On a Class of Functional Equations

The functional equation

$$f(2x - f(x)) = x$$

(for a real-valued function f also defined on the set $\mathbb{R}$ of the real numbers) has been long studied since Euler. One obvious solution of it is the identity function ($f(x) = x$ for any real x); moreover, any function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + c$ for any real x is a solution of the considered equation, where c is an (arbitrarily) fixed real number. (Although less evident, these solutions can be easily found, for instance by looking for solutions among the affine functions $f(x) = ax + c$; one finds that such a function satisfies $f(2x - f(x)) = x$ for all x if and only if $a(2 - a) = 1$ and $c(1 - a) = 0$, that is, if and only if $a = 1$, while c can be any real number.)

One immediately sees that, if $f : \mathbb{R} \to \mathbb{R}$ verifies $f(2x - f(x)) = x$ for all $x \in \mathbb{R}$, than f is surjective. It is also clear that, for such f, the mapping $x \mapsto 2x - f(x)$ is injective, but we need to know more about f in order to solve the equation. Even though the injectivity of f is assumed, there are solutions defined as

$$f(x) = \begin{cases} x + m, & x \in \mathbb{Z} \\ x + n, & x \notin \mathbb{Z} \end{cases}$$

(with distinct integers m and n) which show that we cannot hope to find *all* the solutions without imposing other restrictions. Note that, if we assume f to be injective, then f is actually bijective, hence it has an inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$, therefore the equation can be equivalently written (for injective functions) as

$$f(x) + f^{-1}(x) = 2x, \quad \forall x \in \mathbb{R}.$$

We can find this functional equation in numerous mathematical contests and magazines; for instance, it appeared in a Romanian TST fom 1984 as

Problem 1. Determine the bijective and (strictly) monotone functions $f : \mathbb{R} \to \mathbb{R}$ for which we have

$$f(x) + f^{-1}(x) = 2x, \quad \forall x \in \mathbb{R}.$$

Solution. The required functions are those of the form $f(x) = x + c$, $\forall x \in \mathbb{R}$, for some real number c. In order to prove that, note that, by replacing x with $f(x)$ in the given equation, we find

$$f(f(x)) = 2f(x) - x = x + 2(f(x) - x), \ \ \forall x \in \mathbb{R},$$

whence, if we replace once more x with $f(x)$, we get

$$f(f(f(x))) = 2f(f(x)) - f(x) = 3f(x) - 2x = x + 3(f(x) - x), \ \ \forall x \in \mathbb{R}.$$

We also have

$$f^{-1}(x) = x - (f(x) - x), \ \ \forall x \in \mathbb{R},$$

by only rewriting the initial equation — and a general formula can now be guessed. Namely, let us denote, for a positive integer n,

$$f^{[n]} = f \circ \cdots \circ f$$

(with n appearances of f; $f^{[n]}$ is called the nth iterate of f), and let also $f^{[-n]}$ be defined by

$$f^{[-n]} = f^{-1} \circ \cdots \circ f^{-1} = (f \circ \cdots \circ f)^{-1}$$

(the nth iterate of the inverse f^{-1}; or, we may call it the $(-n)$th iterate of f). We also denote by $f^{[0]}$ the identity of the set of the real numbers, defined by $f^{[0]}(x) = x$ for any $x \in \mathbb{R}$. Thus we have $f^{[n]}$ defined for any integer n whenever f is a bijective (equivalently invertible) real function (and only for $n \geq 0$ if f has not this property). These notations will be kept throughout this note for any $f : \mathbb{R} \to \mathbb{R}$.

Coming back to solving our problem, we see that we obtained $f^{[n]}(x) = x + n(f(x) - x)$ for all $x \in \mathbb{R}$ and any $n \in \{-1, 0, 1, 2, 3\}$ (for $n = 0$ and $n = 1$ the equalities are obvious). We leave the reader to prove that this holds for any n.

Exercise 1. For any invertible solution $f : \mathbb{R} \to \mathbb{R}$ of the functional equation $f(x) + f^{-1}(x) = 2x$ we have

$$f^{[n]}(x) = x + n(f(x) - x), \ \ \forall x \in \mathbb{R}, \ \ \forall n \in \mathbb{Z}.$$

One way to prove this is by inducting on n (in both directions). Also, the result can be inferred by using the theory of linear and homogeneous recurrences of second order, since we have

$$f^{[n]}(x) - 2f^{[n-1]}(x) + f^{[n-2]}(x) = 0, \ \ \forall x \in \mathbb{R}, \ \ \forall n \in \mathbb{Z},$$

that is (for each $x \in \mathbb{R}$), the sequence $(f^{[n]}(x))_{n\in\mathbb{Z}}$ verifies such a recurrence.

Now, because f is monotone (actually, strictly monotone, since it is also injective), so is the inverse f^{-1}; moreover, f and f^{-1} have the same type of monotony (they are either both increasing, or both decreasing), consequently their sum also has the same type of monotony. As their sum $f + f^{-1}$ is the strictly increasing function $x \mapsto 2x$, f is strictly increasing, too, which implies that any iterate $f^{[n]}$ ($n \in \mathbb{Z}$) is also strictly increasing. Consequently, for (arbitrary, but fixed for the moment) $x, y \in \mathbb{R}$ with $x < y$, we have $f^{[n]}(x) < f^{[n]}(y)$, for every $n \in \mathbb{Z}$. This yields

$$x + n(f(x) - x) < y + n(f(y) - y), \ \ \forall n \in \mathbb{Z},$$

and hence

$$\frac{x}{n} + f(x) - x < \frac{y}{n} + f(y) - y, \ \ \forall n \in \mathbb{Z}, \ \ n > 0,$$

while

$$\frac{x}{n} + f(x) - x > \frac{y}{n} + f(y) - y, \ \ \forall n \in \mathbb{Z}, \ \ n < 0.$$

By passing to the limit for $n \to \infty$ in the first inequality, and for $n \to -\infty$ in the second, we get

$$f(x) - x \leq f(y) - y,$$

and

$$f(x) - x \geq f(y) - y$$

respectively. Thus we have $f(x) - x = f(y) - y$ for any $x, y \in \mathbb{R}$ with $x < y$, meaning that the mapping $x \mapsto f(x) - x$ is actually constant on $\mathbb{R}$, whence the conclusion $f(x) = x + c$, $\forall x \in \mathbb{R}$ (for some $c \in \mathbb{R}$) is immediate. As we already have seen, all these functions are solutions to the considered functional equation, and Problem 1 is completely solved.

The following variation on the same theme is also well-known.

Problem 2. Determine all the continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(f(x)) - 2f(x) + x = 0,$$

for any $x \in \mathbb{R}$.

Solution. We first note that any such function is injective, since $f(x) = f(y)$ implies $f(f(x)) = f(f(y))$ and, because of

$$f(f(x)) - 2f(x) + x = f(f(y)) - 2f(y) + y (= 0),$$

we see that $f(x) = f(y)$ implies $x = y$. On the other hand, any such solution f is also surjective. Indeed, being continuous and injective f is strictly monotone. Due to monotony, we can consider $m = \inf_{x\in\mathbb{R}} f(x)$ — which exists in $\overline{\mathbb{R}}$. Let then $(x_n)_{n\geq 1}$ be such a sequence of real numbers that $\lim_{n\to\infty} f(x_n) = m$. Suppose that m is a real number. Because f is continuous, $f(m) = \lim_{n\to\infty} f(f(x_n))$, therefore, by

$$x_n = -f(f(x_n)) + 2f(x_n), \quad \forall n \in \mathbb{N}^*,$$

we deduce that $(x_n)_{n\geq 1}$ is convergent. If $l = \lim_{n\to\infty} x_n$, then we get (also by the continuity of f) that $m = \lim_{n\to\infty} f(x_n) = f(l)$. Nevertheless, this is not possible for a monotone function $f : \mathbb{R} \to \mathbb{R}$. (For instance, if f was strictly increasing, then $f(t) < f(l) = m = \inf_{x\in\mathbb{R}} f(x)$ would follow for every $t < l$.) So, assuming that $m \in \mathbb{R}$ is wrong, leaving only the possibilities $\inf_{x\in\mathbb{R}} f(x) = \infty$ or $\inf_{x\in\mathbb{R}} f(x) = -\infty$. Similarly we see that either $\sup_{x\in\mathbb{R}} f(x) = \infty$, or $\sup_{x\in\mathbb{R}} f(x) = -\infty$. But f is continuous and strictly monotone, hence $f(\mathbb{R}) = \mathbb{R}$ follows, that is, f is surjective (and, finally, bijective). Thus f has an inverse f^{-1} that verifies

$$f(x) + f^{-1}(x) = 2x, \quad \forall x \in \mathbb{R}$$

(which follows by putting $f^{-1}(x)$ in the place of x in the functional equation satisfied by f), and so we find ourselves in the conditions of the first problem, which we already solved. Thus any continuous solution of the functional equation

$$f(f(x)) - 2f(x) + x = 0$$

is of the form $f(x) = x + c$ for some real number c.

We invite the reader to check the following generalization of the results from the beginning of the solution of Problem 2.

Exercise 2. Let $a_n, \ldots, a_0$ be real numbers ($n \geq 1, a_n \neq 0$), and let $f : \mathbb{R} \to \mathbb{R}$ be a function that verifies the functional equation

$$a_n f^{[n]}(x) + a_{n-1} f^{[n-1]}(x) + \cdots + a_1 f(x) + a_0 x = 0,$$

for any real number x.

(a) Prove that, if $a_0 \neq 0$, then f is injective. Moreover, if f is continuous, then f is strictly monotone. In this case, if $(-1)^j a_j \geq 0$ for each $j \in \{0, \ldots, n\}$, then f is strictly increasing, and if $a_j \geq 0$ for each $j \in \{0, \ldots, n\}$, then f is strictly decreasing.

(b) Prove that, if f is continuous and $a_0 \neq 0$, then f is surjective.

(c) If the equation

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 = 0$$

has a real root $r \in \mathbb{R}$, then the linear function f defined by $f(x) = rx$ is a solution of the above functional equation.

(d) If the equation

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 = 0$$

has no real roots, then the functional equation

$$a_n f^{[n]}(x) + a_{n-1} f^{[n-1]}(x) + \cdots + a_1 f(x) + a_0 x = 0, \quad x \in \mathbb{R}$$

has neither monotone, nor continuous solutions.

The proofs of (a) and (b) are, as we said, in the vein of the above ideas (or of those from [1,2]). Part (c) is a simple checking, while part (d) is an interesting exercise for the olympiad amateurs. The algebraic equation

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 = 0$$

is usually named the *characteristic equation* of the functional equation

$$a_n f^{[n]}(x) + a_{n-1} f^{[n-1]}(x) + \cdots + a_1 f(x) + a_0 x = 0.$$

To end this digression, it is worth mentioning that (as one can see in [1], or [2]) the linear solutions shown in part (c) are far from being the only (even continuous) solutions of this functional equation. For example, $f(x) = x + c$ (and not only $f(x) = x$) is a (continuous, monotone) solution of the functional equation whenever its characteristic equation admits 1 as a double root. Also, for $c \in \mathbb{R}$, functions as

$$f(x) = \begin{cases} x, & x \in (-\infty, c) \\ 2x - c, & x \in [c, \infty) \end{cases}$$

are solutions (again: continuous and monotone) of the functional equation $f(f(x)) - 3f(x) + 2x = 0$, and so on. Thus, as we saw and as we will see further, even in simple particular cases there are many more continuous solutions of such a functional equation than those of the form $f(x) = rx$, with r being a root of the characteristic equation (usually at least an uncountable class of such solutions — although there are also exceptions from this rule; for instance, the equation $f(f(x)) + f(x) - x = 0$ has only two continuous solutions, namely $f_k(x) = r_k x$, $k = 1, 2$, where r_1 and r_2 are the solutions of $t^2 + t - 1 = 0$). In the papers [1] and [2] the problem of finding the continuous solutions of such a functional equation of order 2 (that is, for $n = 2$) is completely solved.

Starting from Problem 2, the problem of finding the continuous solutions of the functional equation

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}f^{[j]}(x)=0,\quad \forall x\in\mathbb{R}$$

arises in a natural way, and seeking for these solution is our actual purpose in this note. The first case that follows is, of course, $n=3$, when (as for $n=2$) we have the complete (and similar) answer.

Problem 3. Determine all continuous functions $f:\mathbb{R}\to\mathbb{R}$ for which we have

$$f(f(f(x)))-3f(f(x))+3f(x)-x=0$$

for any real number x.

Solution. As in the case $n=2$ we show that all the continuous solutions are the functions $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x+c$ for any $x\in\mathbb{R}$ (and some fixed real number c).

According to the previous exercise, any continuous solution f of this functional equation is bijective and strictly monotone. Moreover, any such f is strictly increasing (since assuming that f was strictly decreasing, would lead to the contradiction that $0=f^{[3]}-3f^{[2]}+3f^{[1]}-f^{[0]}$ is also strictly decreasing, as long as $f^{[2k+1]}$ is decreasing, and $f^{[2k]}$ is increasing for any integer k). Replacing x with $f^{[n-3]}(x)$ in the equation $f^{[3]}(x)-3f^{[2]}(x)+3f^{[1]}(x)-f^{[0]}(x)=0$, $\forall x\in\mathbb{R}$ yields

$$f^{[n]}(x)-3f^{[n-1]}(x)+3f^{[n-2]}(x)-f^{[n-3]}(x)=0,$$

for any integer n, and any real number x. Using induction (or the theory of linear and homogeneous recurrences) we get

$$f^{[n]}(x)=x-\frac{f^{[2]}(x)-4f(x)+3x}{2}n+\frac{f^{[2]}(x)-2f(x)+x}{2}n^2$$

for all $n\in\mathbb{Z}$ and all $x\in\mathbb{R}$.

Let x be a real number (arbitrary, but fixed for the moment). Obviously, $f(x)=x$ implies $f(f(x))=f(x)=x$, and hence $f(f(x))-2f(x)+x=0$. On the other hand, if $x<f(x)$ we get (by repeatedly applying either f, or its inverse f^{-1}, and by using the fact that both f and f^{-1} are strictly increasing)

$$\cdots<f^{[-2]}(x)<f^{[-1]}(x)<f^{[0]}(x)<f^{[1]}(x)<f^{[2]}(x)<\cdots,$$

and, analogously, the assumption that $f(x)<x$ leads to the conclusion that the sequence $(f^{[n]}(x))_{n\in\mathbb{Z}}$ is strictly decreasing. However, the above expression of $f^{[n]}(x)$ shows that both limits $\lim_{n\to\infty}f^{[n]}(x)$ and $\lim_{n\to-\infty}f^{[n]}(x)$

are simultaneously equal to ∞, or to $-\infty$ according to whether we have $f^{[2]}(x)-2f(x)+x>0$, or $f^{[2]}(x)-2f(x)+x<0$. This comes in contradiction with the monotony of the sequence $(f^{[n]}(x))_{n\in\mathbb{Z}}$ whenever $f^{[2]}(x)-2f(x)+x\neq 0$, therefore $f^{[2]}(x)-2f(x)+x=0$ remains the only acceptable possibility in any of the situations $f(x)=x$, $f(x)>x$, or $f(x)<x$ — that is, for any real number x.

Consequently, the equality

$$f(f(x))-2f(x)+x=0$$

holds for any $x\in\mathbb{R}$, and thus, according to the previous Problem 2, it follows that

$$f(x)=x+c$$

for some fixed $c\in\mathbb{R}$ and any $x\in\mathbb{R}$ — which is what we intended to prove.

Posing the general problem seems to be more and more meaningful, and this is as follows.

Problem 4. Let $n\geq 2$ be a natural number. Is it true that any continuous function that verifies

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}f^{[j]}(x)=0,\quad \forall x\in\mathbb{R}$$

must be of the form $f(x)=x+c$ for some real number c?

As we just saw, this is the case for $n\in\{2,3\}$. (No effort is needed to see that, for $n=1$, $f(x)=x$ follows for any $x\in\mathbb{R}$ — even without any supplementary condition at all.) Nevertheless, somehow surprisingly, for $n\geq 4$ the answer to the question from Problem 4 is in the negative. Namely, due to the (easy to check and well-known) identity

$$(a+4b)^3-4(a+3b)^3+6(a+2b)^3-4(a+b)^3+a^3=0,\quad \forall a,b\in\mathbb{R},$$

one immediately sees that $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=(\sqrt[3]{x}+k)^3$ (for some fixed real number k) verifies

$$f^{[4]}(x)-4f^{[3]}(x)+6f^{[2]}(x)-4f^{[1]}(x)+f^{[0]}(x)=0$$

for every $x\in\mathbb{R}$. Moreover, by the inductive way (as in the proof of the binomial expansion) we see that such a function also satisfies

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}f^{[j]}(x)=0,\quad \forall x\in\mathbb{R},$$

for any $n \geq 4$ and any $x \in \mathbb{R}$. For instance, we have

$$f^{[5]}(x) - 5f^{[4]}(x) + 10f^{[3]}(x) - 10f^{[2]}(x) + 5f^{[1]}(x) - f^{[0]}(x) =$$

$$= f^{[5]}(x) - 4f^{[4]}(x) + 6f^{[3]}(x) - 4f^{[2]}(x) + f^{[1]}(x) -$$
$$-(f^{[4]}(x) - 4f^{[3]}(x) + 6f^{[2]}(x) - 4f^{[1]}(x) + f^{[0]}(x)) =$$
$$= f^{[4]}(f(x)) - 4f^{[3]}(f(x)) + 6f^{[2]}(f(x)) - 4f^{[1]}(f(x)) + f^{[0]}(f(x)) -$$
$$-(f^{[4]}(x) - 4f^{[3]}(x) + 6f^{[2]}(x) - 4f^{[1]}(x) + f^{[0]}(x)) = 0.$$

Evidently (more or less — we invite the reader to prove this affirmation), no such function with $k \neq 0$ has the form $f(x) = x + c$, for some real number c, hence Problem 4 has (as we said) a negative answer for $n \geq 4$ (and we believe that finding *all* the continuous solutions is not an easy task in this case).

It is worth noting how one can find such solutions for $n \geq 4$. We thank Gabriel Dospinescu for pointing out to us the following result. Remember that a *homeomorphism* f of $\mathbb{R}$ is a continuous and bijective function $f : \mathbb{R} \to \mathbb{R}$, whose inverse is also continuous. The result says that, if f is homeomorphism of $\mathbb{R}$ such that $f(x) > x$ for all $x \in \mathbb{R}$, then f is *conjugated* to $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 1$. In other words, for such a homeomorphism f, there exists $h : \mathbb{R} \to \mathbb{R}$ (also a homeomorphism of $\mathbb{R}$) such that $f = h \circ g \circ h^{-1}$, therefore such that

$$f(h(x)) = h(x + 1), \quad \forall x \in \mathbb{R}.$$

Since f from our functional equation is a homeomorphism and (apart from the trivial situation $f(x) = x$ for all x) we may assume it satisfies $f(x) > x$ for all $x \in \mathbb{R}$), such a h must exist, so, by replacing x with $h(x)$ we get

$$f^{[4]}(h(x)) - 4f^{[3]}(h(x)) + 6f^{[2]}(h(x)) - 4f^{[1]}(h(x)) + f^{[0]}(h(x)) = 0, \quad \forall x \in \mathbb{R},$$

that is,

$$h(x + 4) - 4h(x + 3) + 6h(x + 2) - 4h(x + 1) + h(x) = 0$$

which, in connection to the identity

$$(a + 4b)^3 - 4(a + 3b)^3 + 6(a + 2b)^3 - 4(a + b)^3 + a^3 = 0, \quad \forall a \in \mathbb{R},$$

for $h(x) = (kx + 1)^3$ (which, being a third degree polynomial function verifies $h(x+4) - 4h(x+3) + 6h(x+2) - 4h(x+1) + h(x) = 0$), leads to the examples

$f(x) = (\sqrt[3]{x} + k)^3$ (use $a = kx + 1$ and $b = k$). Even so, the explanation of the fact that the functional equation

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} f^{[j]}(x) = 0, \quad \forall x \in \mathbb{R}$$

has only the solutions $f(x) = x + c$ in the cases $n = 2$ and $n = 3$, while for $n \geq 4$ much more solutions are presumable (except for those of form $f(x) = x + c$) remains (at least for us, at least for the moment) a mistery.

References:

1. Ion Chiţescu, Traian Gâdea: *The Functional Equation* $f \circ f + af + b1_{\mathbb{R}} = 0$, Analele Ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi (S. N.), Matematică, Tomul LVI, 2010, f.2

2. Janusz Matkowski, Zhang Weinian: *Method of Characteristic for Functional Equations in Polynomial Form*, Acta Mathematica Sinica, New Series, 1997, July, Vol. 13, No. 3, pp. 421-432

Titu Andreescu, University of Texas at Dallas
Marian Tetiva, National College Gheorghe Roşca Codreanu, Bârlad, Romania