Junior Problems

J631. Find the least positive integer n for which $n^4 - 2023n^2 + 1$ is a product of two primes.

Proposed by Adrian Andreescu, Dallas, USA

J632. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a(b+c)^5} + \frac{1}{b(c+a)^5} + \frac{1}{c(a+b)^5} \geq \frac{3}{32}.$$

Proposed by Mihaly Bencze, Braşov and Neculai Stanciu, Buzău, România

J633. Let a, b, c, t be positive real numbers with $t \geq 1$. Prove that

$$\frac{ta^3 + a^2b}{a+b} + \frac{tb^3 + b^2c}{b+c} + \frac{tc^3 + c^2a}{c+a} \geq \frac{t+1}{2}(ab + bc + ca).$$

Proposed by Mihaela Berindeanu, Bucharest, România

J634. Find all triples (x, y, n) of integers where n is a positive integer, satisfying the equation

$$x^2 + xy + y^2 = (xy)^n.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J635. Let a, b, c be positive real numbers such that $a^4 - 23a^2 + 1 = 0$, $b^4 - 223b^2 + 1 = 0$, and $c^4 - 2023c^2 + 1 = 0$. Prove that

$$a^2b^2c^2 - nabc + 1 = (ab+1)(bc+1)(ca+1),$$

for some integer n.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J636. Let a, b, c be positive numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \leq \frac{a+b+c}{2}.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S631. Find all positive integers n for which $(n-1)!+(n+1)^2=\left(n^2-41\right)\left(n^2+49\right)$.

Proposed by Adrian Andreescu, Dallas, USA

S632. Solve in real numbers the system of equations

$$\begin{aligned}238^x+2016^y &= 2030^x\\238^y+2016^z &= 2030^y\\238^z+2016^x &= 2030^z.\end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

S633. Let $ABCD$ be a convex quadrilateral with $CD=CB$ and $\angle BCD=180^\circ-2(\angle BAD)$. The orthogonal projection of A on BD is E and the orthogonal projections of the point E on AD and AB are F and K, respectively. Let O be the midpoint of the segment AE and let X be the intersection of AC and FK. Prove that $OX=AO\cdot\cos(\angle BAD)$.

Proposed by Mihaela Berindeanu, Bucharest, România

S634. Prove that there are no integers a,b,c such that

$$a^3-b^2-c^2+abc=5.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

S635. Find all rational numbers x such that

$$x^3-\lfloor x\rfloor^3-\{x\}^3=\frac{162}{5},$$

where $\lfloor x\rfloor$ and $\{x\}$ are the greatest integer less than or equal to x and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S636. Prove that for any prime p the sum of the digits of $7^p+13^p+2023^p$ is not a prime.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Undergraduate Problems

U631. Evaluate

$$\int_2^3 \frac{(x^2+2)\sqrt{x^4-x^2+4}}{x^3}\,dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U632. Let $H_n = \sum_{k=1}^{n} 1/k$. Evaluate

$$S = \sum_{n=1}^{\infty} \frac{H_{n+2}}{n(n+1)}.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U633. Evaluate

$$\lim_{n\to\infty} \frac{\ln\sqrt[3]{n}\cdot\ln(n+3)}{\sum\limits_{1\le i<j\le n}\frac{1}{ij}}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

U634. Let $A_1, A_2, \ldots, A_n$ be points lying on a circle with radius 1. Prove that there is a point P on this circle such that

$$PA_1 + PA_2 + \cdots + PA_n \ge \frac{4n}{\pi}\,.$$

Proposed by Karol Janowicz and Waldemar Pompe, Warsaw, Poland

U635. Evaluate

$$\int_0^1 \frac{\sin x \sin \pi x}{\cos\frac{2x-1}{2}}\,dx.$$

Proposed by Vasile Lupulescu, University of Târgu Jiu, România

U636. Evaluate

$$\iiint_D e^{\sqrt{x^2+y^2}/2}\frac{zy^2}{(x^2+y^2)^{\frac{3}{2}}}\frac{(\frac{1}{2}(x^2+y^2+z^2)-1)^3}{\sqrt{4-x^2-y^2-z^2}}\frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}},$$

where $D = \{(z-1)^2+y^2+x^2 \le 1\}$.

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Olympiad Problems

O631. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{x}{4y^2+yz+4z^2}+\frac{y}{4z^2+xz+4x^2}+\frac{z}{4x^2+xy+4y^2}\geq\frac{1}{45}+\frac{14(xy+yz+zx)}{135},$$

Proposed by Marius Stănean, Zalău, România

O632. Find the largest integer $n\geq 8$ with the following property: it is possible to mark 64 cells of an $n\times n$ board such that each 2×3 rectangle and each 3×2 rectangle contains at least one marked cell.

Proposed by Josef Tkadlec, Czech Republic

O633. Let $ABCDEF$ and $A'B'C'D'E'F'$ be regular hexagons with the same orientation. Let $X=AA'\cap BB'$, $Y=DD'\cap EE'$, $Z=CC'\cap FF'$. Prove that points X, Y, Z are collinear.

Proposed by Waldemar Pompe, Warsaw, Poland

O634. Let $200<a_1<\cdots<a_n$ be positive integers such that for each positive integer d, there are at most $d-1$ consecutive terms with difference d. Prove that

$$\frac{1}{a_1}+\cdots+\frac{1}{a_n}\leq\frac{1}{2}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O635. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{128(ab+bc+ca)^2}{(a+b)(b+c)(c+a)}+\frac{81}{abc}\geq 225.$$

Proposed by Marius Stănean, Zalău, România

O636. Prove that there are no nonzero polynomials $P(x)$ with real coefficients such that

$$P(-a+b+c)+P(a-b+c)+P(a+b-c)=0,$$

for all real numbers a, b, c which satisfy the condition $a^4+b^4+c^4=2$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran