Projective plane, extended complex plane

Nguyễn Minh Hà

Abstract. *First, I will introduce the basic definitions of two concepts: projective plane and extended complex plane. Then, I prove that the concept of cross ratio in the extended complex plane is the extension of cross ratio in the projective plane. Finally, I discuss the meaning of two concepts: the projective plane, and the extended complex plane.*

1. Introduction

First, without using the coordinate system nor the projection that turns the sphere into a plane, I will introduce the following concepts in a rudimentary way: the projective plane and the extended complex plane. Then, I prove that the concept of cross ratio in the extended complex plane is the extension of cross ratio in the projective plane. Finally, using Pappus theorem and Fuerbach theorem, I will talk about the meaning of two concepts: the projective plane and the extended complex plane.

For easy to follow, please note that: the normal plane is called the real plane; the ordinary plane on which a complex coordinate system is attached is called the complex plane.

2. Projective plane, point at infinity, line at infinity

The set of lines in the real plane which are pairwise parallel or coincide $a \,//\, b \,//\, c \,//\dots$ is noted as a point and such point is called point at infinity, denoted as ∞_a or ∞_b or ∞_c … Obviously $\infty_a = \infty_b = \infty_c$ …

The real plane equipped with additional points at infinity is called the projective plane.

The line a (in the real plane) equipped with additional point at infinity ∞_a - denoted as a_∞ - is called the projective line (in the projective plane) generated by line a.

In the projective plane there are no parallel lines.

If the lines a and b are parallel (in the real plane) then a_∞ and b_∞ intersect at the point at infinity (in the projective plane), say $a_\infty \cap b_\infty = \infty_a = \infty_b$.

For projective plane, the axioms of connection in Euclidean plane geometry are still valid.

Since each line belongs to only one set of lines which are pairwise parallel or coincide, so each projective line cannot not have two different points at infinity.

Let a and b be two intersecting lines (in the real plane). Since a intersects b then $\infty_a \neq \infty_b$. Since two distinct points of a straight line completly determine that line, then there exists a line Δ (in the projective plane) goes through ∞_a and ∞_b. Suppose there exists a point C onΔ and C is not a point at infinity. Let c be the line containing C and it is parallel to the line a (in the real plane). Since $a \,//\, c$ so $\infty_a = \infty_c \in c_\infty$. Hence c_∞ and Δ both contains ∞_a and C. As a result, $c_\infty \equiv \Delta$. In other words, c_∞ contains ∞_a and ∞_b. Therefore $\infty_a = \infty_b$: a contradiction. So the line Δ contains only the points at infinity.

Let d be a line in the real plane. Since there are no parallel lines in the projective plane so d_∞ intersects Δ. Let E be the intersection point of d_∞ and Δ. Note that the line Δ contains only the points at infinity, hence E is a point at infinity. Therefore, with remark that d_∞ cannot have two distinct points at infinity, we have $\infty_d = E \in \Delta$.

In summary, the set of all points at infinity in the projective plane is a line, and that line is called the line at infinity.

3. Extended complex plane, point at infinity

The set of all lines in a complex plane is regarded as a point, and that set is called the point at infinity, and it is denoted as ∞.

The complex plane on which the point ∞ is attached is called the extended complex plane.

There are no line on the extended complex plane.

The line *a* (in the complex plane) on which the point ∞ is attached is noted as a^{∞} and it is called the extended complex circle (in the extended circle plane) generated by the the line *a*.

If two lines *a* and *a* are parallel (in the complex plane) then two corresponding extended complex circle a^{∞} and b^{∞} tangent to each other at point ∞ (in the extended complex plane). If two lines *a* and *b* intersect at a point *P* (in the complex plane) then two corresponding extended complex circle a^{∞} and b^{∞} intersect at two points *P* and ∞ (in the extended complex plane).

4. Cross ratio in the complex plane and cross ratio the projective plane

4.1.Cross ratio in the real plane

The cross ratio in the real plane is a familiar concept. However, for convenience, I still reiterate this concept.

Definition 1. Let *A*, *B*, *C* and *D* be four distinct points on a line (in the real plane), then the cross ratio of the quadruple *A*, *B*, *C*, *D* (with order), is denoted as (*A B C D*) , and it is defined as follow:

$$(ABCD) = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$$

4.2. Cross ratio in the projective plane

Definition 2. Let *A*, *B*, *C*, *D* be four distinct points on a line Δ_{∞} (in the projective plane), then the cross ratio of the quadruple *A*, *B*, *C*, *D* (with order) , is denoted as (*ABCD*), and it is defined as follow:

1) $(ABCD) = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$ if $A, B, C, D \neq \infty_{\Delta}$.

2) $(ABCD) = \frac{\overline{DB}}{\overline{CB}}$ if $A = \infty_{\Delta}$.

3) $(ABCD) = \frac{\overline{CA}}{\overline{DA}}$ if $B = \infty_{\Delta}$.

4) $(ABCD) = \frac{\overline{DB}}{\overline{DA}}$ if $C = \infty_{\Delta}$.

5) $(ABCD) = \frac{\overline{CA}}{\overline{CB}}$ if $D = \infty_{\Delta}$.

Definition 3. If ∞_a, ∞_b, ∞_c and ∞_d be four distinct points on a line at infinity (in the projective plane), then the cross ratio of the quadruple ∞_a, ∞_b, ∞_c, ∞_d (with order), is denoted as (∞_a, ∞_b, ∞_c, ∞_d), and it is defined as follow:

$$(\infty_a\infty_b\infty_c\infty_d) = (A'B'C'D').$$

In the above equation, *A', B', C'* and *D'* are the intersection points of the line Δ' and the lines $O\infty_a$, $O\infty_b$, $O\infty_c$ and ∞_d , respectively, where *O* is an arbitrary point not belonging to the line Δ; Δ' is an arbitrary line which is not go through the point *O* and it is different from Δ.

It has been proved that (*A' B' C' D'*) does not depend on the choice of the point *O* and the line Δ'.

It is easy to prove that the properties of cross ratio in the real plane are still valid on the projective plane.

5. Cross ratio on the complex plane and extended complex plane

5.1. Cross ratio in the complex plane

Each point *M* in the complex plane defines a complex number *z, z* is called the affix of *M*. To denote that the point *M* has *z* as its affix, we write *M(z)*.

Definition 4. Let *A(a), B(b), C(c)* and *D(d)* be four distinct points in the complex plane, then the cross ration of the quadruple *A, B, C, D* (with order) is denoted as [*ABCD*] and it is defined as:

$$[ABCD] = \frac{a-c}{b-c} : \frac{a-d}{b-d}$$

It has been proved that:

- [*ABCD*] does not depend on the choice of the complex coordinate.
- [*ABCD*] is a real number if and only if *A, B, C*, and *D* are on same line or they are on the same circle [1].

5.2. Cross ratio in the extended complex plane

Definition 5. Let *A, B, C* and *D* be four distinct points on an extended complex circle Δ^∞ (in the extended complex plane), then the cross ration of the quadruple *A, B, C* and D (with order) is denoted as [*ABCD*] and it is defined as follow:

1) $[ABCD] = \frac{a-c}{b-c} : \frac{a-d}{b-d}$ if $A, B, C, D \neq \infty.$
2) $[ABCD] = \frac{b-d}{b-c}$ if $A = \infty.$
3) $[ABCD] = \frac{a-c}{a-d}$ if $B = \infty.$
4) $[ABCD] = \frac{b-d}{a-d}$ if $C = \infty.$
5) $[ABCD] = \frac{a-c}{b-c}$ if $D = \infty.$

It has been proved that:

- [*ABCD*] does not depend on the choice of the complex coordinate.

• [*ABCD*] is a real number if and only if *A*, *B*, *C* and *D* are on the same circle (in the complex plane) or they are on the same extended complex circle (in the extended complex plane).

It is easy to prove that the properties of cross ratio in the complex plane are still valid in the extended complex plane.

6. The basic results

Theorem 1. If the points *A*, *B*, *C* and *D* are on the same line (in the complex plane and it can be regarded as in the real plane) then [*ABCD*] = (*ABCD*).

Proof.

Without lost of generality, suppose that *A*, *B*, *C* and *D* are on the real axis in the complex plane with A(a), B(b), C(c), D(d).

Since *a, b, c* and *d* are real numbers then

$$\left[ABCD\right]=\frac{a-c}{b-c}:\frac{a-d}{b-d}=\frac{\overline{CA}}{\overline{CB}}:\frac{\overline{DA}}{\overline{DB}}=(ABCD).$$

Theorem 2. If the points *A*, *B* and *C* are on a line Δ(in the real plane and it can be regarded as in the complex plane) then

1) $\left[\infty ABC\right]=(\infty_\Delta ABC).$

2) $\left[A\infty BC\right]=(A\infty_\Delta BC).$

3) $\left[AB\infty C\right]=(AB\infty_\Delta C).$

4) $\left[ABC\infty\right]=(ABC\infty_\Delta).$

Proof.

Without lost of generality, suppose that Δis the real axis in the complex plane and *A(a), B(b), C(c).*

By definitons 2 and 5, we have:

1) $\left[\infty BCD\right]=\frac{b-d}{b-c}=\frac{\overline{DB}}{\overline{CB}}=(\infty_\Delta BCD).$

2) $\left[A\infty CD\right]=\frac{a-c}{a-d}=\frac{\overline{CA}}{\overline{DA}}=(A\infty_\Delta CD).$

3) $\left[AB\infty D\right]=\frac{b-d}{a-d}=\frac{\overline{DB}}{\overline{DA}}=(AB\infty_\Delta D).$

4) $\left[ABC\infty\right]=\frac{a-c}{b-c}=\frac{\overline{CA}}{\overline{CB}}=(ABC\infty_\Delta).$

Theorem 1 asserts that the concept of cross ratio in the complex plane is the extension of cross ratio in the real plane.

The theorems 1 and 2 assert that basically the concept of cross ratio in the extended complex plane is the extension of cross ration in the projective plane.

7. Applications

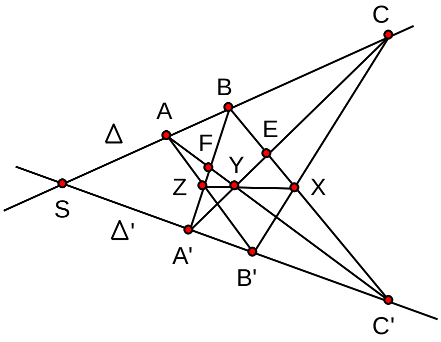
For some geometry theorems in real plane (complex plane), if we treat them as theorems in projective plane (extended complex plane), not only we get simpler proofs for those theorems, but also get new theorems, different from those in the real planes (complex plane). Pappus theorem and Feuerbach theorem are two of them.

Theorem 3. Given two lines Δ and Δ'. The points *A, B* and *C* are on Δ. The points *A', B'* and *C'* are on Δ'. Let *X = BC'* ∩ *B'C, Y = CA'* ∩ *C'A, Z = AB'* ∩ *A'B*. Hence *X, Y* and *Z* are colinear (Pappus).

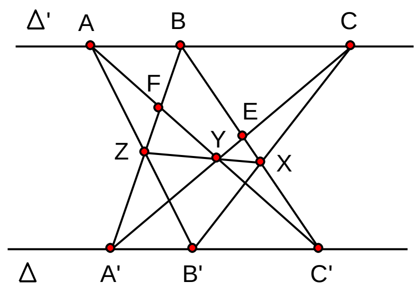
Theorem 3 is a theorem in real plane geometry. If theorem 3 is considered as a theorem in the projective plane, then the proof of theorem 3 in the projective plane is much more easier that the proof of theorem 3 in the real plane.

Here is a proof of theorem 3 in the projective plane.

Let *S* = Δ∩Δ' , *E = BC'* ∩ CA', *F = AC'* ∩ *BA'* (fig.1, fig.2).



(fig.1)



(fig.2)

It's obvious that:

(BEXC') = C(BEXC') = C(SA'B'C') = (SA'B'C') = A(SA'B'C') = A(BA'ZF) = (BA'ZF).

Hence *EA', XZ* and *C'F* are concurrent.

Therefore *X, Y* and *Z* are colinear.

Remark.

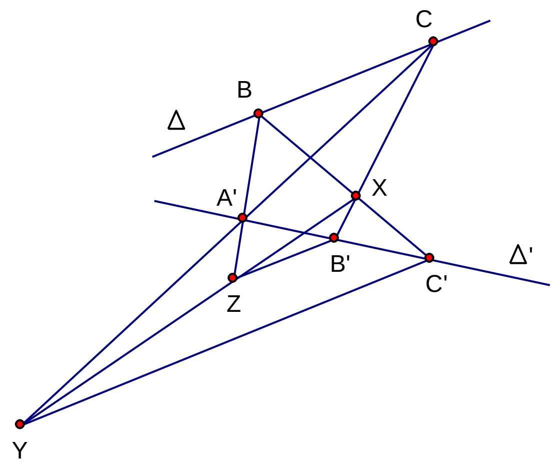
+ In Fig. 2, S = $\infty_{\Delta} = \infty_{\Delta'}$.

+ In the real plane there are no point *S* .

- If A = ∞_{Δ} then, in the real plane, the line *AB'* goes through *B'* and it is parallel with the line Δ and the line *AC'* goes through *C'* and it is parallel with the line Δ.

With the remarks mentioned above, we have new theorem in the real plane.

Theorem 4. Given two lines Δand Δ'. The points *B* and *C* are on Δ. The points *A', B'* and *C'* are on Δ'. Let *X = BC'* ∩ *B'C*. Let *Y* be the intersection of *CA'* and the line that goes through *C'* and it is parallel with Δ. Let *Z* be the intersection of *BA'* and the line that goes through *B'* and it is parallel with Δ. Therefore *X, Y* and *Z* are colinear (fig.3).

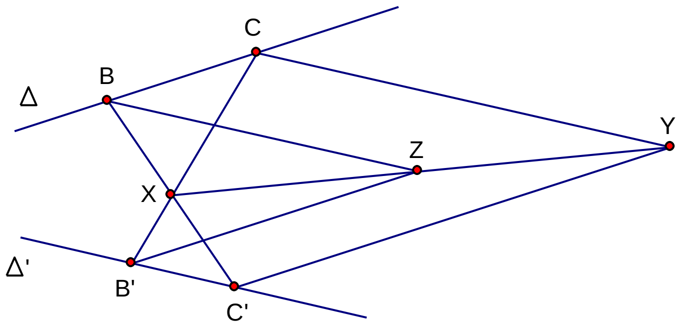


(fig.3)

- If $A = \infty_\Delta$ and $A' = \infty_{\Delta'}$, then, in the real plane, the line *AB'* is the line going through *B'* and it is parallel with Δ, the line *BA'* is the line going through *B* and it is parallel with Δ', the line *AC'* is the line going through *C'* and it is parallel with Δ, the line *CA'* is the line going through *C* and it is parallel with Δ'.

With the remark mentioned above, we have a new theorem in the real plane.

Theorem 5. Given two lines Δand Δ'. The points *B* and *C* are on Δ. The points *B'* and *C'* are on Δ'. Let $X = BC' \cap B'C$. Let *Y* be the intersection of the line passing through *C*, parallel with Δ' and the line passing through *C'*, parallel with Δ. Let *Z* be the intersection of the line passing through *B*, parallel with Δ' and the line passing through *B'*, parallel with Δ. Therefore, *X*, *Y* and *Z* are colinear (fig.4).



(fig.4)

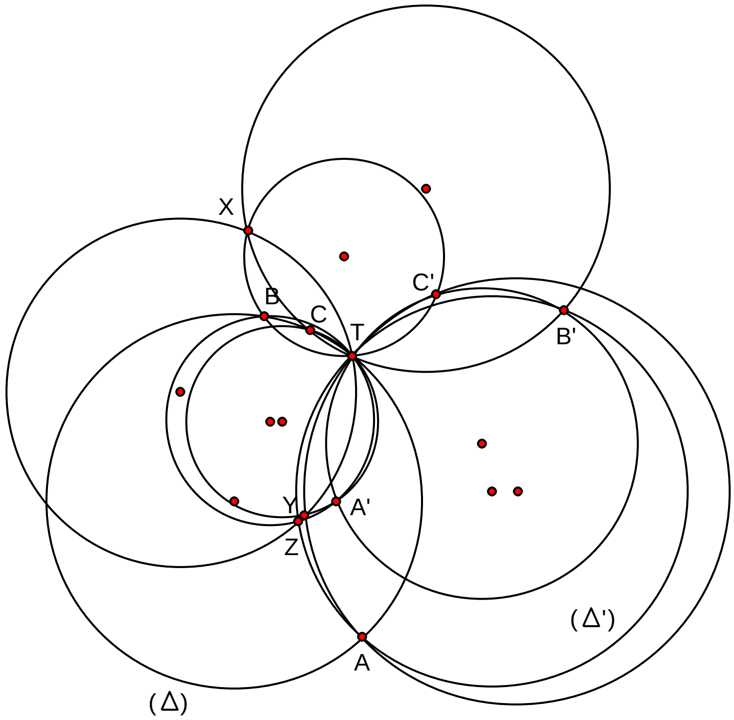
Remark.
The line AA' $\equiv \infty_\Delta\infty_{\Delta'}$ is the line at infinity.

In the projective plane, the theorems 3, 4 and 5 are the same.

Similarly, if we change one or two points among the points *A*, *B*, *C*, *A'*, *B'*, *C'*, *X*, *Y* and *Z* by the point(s) at infinity on the projective line containing them, then from theorem 3 we have a lot of new theorems (different from theorem 3) in the real plane.

- Theorem 3 is also a theorem in the complex plane. If theorem 3 is considered as a theorem in the extended complex plane, then eight circles in theorem 3 are eight extended complex circles.. If some circles in eight extended complex circles in theorem 3 are circles, then we have a lot of new theorems (different from theorem 3) in the complex plane. The next theorem is derived from theorem 3 by replacing all eight extended complex circles into circles.

Theorem 6. Given two circles (Δ) and (Δ') that have a common point *T*. Let the points *A*, *B* and *C* be on (Δ). Let the points *A', B'* and *C'* be on (Δ'). Let {T, X} = (TBC') ∩ (TB'C), {T, Y} = (TCA') ∩ (TB'C), {T, Z} = (TAB') ∩(TA'B). Therefore *X, Y, Z* and *T* are on the same circle or they are colinear (fig.5).



(fig.5)

In theorem 6, the symbol *(UVW)* means that the circle containing three points *U*, *V* and *W*.

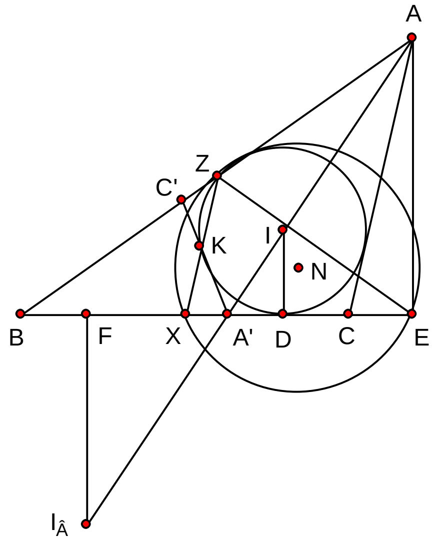
In the high school geometry curriculum there is no extended complex plane concept. Hence, theorem 6 still needs to be proved. One of the ways to prove theorem 6 is to use theorem 3 and an inversion of center *T* with any radius, where *T* does not lie on any eight lines mentioned in theorem 3.

Theorem 7. Given triangle *ABC*, let *(I)* and *(N)* be its inscribed circle and Euler circle, respectively. Therefore *(I)* and *(N)* are tangent to each other (Feuerbach).

Theorem 7 is a theorem in the real plane as well as a theorem in the complex plane. If theorem 7 is considered as a theorem in the extended complex plane then the proof for theorem 7 in the extended complex plane is much more simpler than the proof of theorem 7 in the complex plane.

Here is the proof of theorem 7 in the extended complex plane.

Let *A'* be the intersectiong of *AI* and *BC*; let *C'* be the symmetric point of *C* with respect to *AI*; let *X* and *Z* are the midpoints of *BC* and *AB*, respectively; let *K* be the intersection of *A'C'* and *XZ*; let I_A be the excenter relative to the vertex *A* of the triangle *ABC*; let *D, E* and *F* be the orthogonal projections of *I, A,* and I_A onto BC, respectively (fig.6).



(fig.6)

Since *A'C* is a tangent to (I) then *A'C'* is a tangent to *(I)* (1).

It is clearly that $(II_A AA') = -1$.

Combine with $ID // I_A F // AE$, we have $(FDEA') = -1$.

It is clearly that $XD = XF$.

Then, by Newton formula, we have $XD^2 = \overline{XE}.\overline{XA'}$.

Note that *C'A* and *AC* are the symmetries of *CA'* and *AB* with respect to *AI*; *KP // AC* and the triangle *ZBE* is isoceles with *ZB=ZE*, we have

$$(KA', KZ \equiv (KA', AI) + (AI, KZ) \equiv (C'A', AI) + (AI, AC) (mod\,\pi)$$

$$\equiv (AI, CA') + (AB, AI) \equiv (AB, CA') \equiv (ZB, EB) \equiv (BE, ZE) \equiv (EA', EZ) (mod\,\pi).$$

Therefore the points *K, A', Z* and *E* lie on a circle.

So $\overline{XE}.\overline{XA'} = \overline{XZ}.\overline{XK}$.

Hence $XD^2 = \overline{XE}.\overline{XA'} = \overline{XP}.\overline{XK} = k$.

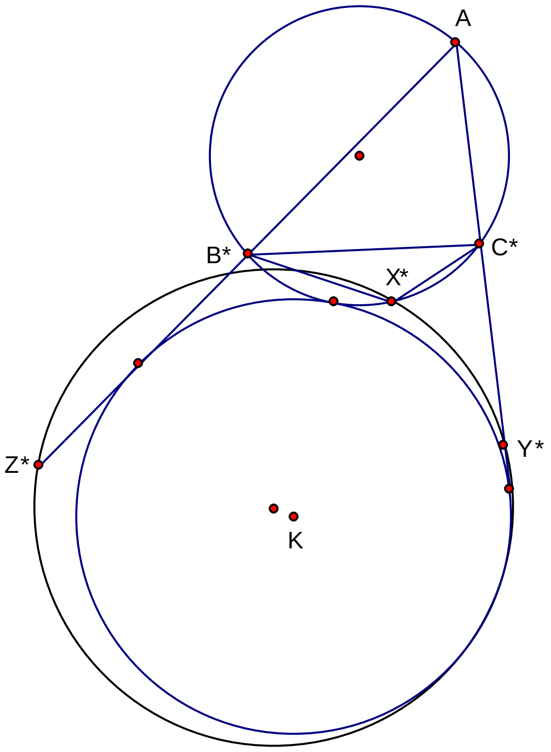
Therefore, by the inversion N_X^k, the points *X, D, E* and *Z* are transformed into the points ∞, *D, A', K*.

Combine with the fact that *XD* is a tangent to (I), note that the circle *(N)* contains *X, E* and *Z*, hence, by the inversion N_X^k, the circle *(I)* is transformed into the circle *(I)* and the circle (*N*) is transformed into the extended complex circle (∞*A'C'*) (2).

Using (1) and (2), we have the circles *(I)* and *(N)* are tangent to each other.

- Let *X, Y* and *Z* be the mitpoints of *BC, CA* and *AB*, respectively; let N_A^k be the inversion of center *A* with radius k ($k \in \mathbb{R}/\{0\}$).

Using N_A^k, the points ∞ *, B, C, X, Y* and *Z* are transformed into the points *A, B*, C*, X*, Y** and *Z**.



(h.7)

Using N_A^k , the circle *(I)* is transformed into the circle *(K)*.

Since *X* is the midpoint of *BC* then $(BCX\infty_{BC})=-1$.

Hence, by theorem 2, $[BCX\infty]=-1$.

Therefore $[B^*C^*X^*A]=-1$.

In other words, *AB*X*C** is a harmonic quadrilateral.

So $\overline{AY}.\overline{AY^*}=\overline{AC}.\overline{AC^*}=2\overline{AY}.\overline{AC^*}$.

Hence $\overline{AY^*}=2\overline{AC^*}$.

Similarly $\overline{AZ^*}=2\overline{AB^*}$.

Since *(I)* is inscribed the triangle *ABC* then *(K)* is the A-mixtilinear excircle of triangle *ABC*.

By theorem 7, the circles *(XYZ)* and *(I)* are tangent to each other.

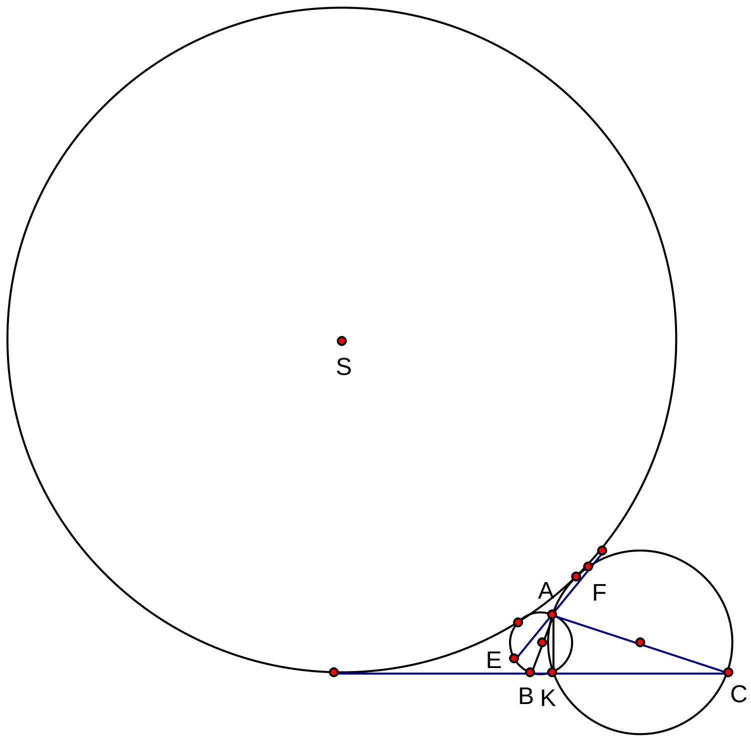
Hence the circles *(X*Y*Z*)* and (K) are tangent to each other (fig.7).

Therefore, by using the inversion N_a^k in the complex plane, theroem 7 is transformed in to a new theorem, which is different from theorem 7.

Theorem 8. Given a harmonic quadrilateral ABXC. Let *Y* and *Z* be the reflecting points of *A* over *C* and *B*, respectively. Let *(K)* be the A-mixtilinear excircle of triangle *ABC*. Therefore, the circles *(K)* and *(XYZ)* are tangent to each other.

- Let *K* be the orthogonal projection of *A* onto *BC*. Let N_k^k be the inversion of center *K* with radius *k* ($k\in\mathbb{R}/\{0\}$). Same as above, using the inversion N_k^k in the complex plane, theroem 7 is transformed in to a new theorem, which is different from theorem 7.

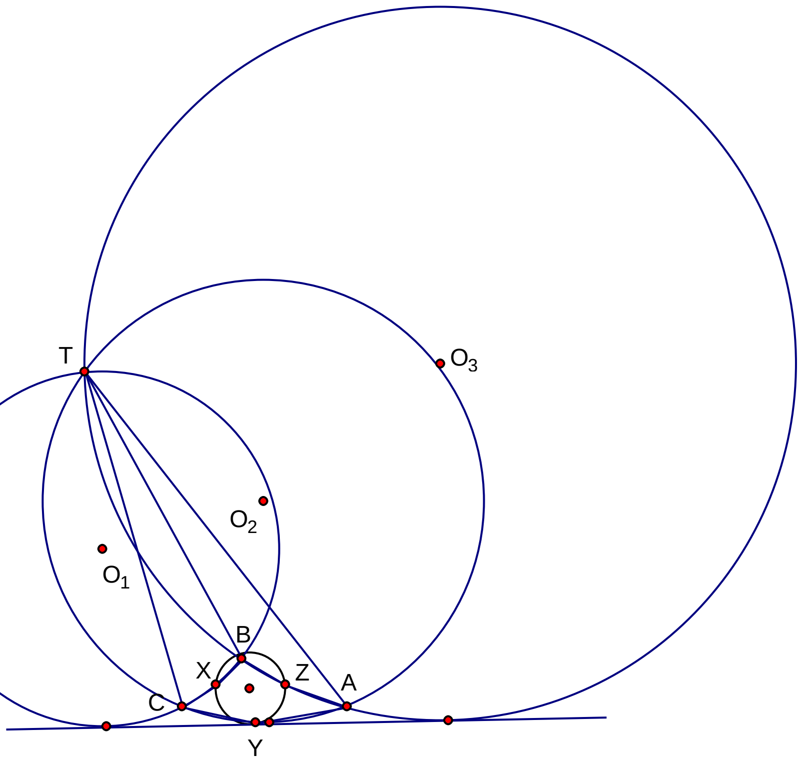
Theorem 9. Given triangle *ABC*. Let *(S)* be the circle in the half-plane with *BC* as boundary line, containing *A*, having *BC* as tangent line and it is tangent with the circles with diameters *AB* and *AC*. Let *K* be the orthogonal projection of *A* onto *BC*. Let *E* and *F* be the reflecting points of *K* over *AB* and *AC*, respectively. Then the circle *(S)* has *EF* as a tangent line (fig.8).



(h.8)

- Let *T* be an arbitrary point of tangency on *(I)* and it is different to other points of tangency of *(I)* with *BC, CA* and *AB*. Let N_T^k be the inversion of center *T* and ratio *k* ($k \in \mathbb{R} / \{0\}$). Same as above, using the inversion N_t^k in the complex plane, theorem 7 is transformed into another new theorem which is different from theorem 7.

Theorem 10. Given three circles (O_1), (O_2) and (O_3) going through a point *T* and they are tangent with a line Δ. Let *A*, *B* and *C* are the second point of intersections ot the pairs of circles ((O_2), (O_3)), ((O_3), (O_1)) and ((O_1), (O_2)), respectively. Let *X, Y* and *Z* be the points in (O_1), (O_2) and (O_3), respectively, such that the quadrilaterals *TBXC*, *TCYA* and *TAZB* are harmonic quadrilaterals. Therefore the points *X*, *Y* and *Z* are in a circle which is tangent with Δ or the points *X*, *Y* and *Z* are in a line which is parallel with Δ (fig.9).



(fig.9)

In the extended complex plane, the theorems 7, 8, 9, 10 are the same.

If we choose the center and radius of an inversion appropriately, then in the complex plane, we can derive many new theorems, different from theorem 7.

Acknowledgements.

We thank Associate Professor – Doctor Nguyễn Doãn Tuấn for his valuable comments which help me complete this paper.

We thank Vũ Thiện Căn for translating this paper to English.

REFERENCES

[1] Đoàn Quỳnh, Complex number with plane geometry, Viet Nam Education Publishing House, 1997, p 157.

HA NOI UNIVERSITY EDUCATION
HA NOI VIET NAM

Email address: minhha27255@yahoo.com