, USA; Soham Bhadra, India; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Sundaresh H R, Shivamogga, India; Theo Koupelis, Clark College, WA, USA.

J629. Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$bc\sqrt{2a+b+c}+ca\sqrt{2b+c+a}+ab\sqrt{2c+a+b}\geq\frac{2}{\sqrt{a+b+c}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

First, we have

$$\left(\sum_{\text{cyc}}bc\sqrt{2a+b+c}\right)^2\left(\sum_{\text{cyc}}\frac{bc}{\sqrt{2a+b+c}}\right)\geq(bc+ca+ab)^3=1.$$

On the other hand, the Cauchy-Schwarz inequality also yields

$$\begin{aligned}\sum_{\text{cyc}}\frac{bc}{\sqrt{2a+b+c}}&\leq(bc+ca+ab)\left(\sum_{\text{cyc}}\frac{bc}{2a+b+c}\right)\\&=\sum_{\text{cyc}}\frac{bc}{(a+b)+(a+c)}\\&\leq\sum_{\text{cyc}}\frac{bc}{4}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\\&=\frac{a+b+c}{4}.\end{aligned}$$

Combining these two ralations we get the desired inequality.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Batakogias Panagiotis, High School of Velestino, Greece; Jiang Lianjun,Quanzhou Middle School,Guilin,China; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Theo Koupelis, Clark College, WA, USA.

J630. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{7}{a}+\frac{7}{b}+\frac{7}{c}+\frac{16}{a+b}+\frac{16}{b+c}+\frac{16}{c+a}+\frac{27}{a+b+c} \geq 54.$$

Proposed by Marius Stănean, Zalău, România

Solution by Theo Koupelis, Clark College, WA, USA

Without loss of generality let $a \geq b \geq c$, and

$$f(a,b,c) := \frac{7}{a}+\frac{7}{b}+\frac{7}{c}+\frac{16}{a+b}+\frac{16}{b+c}+\frac{16}{c+a}+\frac{27}{a+b+c}.$$

For $b = c$ and $ab^2 = 1$ we get

$$\begin{aligned} f(a,b,b) &= \frac{7}{a}+\frac{22}{b}+\frac{32}{a+b}+\frac{27}{a+2b} = 7b^2+\frac{22}{b}+\frac{32b^2}{1+b^3}+\frac{27b^2}{1+2b^3} \\ &= \frac{2(7b^9+78b^6+66b^3+11)}{b(1+b^3)(1+2b^3)}, \end{aligned}$$

and thus

$$f(a,b,b) \geq 54 \Longleftrightarrow 2(b-1)^4(7b^5+28b^4+16b^3+2b^2+17b+11) \geq 0,$$

which is obvious. Also,

$$f(a,b,c)-f(a,b,b) = (b-c)\cdot\left[\frac{7}{bc}+\frac{16}{2b(b+c)}+\frac{16}{(a+b)(a+c)}+\frac{27}{(a+b+c)(a+2b)}\right] \geq 0.$$

Thus, $f(a,b,c) \geq 54$, with equality when $a = b = c = 1$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Austin Lee, The Lawrenceville School, NJ, USA; Arkady Alt, San Jose, CA, USA; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Senior problems

S625. Prove that there are infinitely many positive integers n such that $n^2 + 5$ has a proper divisor greater than $8n/5$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Use the identity

$$(L_m)^2 - L_{m-1}L_{m+1} = 5(-1)^{m-1},$$

where $(L_m)_{m\geq 0}$ is the Lucas sequence: $L_0 = 2, L_1 = 1, L_{m+1} = L_m - L_{m-1}, m = 1, 2, 3, \dots$ to get

$$(L_{2k})^2 + 5 = L_{2k-1}L_{2k+1}.$$

Take $n = L_{2k}$ and use the fact that $L_{m+1} > \frac{8}{5}L_m$, for $m \geq 5$, which can be proven by an easy strong induction.

Second solution by Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia

Let's call $n \in \mathbb{N}$ good if n^2+5 has a proper divisor greater than $\frac{8n}{5}$. We will show that all odd positive integers are good.

If $n \in \{1, 3\}$ then $\frac{n^2+5}{2}$, which is the greatest proper divisor of $n^2 + 5$, is clearly greater than $\frac{8n}{5}$.

If odd $n \geq 5$, let $n = 2k + 1$, $k \geq 2$. We still consider $\frac{n^2+5}{2} = 2k^2 + 2k + 3$ as the greatest proper divisor of $n^2 + 5$. In fact, $\forall k \geq 2$, we have

$$\frac{8n}{5} = \frac{16k + 8}{5} \leq 4k < 4k + 3 + 2k(k - 1) = \frac{n^2 + 5}{2}.$$

The proof is complete.

Also solved by Soham Dutta, West Bengal, India; Kay Yeon Lee, Choate Rosemary Hall, Wallingford, CT, USA; Austin Lee, The Lawrenceville School, NJ, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Theo Koupelis, Clark College, WA, USA; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Chayan Kumar Mandal, Malda, West Bengal, India; Sundaresh H R, Shivamogga, India.

S626. Let a, b, c, d, e be nonnegative real numbers such that $ab + bc + cd + de + ea = 1$. Prove that

$$3 < \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} + \frac{1}{e+1} \le 4.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

Solution by the author

Let $(x_1, x_2, x_3, x_4, x_5)$ be a permutation of (a, b, c, d, e) such that $x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5$. Due to symmetry, the desired inequality is equivalent to

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \frac{1}{x_3+1} + \frac{1}{x_4+1} + \frac{1}{x_5+1} \le 4,$$

that can be written in the form

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} + \frac{x_3}{x_3+1} + \frac{x_4}{x_4+1} + \frac{x_5}{x_5+1} \ge 1.$$

For $x_1 x_2 \ge 1$, the inequality is true because

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} - 1 = \frac{x_1 x_2 - 1}{(x_1+1)(x_2+1)} \ge 0.$$

Consider further $x_1 x_2 \le 1$. For $x_3 + x_4 + x_5 > 0$, by the Cauchy-Schwarz inequality we have

$$\frac{x_3}{x_3+1} + \frac{x_4}{x_4+1} + \frac{x_5}{x_5+1} \ge \frac{(x_3+x_4+x_5)^2}{x_3(x_3+1) + x_4(x_4+1) + x_5(x_5+1)}$$

$$\ge \frac{(x_3+x_4+x_5)^2}{(x_3+x_4+x_5)^2 + x_3+x_4+x_5} = \frac{x_3+x_4+x_5}{x_3+x_4+x_5+1} = 1 - \frac{1}{x_3+x_4+x_5+1}\ ,$$

hence

$$\frac{x_3}{x_3+1} + \frac{x_4}{x_4+1} + \frac{x_5}{x_5+1} \ge 1 - \frac{1}{x_3+x_4+x_5+1}.$$

We can see that this inequality is also true for $x_3 = x_4 = x_5 = 0$. So, it suffices to prove that

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} - \frac{1}{x_3+x_4+x_5+1} \ge 0.$$

By Lemma below, we have

$$x_3 + x_4 + x_5 \ge \frac{1 - x_1 x_2}{x_1 + x_2},$$

hence

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} - \frac{1}{x_3+x_4+x_5+1} \ge \frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} - \frac{1}{\frac{1-x_1x_2}{x_1+x_2}+1}$$

$$= \frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} - \frac{x_1+x_2}{1 - x_1x_2 + x_1 + x_2} = \frac{2x_1x_2(1-x_1x_2)}{(x_1+1)(x_2+1)(1-x_1x_2+x_1+x_2)} \ge 0.$$

The proof is completed. The equality is an equality for $ab = 1$ and $c = d = e = 0$ (or any cyclic permutation).

Lemma: Let a, b, c, d, e be nonnegative real numbers, and let $(x_1, x_2, x_3, x_4, x_5)$ be a permutation of (a, b, c, d, e) such that $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$. Then,

$$x_1x_2 + (x_1 + x_2)(x_3 + x_4 + x_5) \geq ab + bc + cd + de + ea.$$

Proof: Assume that $a = \max\{a, b, c, d, e\}$, hence $a = x_1$ and $x_2 + x_3 + x_4 + x_5 = b + c + d + e$. Since

$$x_1x_2 + (x_1 + x_2)(x_3 + x_4 + x_5) = x_1(x_2 + x_3 + x_4 + x_5) + x_2(x_3 + x_4 + x_5)$$

$$= a(b + c + d + e) + x_2(x_3 + x_4 + x_5) \geq a(b + c + d + e) + x_2x_3,$$

it suffices to show that

$$a(b + c + d + e) + x_2x_3 \geq ab + bc + cd + de + ea,$$

that is

$$a(c + d) + x_2x_3 \geq bc + cd + de,$$

which is equivalent to the obvious inequality

$$c(a - b) + d(a - c) + (x_2x_3 - de) \geq 0.$$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S627. Let a, b, c be positive real numbers. Prove that

$$2a\sqrt{9b^2+16c^2}+2b\sqrt{9c^2+16a^2}+2c\sqrt{9a^2+16b^2}+$$
$$+15abc\left(\frac{1}{2b+3c}+\frac{1}{2c+3a}+\frac{1}{2a+3b}\right)\geq 13(ab+bc+ca)$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution by the author

$$\sqrt{9b^2+16c^2}\geq 3b+4c-\frac{10bc}{2b+3c}\iff\frac{24bc(b-c)^2}{(2b+3c)^2}\geq 0$$

hence we come to prove

$$2\sum_{\text{cyc}}(3ab+4ac)-20abc\sum_{\text{cyc}}\frac{1}{2b+3c}+15abc\sum_{\text{cyc}}\frac{1}{2b+3c}\geq 13(ab+bc+ca)$$

that is

$$ab+bc+ca\geq 5abc\left(\frac{1}{2b+3c}+\frac{1}{2c+3a}+\frac{1}{2a+3b}\right)$$

The AGM yields

$$ab+bc+ca\geq 5abc\left(\frac{1}{5b^{2/5}c^{3/5}}+\frac{1}{5c^{2/5}a^{3/5}}+\frac{1}{5a^{2/5}b^{3/5}}\right)=b^{3/5}c^{2/5}a+c^{3/5}a^{2/5}b+a^{3/5}b^{2/5}c$$

and the AGM again finishes the proof

$$(ab+ab+ab+ac+ac)/5\geq ab^{3/5}c^{2/5}\text{ and cyclic}$$

Also solved by Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S628. Prove that there are infinitely many positive integers n such that precisely two of the numbers $n-2$, $n+2$, $5(n-2)$, $5(n+2)$ are perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

We start with two observations. For $n > 2$,

- $n-2$ and $n+2$ cannot simultaneously be perfect squares; and
- if $n-2$ is a perfect square, then $5(n-2)$ cannot be a perfect square as well.

Thus, it suffices to show that there are infinitely many positive integers n such that $n-2$ and $5(n+2)$ are perfect squares. Suppose $n-2 = m^2$ and $5(n+2) = \ell^2$ for positive integers m and ℓ. This leads to the Pell equation $\ell^2 - 5m^2 = 20$, which has solutions

$$(\ell_k, m_k) = (5F_{2k-1}, L_{2k-1})$$

for $k = 1, 2, 3, \ldots$, where F_k and L_k are the kth Fibonacci number and the kth Lucas number, respectively. Thus, for $n = L_{2k-1}^2 + 2$ with $k = 1, 2, 3, \ldots$, $n-2$ and $5(n+2)$ are perfect squares.

Also solved by Soham Dutta, West Bengal, India; Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia; Theo Koupelis, Clark College, WA, USA; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Batakogias Panagiotis, High School of Velestino, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S629. Let $ABCD$ be a rectangle with area $[ABCD]$ and O a point in its plane. Prove that

$$|OA \cdot OC - OB \cdot OD| \le [ABCD] \le OA \cdot OC + OB \cdot OD.$$

Proposed by Jozsef Tkadlec, Czech Republic

Solution by Theo Koupelis, Clark College, WA, USA

Let the center K of the rectangle be at the origin of a Cartesian coordinate system so that $A = (-a, b)$, $B = (-a, -b)$, $C = (a, -b)$, and $D = (a, b)$, where $a, b > 0$. Then $[ABCD] = 4ab$. If $O = (x, y)$, then using Cauchy-Schwarz we get

$$OA \cdot OC = \sqrt{(x+a)^2 + (y-b)^2}\sqrt{(x-a)^2 + (y+b)^2} \ge |(x+a)(y+b)| + |(x-a)(y-b)|,$$

and

$$OB \cdot OD = \sqrt{(x+a)^2 + (y+b)^2}\sqrt{(x-a)^2 + (y-b)^2} \ge |(x+a)(y-b)| + |(x-a)(y+b)|.$$

Thus,

$$OA \cdot OC + OB \cdot OD \ge (|x+a| + |x-a|)(|y+b| + |y-b|)$$

Setting $f(x) = |x+a| + |x-a|$, we easily find that $f(x) = 2|x|$ when $|x| \ge a$, and $f(x) = 2a$ when $|x| < a$. Similarly, setting $g(y) = |y+b| + |y-b|$, we find $g(y) = 2|y|$ when $|y| \ge b$, and $g(y) = 2b$ when $|y| < b$. Thus, in all cases $OA \cdot OC + OB \cdot OD \ge 4ab = [ABCD]$.

The inequality on the left-hand-side is equivalent to

$$OA \cdot OC + OB \cdot OD \ge \frac{|OA^2 \cdot OC^2 - OB^2 \cdot OD^2|}{4ab} \Longleftrightarrow$$
$$OA \cdot OC + OB \cdot OD \ge \frac{|16abxy|}{4ab} = 4|xy|,$$

which is obvious from the above analysis of the behavior of the functions $f(x)$ and $g(y)$. Both equalities occur when O is one of the vertices of the rectangle. The equality on the right-hand-side also occurs when $O = K$ in the case when $a = b$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S630. Let $n \geq 2$ and $x_1, \ldots, x_n$ be real numbers, not all zero, adding up to zero. Moreover, for each positive real number t there are at most $\frac{1}{t}$ pairs (i, j) such that $\mid x_i - x_j \mid \geq t$. Prove that

$$x_1^2 + \cdots + x_n^2 < \frac{1}{n}\left(\max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i\right).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Notice that $n\sum_{i=1}^n x_i^2 = (\sum_{i=1}^n x_i)^2 + \sum_{1\leq i<j\leq n}(x_i - x_j)^2$. It then suffices to prove

$$\sum_{1\leq i<j\leq n}(x_i - x_j)^2 \leq \max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i.$$

Let us denote $\{|x_i - x_j| : 1 \leq i < j \leq n\} = \{y_1, \ldots, y_m\}$, then assume $y_1 < y_2 < \cdots < y_m$. Then assume that the total number of pairs (i, j) such that $|x_i - x_j| = y_k$ is a_k. Letting $t = y_k$ then there are at most $\frac{1}{y_k}$ pairs (i, j) such that $|x_i - x_j| \geq t$. Hence,

$$a_k + a_{k+1} + \cdots + a_m \leq \frac{1}{y_k}.$$

Letting $S_k = a_k + a_{k+1} + \cdots + a_m$ it follows that $S_k \leq \frac{1}{y_k}$.
Hence,

$$\sum_{1\leq i<j\leq n}(x_i - x_j)^2 = \frac{1}{2}\sum_{k=1}^m a_k y_k^2 = \frac{1}{2}\sum_{k=1}^m S_k(y_k^2 - y_{k-1}^2) \leq \frac{1}{2}\sum_{k=1}^m \frac{y_k^2 - y_{k-1}^2}{y_k}$$

Putting $y_0 = 0$ and $y_{k-1} < y_k$ it follows that $\frac{y_k^2 - y_{k-1}^2}{y_k} < 2(y_k - y_{k-1})$. Hence,

$$\sum_{1\leq i<j\leq n}(x_i - x_j)^2 < 2\sum_{k=1}^m (y_k - y_{k-1}) = y_m = \max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i.$$

Also solved by Daniel Pascuas, Barcelona, Spain.

U625. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi\sin\frac{\pi k}{2n}}{2n(\sin\frac{\pi k}{2n}+\cos\frac{\pi k}{2n})}.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by Henry Ricardo, Westchester Area Math Circle

Rewriting the given summation, we recognize it as a right Riemann sum so that

$$\begin{aligned}\lim_{n\to\infty}\frac{\pi}{2}\cdot\sum_{k=1}^{n}\frac{\sin\frac{\pi}{2}\cdot\frac{k}{n}}{\sin\frac{\pi}{2}\cdot\frac{k}{n}+\cos\frac{\pi}{2}\cdot\frac{k}{n}}\cdot\frac{1}{n} &= \frac{\pi}{2}\int_0^1\frac{\sin\frac{\pi x}{2}}{\sin\frac{\pi x}{2}+\cos\frac{\pi x}{2}}\,dx\\ &\overset{u=\pi x/2}{=} \int_0^{\pi/2}\frac{\sin u}{\sin u+\cos u}\,du\\ &= \frac{1}{2}\left(\int_0^{\pi/2}\frac{\sin u+\cos u}{\sin u+\cos u}\,du - \int_0^{\pi/2}\frac{\cos u-\sin u}{\sin u+\cos u}\,du\right)\\ &= \frac{\pi}{4}.\end{aligned}$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia; Daniel Pascuas, Barcelona, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Theo Koupelis, Clark College, WA, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Etisha Sharma, Agra College, Agra, India; Matthew Too, Brockport, NY, USA; Soham Dutta, India; Sundaresh H R, Shivamogga, India; Suvankar Saha, Indian Statistical Institute, India; Swastika Dey, India; Toyesh Prakash Sharma, Agra College, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Prajnanaswaroopa S, Bengaluru, Karnataka, India.

U626. Let

$$P(x) = 99x^6 + 3x^5 + x^4 + 2x^3 + 4x^2 - 1.$$

Prove that there is a prime q such that $P(q) = (q-1)!$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Theo Koupelis, Clark College, WA, USA

For $q \geq 19$ we have $(q-1)! = 5! \cdot 6 \cdot 7 \cdots (q-2)(q-1) > 100q^6$, because $5! > 100$ and $k(q-k+1) > q \Leftrightarrow q > k$, where $k \in \{6, 7, 8, 9, 10, 11\}$. But for any prime $p \geq 11$ we have $99p^6 + 3p^5 + p^4 + 2p^3 + 4p^2 - 1 < 100p^6$ because $p^6 \geq 11p^5 > (3+1+2+4)p^5 > 3p^5+p^4+2p^3+4p^2-1$. Therefore, there is no solution for $q \geq 19$. On the other hand, it is clear that $P(x) > 99x^6$ for any positive integer, and thus $P(q) > 99 \cdot 2^6 = 6336$, while $(8-1)! = 7! = 5040$. Therefore, there is no solution for $q \leq 7$. Thus, we only need examine the cases $q = 11, 13, 17$. By substituting, we find that $P(q) = (q-1)!$ only for $q = 13$.

Also solved by Soham Dutta, West Bengal, India; Sophia Tiffany, SUNY Brockport, NY, USA; Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia; Soham Bhadra, India; Juan José Granier, Universidad de Chile, Chile; Prajnanaswaroopa S, Bengaluru, Karnataka, India; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U627. The author claims that he had a nice solution but the margin was too small :)

U628. Let $Q(x,y)$ be the set of rational functions with rational coefficients and S be the set of symmetric rational functions with rational coefficients. Find all a,b such that

$$Q(x^a+y^a,x^b+y^b)=S$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let $N_a=x^a+y^a$. Notice that if P_1,P_2 are members of $Q(N_a,N_b)$ then $\frac{P_1}{P_2},P_1+P_2,P_i^k$ are members of $Q(N_a,N_b)$. Moreover, all symmetric expressions in two variables can be written as a rational function of $x+y,xy$. So, it suffices to check whether or not $x+y,xy$ can be constructed or not.

Indeed, $S_1=x+y,S_2=xy$. Notice that $S_2=\frac{N_1^2-N_2}{2}$. So, $Q(N_2,N_1)=S$. Further, since $N_1^3-N_3=3N_1S_2$ implies $S_2=\frac{N_1^3-N_3}{3N_1}$. Thus, $Q(N_1,N_3)=S$.

Now, assume $b\geq a\geq 2$. Assume $Q(N_a,N_b)=S$ thus, $x+y=S$. Hence, there must be coprime polynomials P_1,P_2 such that

$$\frac{P_1(x^a+y^a,x^b+y^b)}{P_2(x^a+y^a,x^b+y^b)}=x+y$$

If $\gcd(a,b)=d>1$ then the map $x\rightarrow\omega x$ where ω is the primitive $d-$ th root of unity preserves the left while changes the right. Hence, we can assume that $\gcd(a,b)=1$. Let ξ be the primitive $a-$ th root of $-1;\xi^a=-1$ then the map $y\rightarrow\xi x$ yields;

$$P_1(0,(\xi^b+1)x^b)=(\xi+1)xP_2(0,(\xi^b+1)x^b)$$

Since $a\neq 1,\gcd(a,b)=1$ it follows that $\xi+1,\xi^b+1\neq 0$. Let α be the primitive b - th root of unity, that is $\alpha^b=1$ the map $x\rightarrow\alpha x$ preserves everything except x. It follows that for each x;

$$P_1(0,(\xi^b+1)x^b)=P_2(0,(\xi^b+1)x^b)=0.$$

But then, P_1,P_2 will have common factors. This is impossible. Thus, $x+y$ can not be represented as such.

Finally, if $a=1,b\geq 4$ we shall prove that xy can not be constructed. On the contrary, assume there are coprime polynomials P_1,P_2 such that

$$\frac{P_1(x+y,x^b+y^b)}{P_2(x+y,x^b+y^b)}=xy.$$

That is, $P_1(x+y,x^b+y^b)=xyP_2(x+y,x^b+y^b)$. Let β be some of $b-$ th root of -1, that is, $\beta^b=-1$ such that $\beta\neq-1$.

Then, the map $y\rightarrow\beta x$ leads to the following expression

$$P_1((\beta+1)x,0)=\beta x^2P_2((\beta+1)x,0).$$

That is, $P_1(x,0)=\frac{\beta}{(\beta+1)^2}x^2P_2(x,0)$. This means that for all $b-1\geq 3$ roots of $z^b=-1$ such that $z\neq-1$ the fraction $\frac{\beta}{(\beta+1)^2}$ must be constant. But then, the equation $\frac{x}{(x+1)^2}=C$ must have $b-1$ roots, however, it can at most have two distinct roots. This is a contradiction.

U629. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : D \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \le 1$ and $f(0) = 0$. Prove that for $z \ne 0$ the following inequality holds

$$\frac{|f(z)|}{|z|(1+|f'(0)|)} + \frac{|z||f'(0)|}{|z|+|f(z)|} \le 1.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Theo Koupelis, Clark College, WA, USA

Clearing denominators, the desired inequality is equivalent to

$$|f(z)|^2 - |f(z)| \cdot |z||f'(0)| - |z|^2(1-|f'(0)|^2) \le 0.$$

Thus, solving the quadratic in $|f(z)|$, it is sufficient to show that

$$\frac{1}{2}\left(|z||f'(0)| - |z|\sqrt{4-3|f'(0)|^2}\right) \le |f(z)| \le \frac{1}{2}\left(|z||f'(0)| + |z|\sqrt{4-3|f'(0)|^2}\right).$$

But from the given conditions and the Schwarz lemma we get that $|f(z)| \le |z|$ for all $z \in \mathbb{D}$, and $|f'(0)| \le 1$. Clearly then $|z||f'(0)| \le |z|\sqrt{4-3|f'(0)|^2}$, and thus, it is sufficient to show that

$$|z| \le \frac{1}{2}\left(|z||f'(0) + |z|\sqrt{4-3|f'(0)^2}\right),$$

or, for $z \ne 0$,

$$2 - |f'(0)| \le \sqrt{4-3|f'(0)|^2}.$$

After squaring and simplifying we get $|f'(0)|^2 \le |f'(0)|$, which is obvious.

Also solved by Prajnanaswaroopa S, Bengaluru, Karnataka, India.

U630. Evaluate

$$\int_{-\infty}^{+\infty}\frac{(\tanh y)(\sinh y)^2}{2+(\sinh(2y))^2}y^3\mathrm{d}y$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution by the author

The integral becomes

$$\int_{-1}^{1}\frac{x\frac{x^2}{1-x^2}}{2+\frac{4x^2}{(1-x^2)^2}}\frac{1}{8}\left(\ln\frac{1+x}{1-x}\right)^3\frac{1}{1-x^2}dx=\frac{1}{16}\int_{-1}^{1}\frac{x^3}{1+x^4}\left(\ln\frac{1+x}{1-x}\right)^3dx$$

We employ the residue–theory. Define the complex function

$$\frac{z^3}{1+z^4}\left(\text{Ln}\frac{1+z}{1-z}\right)^4\doteq g(z)f^4(z)$$

defined over all **C** cut on the interval $[-1,1]$ where $\text{Ln}(\rho e^{it})=\ln\rho+it$ (we reserve the notation $\ln y$ for real arguments y). The path of integration runs counterclockwise and encircles $[-1,1]$, close to it. When the maximum distance between the path and the interval $[-1,1]$ goes to zero, the integral becomes

$$\int_{1}^{-1}g(x)f^4(x)dx+\int_{-1}^{1}g(x)(f(x)+2\pi i)^4dx=-2\pi i\sum\left(Res(g(z)f^4(z))\right)$$

The residues are calculated at the points $z_k=e^{i\frac{\pi}{4}+k\frac{\pi}{2}}$, $k=0,1,2,3$ and at $z=\infty$. The points z_k are clearly the zeroes of $z^4=-1$. We get

$$\int_{1}^{-1}gf^4\,dx+\int_{-1}^{1}gf^4\,dx+\int_{-1}^{1}g\left(8\pi if^3+4(2\pi i)^3f+6f^2(2\pi i)^2\right)dx=$$
$$=-2\pi i\sum Res(g(z)f^4(z))$$

that is

$$4I=-\sum Res(g(z)f^4(z))+16\pi^2\int_{-1}^{1}gf\,dx$$

where we have used $\int_{-1}^{1}g\,dx=\int_{-1}^{1}gf^2\,dx=0$ by oddness.

To compute

$$\int_{-1}^{1}g(x)f(x)\,dx$$

we proceed in the same way writing

$$\int_{1}^{-1}gf^2\,dx+\int_{-1}^{1}g(f+2\pi i)^2\,dx=-2\pi i\sum Resg(z)f^2(z)$$

which yields

$$\int_{-1}^{1}gf\,dx=-\frac{1}{2}\sum Res(g(z)f^2(z))$$

To compute the residue at $z=\infty$ of the function $g(z)f^2(z)$ we compute it on the curve $\gamma(t)\in\mathbb{C}$, $z=\gamma(t)=it$ and obtain

$$g(z)f^2(z)\Big|_{z=\gamma(t)}=\frac{1}{it}\frac{1}{1+\frac{1}{t^4}}\left(\text{Ln}\frac{\sqrt{1+t^2}e^{i\arctan t}}{\sqrt{1+t^2}e^{-i\arctan t}}\right)^2=$$
$$=\frac{1}{it}\frac{1}{1+\frac{1}{t^4}}\left(\text{Ln}e^{-2i\arctan\frac{1}{t}+i\pi}\right)^2=\frac{1}{it}\frac{1}{1+\frac{1}{t^4}}\left(-i\left(\frac{1}{t}+O(\frac{1}{t^3})+\pi\right)\right)^2=$$
$$=-\frac{\pi^2}{it}+O(1/t^2)$$

whence the residue equal to π^2. The residue in the other four points are

$$\lim_{z\to z_k} g(z)(z-z_k)f^2(z)$$

because the function has simple poles at any point z_k, $k=0,1,2,3$. Denoting $\varphi_0=\pi/4$ we get

$$\lim_{z\to z_0} g(z)(z-z_0)f^2(z)=\lim_{z\to z_0}\frac{3z^2(z-z_0)+z^3}{4z^3}\left(\mathrm{Ln}\frac{1+z_0}{1-z_0}\right)^2=$$
$$=\frac{1}{4}\left(\mathrm{Ln}\frac{1+e^{i\varphi_0}}{1-e^{i\varphi_0}}\right)^2=\frac{1}{4}\left(\mathrm{Ln}\frac{i}{\tan\frac{\varphi_0}{2}}\right)^2=\frac{1}{4}\left(\frac{1}{2}\pi i+\ln(1+\sqrt{2})\right)^2$$

$\varphi_1=3\pi/4$ we get

$$\lim_{z\to z_1} g(z)(z-z_1)f^2(z)=\lim_{z\to z_1}\frac{3z^2(z-z_1)+z^3}{4z^3}\left(\mathrm{Ln}\frac{1+z_1}{1-z_1}\right)^2=$$
$$=\frac{1}{4}\left(\mathrm{Ln}\frac{1+e^{i\varphi_1}}{1-e^{i\varphi_1}}\right)^2=\frac{1}{4}\left(\mathrm{Ln}\frac{i}{\tan\frac{\varphi_1}{2}}\right)^2=\frac{1}{4}\left(\frac{1}{2}\pi i+\ln(\sqrt{2}-1)\right)^2$$

$\varphi_2=5\pi/4$ we get

$$\lim_{z\to z_2} g(z)(z-z_2)f^2(z)=\lim_{z\to z_2}\frac{3z^2(z-z_2)+z^3}{4z^3}\left(\mathrm{Ln}\frac{1+z_2}{1-z_2}\right)^2=$$
$$=\frac{1}{4}\left(\mathrm{Ln}\frac{1+e^{i\varphi_2}}{1-e^{i\varphi_2}}\right)^2=\frac{1}{4}\left(\mathrm{Ln}\frac{i}{\tan\frac{\varphi_2}{2}}\right)^2=\frac{1}{4}\left(\frac{3}{2}\pi i+\ln(\sqrt{2}-1)\right)^2$$

$\varphi_3=7\pi/4$ we get

$$\lim_{z\to z_3} g(z)(z-z_3)f^2(z)=\lim_{z\to z_3}\frac{3z^2(z-z_3)+z^3}{4z^3}\left(\mathrm{Ln}\frac{1+z_3}{1-z_3}\right)^2=$$
$$=\frac{1}{4}\left(\mathrm{Ln}\frac{1+e^{i\varphi_3}}{1-e^{i\varphi_3}}\right)^2=\frac{1}{4}\left(\mathrm{Ln}\frac{i}{\tan\frac{\varphi_3}{2}}\right)^2=\frac{1}{4}\left(\frac{3}{2}\pi i+\ln(\sqrt{2}+1)\right)^2$$

Their sum is (use $\ln(1+\sqrt{2})=\ln(\sqrt{2}-1)^{-1})$)

$$\frac{1}{4}\Big(-2\frac{9}{4}\pi^2-2\frac{\pi^2}{4}+3\pi i\ln(\sqrt{2}-1)+3\pi i\ln(1+\sqrt{2})+\pi i\ln(\sqrt{2}-1)+$$
$$+\pi i\ln(1+\sqrt{2})+2\ln^2(\sqrt{2}-1)+2\ln^2(1+\sqrt{2})\Big)=$$
$$=\frac{1}{4}\Big(-2\frac{9}{4}\pi^2-2\frac{\pi^2}{4}+2\ln^2(\sqrt{2}-1)+2\ln^2(1+\sqrt{2})\Big)=-\frac{5}{4}\pi^2+\ln^2(1+\sqrt{2})$$

Finally

$$-\frac{1}{2}\sum Res(g(z)f^2(z))=-\frac{\pi^2}{2}-\frac{1}{2}\sum_{k=0}^{3}\lim_{z\to z_k} g(z)(z-z_k)f^2(z)=$$
$$=\frac{\pi^2}{8}-\frac{1}{2}(\ln^2(1+\sqrt{2}))=\int_{-1}^{1} g(x)f^2(x)dx$$

The last step is to compute

$$\sum Res(g(z)f^4(z))\qquad(*)$$

The residue at $z=\infty$ follows by

$$g(z)f^4(z)\Big|_{z=\gamma(t)} = \frac{1}{it}\frac{1}{1+\frac{1}{t^4}}\left(\mathrm{Ln} e^{-i\left(\frac{1}{t}+O(\frac{1}{t^3})-\pi\right)}\right)^4 = \frac{1}{it}(1+O(\frac{1}{t}))\left(\pi^4+O(1/t)\right)$$

whence the residue is equal to $-\pi^4$.

$$Res(gf^4)\Big|_{z=z_0} = \frac{1}{4}\left(\frac{1}{2}\pi i+\ln(1+\sqrt{2})\right)^4, \quad Res(gf^4)\Big|_{z=z_1} = \frac{1}{4}\left(\frac{1}{2}\pi i+\ln(\sqrt{2}-1)\right)^4$$

$$Res(gf^4)\Big|_{z=z_2} = \frac{1}{4}\left(\frac{3}{2}\pi i+\ln(\sqrt{2}-1)\right)^4, \quad Res(gf^4)\Big|_{z=z_3} = \frac{1}{4}\left(\frac{3}{2}\pi i+\ln(\sqrt{2}+1)\right)^4$$

As above the terms multiplied by the imaginary unit i in (*) is equal to zero. By the formula

$$\frac{1}{4}\left(\frac{3}{2}\pi i+\ln(\sqrt{2}-1)\right)^4 = \frac{1}{4}\Big(\frac{81}{16}\pi^4+\ln^4(\sqrt{2}-1)-4\frac{27}{8}\pi^4 i\ln(\sqrt{2}-1)+$$
$$+4\frac{3}{2}\pi i\ln^3(\sqrt{2}-1)+6\frac{-9}{4}\pi^2\ln^2(\sqrt{2}-1\Big)$$

and by summing all the residues, the nonzero terms we get

$$\pi^4\left(-1+\frac{81}{64}+\frac{81}{64}+\frac{1}{64}+\frac{1}{64}\right)+$$
$$\pi^2\left(-\frac{27}{8}\ln^2(\sqrt{2}-1)-\frac{27}{8}\ln^2(\sqrt{2}+1)-\frac{3}{8}\ln^2(\sqrt{2}-1)-\frac{3}{8}\ln^2(\sqrt{2}+1)\right)+$$
$$+\left(\frac{1}{4}\ln^4(\sqrt{2}-1)+\frac{1}{4}\ln^4(\sqrt{2}-1)+\frac{1}{4}\ln^4(\sqrt{2}+1)+\frac{1}{4}\ln^4(\sqrt{2}+1)\right)=$$
$$=\frac{25}{16}\pi^4-\frac{15}{2}\pi^2\ln^2(\sqrt{2}+1)+\ln^4(\sqrt{2}+1)$$

The final computation is

$$I=\frac{1}{4}\left(-\sum Res(g(z)f^4(z))+16\pi^2\int_{-1}^{1} gf\,dx\right)$$

namely

$$\frac{1}{4}\left(-\frac{25}{16}\pi^4+\frac{15}{2}\pi^2\ln^2(\sqrt{2}+1)-\ln^4(\sqrt{2}+1)+16\pi^2\Big(\frac{\pi^2}{8}-\frac{1}{2}(\ln^2(1+\sqrt{2}))\Big)\right)$$

and this yields

$$I=\frac{7}{64}\pi^4-\frac{\pi^2}{8}\ln^2(\sqrt{2}+1)-\frac{1}{4}\ln^4(\sqrt{2}+1)$$

Also solved by Yunyong Zhang, Chinaunicom; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, WA, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

O625. Find all primes p such that

$$\frac{(p-2)!-1}{p^2} = \frac{2(p^4+3p^2-9)}{p-1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, West Bengal, India
We first have from the consequence of the Wilson's Theorem, $(p-2)! \equiv 1 \pmod{p}$ and hence, p divides $(p-2)!-1$. Let $(p-2)!-1 = pk$. Then,

$$\frac{kp}{p^2} = \frac{2(p^4+3p^2-9)}{p-1}$$
$$\iff k(p-1) = 2p(p^4+3p^2-9).$$

Now, since $\gcd(p, p-1) = 1$, k must be divisible by p. So, the left hand side of the given equation is an integer, and hence, the right hand side must be an integer as well. Thus, $p-1 \mid 2(p^4+3p^2-9)$, which can be written as $2(p^2+9)(p^2-1)-10p^2$. Since $p-1 \mid (p^2-1)$, we must have $10p^2$ to be divisible by $p-1$. As $\gcd(p, p-1) = 1$, $p-1 \mid 10$, and hence, $p = 11$ is the only solution.

Also solved by Soham Dutta, West Bengal, India; Batakogias Panagiotis, High School of Velestino, Greece; Anmol Kumar, IISc, Bangalore, India; Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia; Sundaresh H R, Shivamogga, India; Theo Koupelis, Clark College, WA, USA.

O626. Prove that there are infinitely many triples (a, b, k) of positive integers such that

$$\frac{a+1}{b} + \frac{b+1}{a} = k$$

and find all possible values of k.

Proposed by Mircea Becheanu, Canada

Solution by the author
We start by considering particular cases.

Case 1: Assume that $a = b$. The equation becomes

$$2\left(\frac{a+1}{a}\right) = k \Leftrightarrow \frac{2}{a} = k - 2$$

and this requires $a|2$. We get cases:
Case (1.i): $a = b = 1$ giving the triple $(a, b, k) = (1, 1, 4)$.
Case (1.ii): $a = b = 2$ giving the triple $(a, b, k) = (2, 2, 3)$.
Therefore we can assume that $a > b$. In this situation we consider two cases.
Case 2: Assume that $a = b + 1$. The equation becomes

$$\frac{b+2}{b} + 1 = k \Leftrightarrow \frac{2}{b} = k - 2,$$

giving that $b|2$. We get cases
Case (2.i): $b = 1$, $a = 2$ which gives the triple $(a, b, k) = (2, 1, 4)$.
Case (2.ii): $b = 2$, $a = 3$ which gives the triple $(a, b, k) = (3, 2, 3)$.
Case 3: Assume that $a > b + 1$. The given equation can be written as a quadratic equation in a:

$$a^2 - (kb - 1)a + b^2 + b = 0. \tag{1}$$

We apply Fermat descend method. Assume that (a, b, k) is a triple which satisfies (1) in which $a > b + 1$. Such triple exists like, for example $(14, 6, 3)$ or $(21, 6, 4)$, in which both cases $k = 3$ and $k = 4$ appear. The quadratic equation (1) has roots a_1, a_2 in which $a_1 = a$. Then

$$a_2 = kb - 1 - a = \frac{b(b+1)}{a}$$

is also a root of (1). We prove that $a_2 < b$:

$$a_2 < b \Leftrightarrow \frac{b(b+1)}{a} < b \Leftrightarrow b(b+1) < a^2$$

and this is true, because $a > b + 1 > b$. So, the triple (b, a_2, k) is a solution of the problem with $b > a_2$ and in which $b + a_2 < a + b$. If $b > a_2 + 1$ we can apply again descend method to obtain a new triple (a_2, b_1) in which $a_2 > b_1$ and $a_2 + b_1 < b + a_2$. So, starting from $a_1 = a, b_1 = b, k$ we can construct a going down sequence of triples

$$.... \leftarrow (a_3, b_3, k) \leftarrow (a_2, b_2, k) \leftarrow (a_1, b_1, k) \tag{2}$$

in which $a_i > b_i$ and the sums $a_i + b_i$ decrease. The sequence ca be continued as long as $a_i > b_i + 1$. This can not be done indefinitely, so at some step we arrive to a triple (a_i, b_i, k) in which $a_i = b_i + 1$. But in this case we have one of the cases (2.i) or (2.ii) which show that $k = 3$ or $k = 4$.

The transformations from the sequence (2) can also be applied up, so we obtain infinitely many triples.

Also solved by Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O627. Prove that 3 is the largest value of the positive constant k such that

$$\frac{1}{ab+k}+\frac{1}{bc+k}+\frac{1}{cd+k}+\frac{1}{da+k}\ge\frac{4}{1+k}$$

for all nonnegative real numbers a,b,c,d satisfying $ab+ac+ad+bc+bd+cd=6$.

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

Solution by the author

By choosing $b=d=\sqrt{ac}$, the constraint becomes $(a+c)\sqrt{ac}+ac=3$, while the inequality can be written as follows:

$$\frac{1}{a\sqrt{ac}+k}+\frac{1}{c\sqrt{ac}+k}\ge\frac{2}{1+k},$$

$$\frac{(a+c)\sqrt{ac}+2k}{a^2c^2+k(a+c)\sqrt{ac}+k^2}\ge\frac{2}{1+k}.$$

Denoting $x=ac$, we have $3=(a+c)\sqrt{ac}+ac\ge 3ac=3x$, hence $x\in(0,1]$. Since $(a+c)\sqrt{ac}=3-x$, the inequality becomes

$$\frac{3+2k-x}{x^2-kx+k^2+3k}\ge\frac{2}{1+k},$$

which is equivalent to $(x-1)(2x+3-k)\le 0$. Since $x-1\le 0$, the inequality is true if and only if $2x+3-k\ge 0$ for $x\in(0,1]$. So, we get the necessary condition $3-k\ge 0$, that is $k\le 3$.

To show that 3 is the largest positive k, we need to prove the inequality

$$\frac{1}{ab+3}+\frac{1}{bc+3}+\frac{1}{cd+3}+\frac{1}{da+3}\ge 1.$$

By the AM-GM inequality, we have

$$6=ab+ac+ad+bc+bd+cd\ge 6\sqrt[6]{a^3b^3c^3d^3},$$

hence

$$abcd\le 1.$$

Write the required inequality as follows:

$$\left(\frac{1}{ab+3}+\frac{1}{cd+3}\right)+\left(\frac{1}{bc+3}+\frac{1}{da+3}\right)\ge 1,$$

$$\frac{3(ab+cd)+18}{abcd+3(ab+cd)+9}+\frac{3(bc+da)+18}{abcd+3(bc+da)+9}\ge 3,$$

$$1+\frac{9-abcd}{abcd+3(ab+cd)+9}+1+\frac{9-abcd}{abcd+3(bc+da)+9}\ge 3,$$

$$\frac{1}{abcd+3(ab+cd)+9}+\frac{1}{abcd+3(bc+da)+9}\ge\frac{1}{9-abcd}.$$

According to the AM-HM inequality, it is sufficient to show that

$$\frac{4}{[abcd+3(ab+cd)+9]+[abcd+3(bc+da)+9]}\ge\frac{1}{9-abcd},$$

which is equivalent to

$$6\ge ab+bc+cd+da+2abcd,$$

$$ac+bd\ge 2abcd.$$

Indeed, we have

$$ac+bd\ge 2\sqrt{abcd}\ge 2abcd.$$

The proof is completed. For $k\in(0,3]$, the equality occurs when $a=b=c=d=1$.

Also solved by Theo Koupelis, Clark College, WA, USA; Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, West Bengal, India.

O628. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$(x + y + z - 2)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{2}\right) \geq \frac{5}{2}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

The inequality is equivalent to

$$\frac{(x + y + z - 2)\left[2(xy + yz + zx) - xyz\right]}{2xyz} \geq \frac{5}{2},$$

$$(x + y + z - 2)^2(x + y + z + 2) \geq 5xyz.$$

Let $x = 2\cos A$, $y = 2\cos B$, $z = 2\cos C$, where ABC is an acute triangle. The inequality becomes

$$\left(\sum \cos A - 1\right)^2 \left(\sum \cos A + 1\right) \geq 5 \prod \cos A,$$

inequality, which is obviously true for a triangle that is not acute. Using known identities

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}, \qquad \cos A \cos B \cos C = \frac{s^2 - (2R + r)^2}{4R^2}$$

we get

$$\frac{r^2(2R + r)}{R^3} \geq \frac{5(s^2 - r^2 - 4Rr - 4R^2)}{4R^2},$$

$$\frac{5s^2}{R^2} \leq \frac{4r^3}{R^3} + \frac{13r^2}{R^2} + \frac{20r}{R} + 20.$$

Using the Ravi's substitutions i.e. $a = y + z$, $b = z + x$, $c = x + y$, $x, y, z > 0$, the inequality can be rewritten as follows

$$\frac{5 \cdot 16xyz(x + y + z)^3}{(x + y)^2(y + z)^2(z + x)^2} \leq$$

$$\frac{4 \cdot 4^3x^3y^3z^3}{(x + y)^3(y + z)^3(z + x)^3} + \frac{13 \cdot 4^2x^2y^2z^2}{(x + y)^2(y + z)^2(z + x)^2} + \frac{20 \cdot 4xyz}{(x + y)(y + z)(z + x)} + 20,$$

or

$$5P^3 + 20xyzP^2 + 52x^2y^2z^2P + 64x^3y^3z^3 - 20xyz(x + y + z)^3P \geq 0,$$

where we denote $P = (x + y)(y + z)(z + x)$. After expanding and reducing terms, it becomes

$$5[(6,3,0)] - 5[(6,2,1)] + 15[(5,4,0)] - 15[(5,2,2)] +$$

$$5[(4,4,1)] - 13[(4,3,2)] + 8[(3,3,3)] \geq 0.$$

By Muirhead's Inequality, we have $[(6,3,0)] \geq [(6,2,1)]$ and $[(5,4,0)] \geq [(5,2,2)]$, so it remains to show that

$$3[(5,4,0)] - 3[(5,2,2)] + 5[(4,4,1)] - 13[(4,3,2)] + 8[(3,3,3)] \geq 0. \tag{1}$$

Without loss of generality, we may assume that $z = \max\{x, y, z\}$ and let denote $M = (x - y)^2$, $N = (x - z)(y - z) \geq 0$. We have

$$\begin{aligned}&[(5,4,0)] - [(5,2,2)] = \\ &M[z^5(x + y)^2 + z^3(x^4 + x^3y + x^2y^2 + xy^3 + y^4) + z^2xy(x^3 + x^2y + xy^2 + y^3) \\ &- zx^2y^2(x^2 + xy + y^2) - x^3y^3(x + y)] + N\left[x^5(y + z)^2 + y^5(z + x)^2\right].\end{aligned}$$

$$
\begin{aligned}
&5[(4,4,1)] - 13[(4,3,2)] + 8[(3,3,3)] = \\
&10xyz(x^3y^3 + y^3z^3 + z^3x^3 - 3x^2y^2z^2) - 13x^2y^2z^2\left(\sum_{cyc} x(y^2+z^2) - 6xyz\right) = \\
&10xyz(xy+yz+zx)(z^2M + xyN) - 13x^2y^2z^2\left[2zM + (x+y)N\right].
\end{aligned}
$$

Returning to (1), the inequality can be rewritten as follows $\alpha M + \beta N \geq 0$, where

$$
\begin{aligned}
\alpha =& 3z^5(x+y)^2 + 3z^3(x^4 + x^3y + x^2y^2 + xy^3 + y^4) + 3z^2xy(x^3 + x^2y + xy^2 + y^3) \\
&- 3zx^2y^2(x^2 + xy + y^2) - 3x^3y^3(x+y) + 2xyz^3(5yz + 5zx - 8xy) \\
=& 3z^5(x+y)^2 + 3z^2xy(x^3 + x^2y + xy^2 + y^3) + 3(z^3xy - zx^2y^2)(x^2 + xy + y^2) \\
&+ 3x^4(z^3 - y^3) + 3y^4(z^3 - x^3) + 2xyz^3(5yz + 5zx - 8xy) \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta =& 3x^5(y+z)^2 + 3y^5(z+x)^2 + x^2y^2z(10xy - 3zx - 3yz) \\
=& 3z^2(x^5 + y^5 - x^3y^2 - x^2y^3) + 2xyz(3x^4 + 5x^2y^2 + 3y^4) + 3x^2y^2(x^3 + y^3) \geq 0,
\end{aligned}
$$

which is clearly true.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, WA, USA.

O629. Let a, b, c, d be positive integers such that $4(a^2 + b^2) = 5(c^2 + d^2)$ and $ad - bc$ divides $c^2 + d^2$. Prove that

$$2(ac + bd) = L_{2n+1}|ad - bc|$$

for some nonnegative integer n, where L_k is the k^{th} Lucas number defined by $L_0 = 2$, $L_1 = 1$ and $L_{k+1} = L_k + L_{k-1}$, $k = 1, 2, 3, \ldots$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Theo Koupelis, Clark College, WA, USA

Let $c^2 + d^2 = (ad - bc)m$, where m is an integer $\neq 0$. Using the identity $(ad - bc)^2 + (ac + bd)^2 = (a^2 + b^2)(c^2 + d^2)$ we get

$$\frac{(c^2 + d^2)^2}{m^2} + (ac + bd)^2 = \frac{5}{4}(c^2 + d^2)^2,$$

and thus

$$4m^2(ac + bd)^2 = (c^2 + d^2)^2(5m^2 - 4) = (ad - bc)^2 m^2 (5m^2 - 4),$$

or

$$2(ac + bd) = |ad - bc| \cdot \sqrt{5m^2 - 4}.$$

It is well known, however, that the solutions of the Diophantine equation $5m^2 - 4 = s^2$, with s a positive integer, are given by $m = \pm F_{2k+1}$ and $s = L_{2k+1}$, where L_k is the k^{th} Lucas number, and F_k is the k^{th} Fibonacci number, with $L_0 = 2$, $L_1 = 1$, $L_{k+1} = L_k + L_{k-1}$, and $F_0 = 0, F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$, $k = 1, 2, 3, \ldots$. Thus,

$$2(ac + bd) = L_{2n+1}|ad - bc|$$

for some nonnegative integer n.

O630. Prove that there are infinitely many positive integers m such that $m+1, 2m+1, 3m+1$ are all composite and divide $2^m - 1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Taking $m = 6n$, for some positive integer n, we are going to prove that there are infinitely many n such that the three composite numbers $6n+1, 12n+1, 18n+1$ are composite and divide $2^{6n} - 1$.

Let $n = \frac{2^{36}-1}{9}$ then $6n+1 = \frac{2^{37}+1}{3}, 12n+1 = \frac{2^{38}-1}{3}, 18n+1 = 2^{37}-1$ are composite and divide $2^{6n}-1$. Let $N = \frac{2^{12n}-1}{9}$ then $6N = \frac{2(2^{12n}-1)}{3}$ thus $2(6n+1)(12n+1) \mid 6N$. Hence,

$$6N+1 = \frac{2^{12n+1}+1}{3} | 2^{2(12n+1)} - 1 | 2^{6N} - -1,$$

$$12N+1 = \frac{2^{2(6n+1)}-1}{3} | 2^{2(6n+1)} - 1 | 2^{6N} - 1,$$

$$18N+1 = 2^{12n} - 1 \mid 2^{6N} - 1.$$

Since $6N+1, 12N+1, 18N+1$ are all composite and $N = \frac{2^{12n}-1}{9} > n$ we are done.

Notice that

$$6n+1 | 2^{6n} - 1 | 2^{6n+1} - 2,$$

$$12n+1 | 2^{6n} - 1 | 2^{12n+1} - 2,$$

$$18n+1 | 2^{6n} - 1 | 2^{18n} - 1 | 2^{18n+1} - 2.$$

Remark: Let by t_n and ω_n we denote the $n-$th triangular and pentagonal numbers, we can also prove $t_{12n+1} = (6n+1)(12n+1) | 2^{6n} - 1 | 2^{t_{12n+1}} - 2$. Let $\omega_{12n+1} = (12n+1)(18n+1)$ then $\omega_{12n+1} \mid 2^{\omega_{12n+1}} - 2$.

Also solved by Zijin Peng, The Affiliated High School of South China Normal University, Guangzhou, Guangdong, China.