Junior problems

J631. Find the least positive integer n for which $n^4 - 2023n^2 + 1$ is a product of two primes.

Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Monique McKenrick, SUNY Brockport, USA
First we need positive numbers which means

$$n^4 > 2023n^2 \iff n^2 > 2023 \iff n > 44$$

We test the outcome when we plug in $n = 45, 46, ...$ and we see that the lest one for which the outcome is a product of two primes is 50 when we get

$$P(50) = 1192501 = 251 \cdot 4751$$

Thus the answer is 50.

Also solved by Sundaresh Harige, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Konstantinos Nakis, Athens, Greece; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Sophia Tiffany, SUNY Brockport, NY, USA; Theo Koupelis, Clark College, WA, USA; Anderson Torres, Brazil; Alexander Lee, Chadwick International School, South Korea; Soham Bhadra, India; Adam John Frederickson, Utah Valley University, UT, USA; Kausthubh Prasad, Centre for Advanced Learning, Mangalore, India; G. C. Greubel, Newport News, VA, USA.

J632. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a(b+c)^5}+\frac{1}{b(c+a)^5}+\frac{1}{c(a+b)^5} \geq \frac{3}{32}.$$

Proposed by Mihaly Bencze, Braşov and Neculai Stanciu, Buzau, România

First solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
Applying Cauchy-Schwarz inequality we get

$$\left[\frac{1}{a(b+c)^5}+\frac{1}{b(c+a)^5}+\frac{1}{c(a+b)^5}\right]\left[a(b+c)+b(c+a)+c(a+b)\right] \geq \left[\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right]^2.$$

On the other hand, Cauchy-Schwarz inequality also gives us

$$\begin{aligned}\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2} &\geq \frac{1}{3}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)^2\\ &\geq \frac{1}{3}\left[\frac{9}{2(a+b+c)}\right]^2\\ &= \frac{27}{4(a+b+c)^2}\\ &= \frac{3}{4}.\end{aligned}$$

Combining these inequalities we obtain

$$\frac{1}{a(b+c)^5}+\frac{1}{b(c+a)^5}+\frac{1}{c(a+b)^5} \geq \frac{9}{32(ab+bc+ca)} \geq \frac{27}{32(a+b+c)^2} = \frac{3}{32}.$$

The conclusion follows.

Second solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain
The inequality may be written as

$$\frac{1}{a(3-a)^5}+\frac{1}{b(3-b)^5}+\frac{1}{c(3-c)^5} \geq \frac{3}{32}.$$

Function $f(x)=\dfrac{1}{x(3-x)^5}$ is convex for $x \in [0,3)$ because $f''(x)=-\dfrac{6(7x^2-7x+3)}{(x-3)^7x^3}>0$.
By Jensen's inequality $\dfrac{1}{a(3-a)^5}+\dfrac{1}{b(3-b)^5}+\dfrac{1}{c(3-c)^5} \geq 3\dfrac{1}{\frac{a+b+c}{3}\left(3-\frac{a+b+c}{3}\right)^5}=\dfrac{3}{2^5}=\dfrac{3}{32}$.

Also solved by Arkady Alt, San Jose, CA, USA; Sundaresh Harige, India; Anderson Torres, Brazil; Batakogias Panagiotis, High School of Velestino, Greece; Alexander Lee, Chadwick International School, South Korea; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Konstantinos Nakis, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA; Matthew Too, Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Jiang Lianjun, Quanzhou second Middle School, GuiLin, China; Daniel Pascuas, Barcelona, Spain; Monil Patel, University of Calgary, Canada.

J633. Let a, b, c, t be positive real numbers with $t \geq 1$. Prove that

$$\frac{ta^3 + a^2b}{a+b} + \frac{tb^3 + b^2c}{b+c} + \frac{tc^3 + c^2a}{c+a} \geq \frac{t+1}{2}(ab + bc + ca).$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
Applying Cauchy-Schwarz inequality we get

$$\sum_{\text{cyc}} \frac{ta^3}{a+b} = \sum_{\text{cyc}} \frac{ta^4}{a(a+b)} \geq \frac{t(a^2+b^2+c^2)^2}{a^2+b^2+c^2+ab+bc+ca},$$

$$\sum_{\text{cyc}} \frac{a^2b}{a+b} = \sum_{\text{cyc}} \frac{a^2b^2}{b(a+b)} \geq \frac{(ab+bc+ca)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

Adding them up we obtain

$$\sum_{\text{cyc}} \frac{ta^3 + a^2b}{a+b} \geq \frac{t(a^2+b^2+c^2)^2 + (ab+bc+ca)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

It's enough to show that

$$\frac{t(a^2+b^2+c^2)^2 + (ab+bc+ca)^2}{a^2+b^2+c^2+ab+bc+ca} \geq \frac{t+1}{2}(ab+bc+ca).$$

Let $a^2 + b^2 + c^2 = x$ and $ab + bc + ca = y$ with note that $x \geq y$, the inequality becomes

$$\frac{tx^2 + y^2}{x+y} \geq \frac{(t+1)x}{2}.$$

Now we have

$$\frac{tx^2 + y^2}{x+y} - \frac{(t+1)x}{2} = \frac{(x-y)(2tx + (t-1)y)}{2(x+y)} \geq 0$$

which is true because $x \geq y$ and $t \geq 1$. The proof is completed.

Also solved by Arkady Alt, San Jose, CA, USA; Soham Bhadra, India; Jiang Lianjun, Quanzhou second Middle School, GuiLin, China; Daniel Pascuas, Barcelona, Spain; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J634. Prove that for every positive integer n, the equation $x^2+xy+y^2=(xy)^n$ has no integer solution (x,y) except for $(x,y)=(0,0)$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Pascuas, Barcelona, Spain

For $n=1$ the equation is $x^2+y^2=0$, whose only real solution (x,y) is $(x,y)=(0,0)$, so we may assume that $n>1$. Note that if $x^2+xy+y^2=(xy)^n$ and either $x=0$ or $y=0$, then $x=y=0$. Thus we only have to prove that there are nonzero integers x,y satisfying the equation. We proceed by contradiction. Assume that x,y are nonzero integer solutions of the equation. Then y divides x^2 and x divides y^2. It follows that $|x|$ and $|y|$ have the same prime divisors. Let $|x|=p_1^{\alpha_1}\cdots p_m^{\alpha_m}$ and $|y|=p_1^{\beta_1}\cdots p_m^{\beta_m}$ be their prime decompositions. Then $x^2=p_1^{2\alpha_1}\cdots p_m^{2\alpha_m}$, $y^2=p_1^{2\beta_1}\cdots p_m^{2\beta_m}$, and $xy=qp_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}$, where $q=\pm1$. Now note that $p_j^{n(\alpha_j+\beta_j)}$ divides $(xy)^n$, while, if $\alpha_j\neq\beta_j$ and $\gamma_j=\min(\alpha_j,\beta_j)$, then $p_j^{2\gamma_j}$ is the highest power of p_j dividing x^2+xy+y^2 and $2\gamma_j<n(\alpha_j+\beta_j)$. It follows that $\alpha_j=\beta_j$, for $j=1,\ldots,m$, so $|x|=|y|$, and therefore the equation gives that $3x^2=x^{2n}$, if $q=1$, and $2x^2=(-1)^n x^{2n}$, if $q=-1$. Hence we have that $3=x^{2(n-1)}$, if $q=1$, and $2=(-1)^n x^{2(n-1)}$, if $q=-1$. Both identities are impossible because 2 and 3 are prime numbers and $2(n-1)\geq 2$, and that finishes the proof that the equation $x^2+xy+y^2=(xy)^n$ has no integer solution (x,y) except for $(x,y)=(0,0)$.

Also solved by Theo Koupelis, Clark College, WA, USA; Sundaresh Harige, India; Anderson Torres, Brazil; Soham Bhadra, India; Adam John Frederickson, Utah Valley University, UT, USA; Batakogias Panagiotis, High School of Velestino, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Konstantinos Nakis, Athens, Greece; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

J635. Let a, b, c be positive real numbers such that $a^4 - 23a^2 + 1 = 0$, $b^4 - 223b^2 + 1 = 0$, and $c^4 - 2023c^2 + 1 = 0$. Prove that

$$a^2b^2c^2 - nabc + 1 = (ab+1)(bc+1)(ca+1)$$

for some integer n.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Adam John Frederickson, Utah Valley University, UT, USA

The expressions can be factored as

$$\begin{aligned} a^4 - 23a^2 + 1 &= (a^2 - 5a + 1)(a^2 + 5a + 1), \\ b^4 - 223b^2 + 1 &= (b^2 - 15b + 1)(b^2 + 15b + 1), \\ c^4 - 2023c^2 + 1 &= (c^2 - 45c + 1)(c^2 + 45ac + 1). \end{aligned}$$

Since $a, b, c > 0$, only the first factor from each can be 0, and therefore

$$a^2 = 5a - 1, \qquad b^2 = 15b - 1, \qquad c^2 = 45c - 1.$$

Then the following statements are equivalent:

$$\begin{aligned} a^2b^2c^2 - nabc + 1 &= (ab+1)(bc+1)(ca+1) \\ &= a^2b^2c^2 + a^2bc + ab^2c + abc^2 + ab + ac + bc + 1 \end{aligned}$$

$$\begin{aligned} -nabc &= a^2bc + ab^2c + abc^2 + ab + ac + bc \\ &= (5a-1)bc + (15b-1)ac + (45c-1)ab + ab + ac + bc \\ &= 65abc \end{aligned}$$

$$n = -65.$$

Also solved by Sundaresh Harige, India; Anderson Torres, Brazil; Alexander Lee, Chadwick International School, South Korea; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Konstantinos Nakis, Athens, Greece; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA.

J636. Let a, b, c be positive numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \le \frac{a+b+c}{2}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Daniel Pascuas, Barcelona, Spain

By taking common denominators, we have that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = \frac{P(a,b,c)}{Q(a,b,c)},$$

where

$$\begin{aligned} P(a,b,c) &= a^2b^2 + b^2c^2 + c^2a^2 + 2(a^2+b^2+c^2) + 3 = 2(a+b+c)(a+b+c-abc) \quad \text{and} \\ Q(a,b,c) &= (a^2+1)(b^2+1)(c^2+1) = a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2 + 1 \\ &= a^2b^2c^2 + (a+b+c)(a+b+c-2abc) + 4 = (a+b+c-abc)^2 + 4, \end{aligned}$$

since

$$\begin{aligned} a^2+b^2+c^2 &= (a+b+c)^2 - 2(ab+bc+ca) = (a+b+c)^2 - 6 \quad \text{and} \\ a^2b^2+b^2c^2+c^2a^2 &= (ab+bc+ca)^2 - 2abc(a+b+c) = 9 - 2abc(a+b+c). \end{aligned}$$

Therefore we obtain the desired inequality:

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = \frac{a+b+c}{2}\,\frac{4(a+b+c-abc)}{(a+b+c-abc)^2+4} \le \frac{a+b+c}{2}.$$

Also solved by Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Jiang Lianjun, Quanzhou second Middle School, GuiLin, China; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Konstantinos Nakis, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, România.

Senior problems

S631. Find all positive integers n for which

$$(n-1)! + (n+1)^2 = (n^2-41)(n^2+49)$$

Proposed by Adrian Andreescu, Dallas, USA

Solution by Sophia Tiffany, SUNY Brockport, NY, USA

$$(n-1)! + (n+1)^2 = (n^2-41)(n^2+49) \iff (n-1)! = n^4 + 7n^2 - 2n - 2010$$

First we write it as

$$(n-1)! = n^4 - 1 + 7n^2 - 7 - 2n + 2 - 2004 = (n-1)(n+1)(n^2+1) + 7(n-1)(n+1) - 2(n-1) - 2004$$

Therefore $n-1|2004$ and so $n \in \{2, 3, 4, 5, 7, 13, 168, 305, 502, 669, 1003, 2005\}$.
Then we write it as

$$(n-1)! = n^4 - 2^4 + 7n^2 - 28 - 2n + 4 - 1964 = (n-2)(n+2)(n^2+4) + 7(n-2)(n+2) - 2(n-2)$$

Therefore $n-2|1964$ and so $n \in \{3, 4, 7, 12, 199, 396, 985, 1972\}$.
Combining this with the previous set we get that

$$n \in \{3, 4, 7\}$$

Checking the three values above we see that the equality holds only when $n = 7$.

Also solved by Anderson Torres, Brazil; Soham Bhadra, India; Batakogias Panagiotis, High School of Velestino, Greece; Monil Patel, University of Calgary, Canada; G. C. Greubel, Newport News, VA, USA; Sundaresh Harige, India; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Konstantinos Nakis, Athens, Greece; Theo Koupelis, Clark College, WA, USA.

S632. Solve in real numbers the system of equations

$$\begin{aligned} 238^x + 2016^y &= 2030^x \\ 238^y + 2016^z &= 2030^y \\ 238^z + 2016^x &= 2030^z. \end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

The given system is equivalent to the system of equations

$$\begin{aligned} y &= f(x) \\ z &= f(y) \\ x &= f(z), \end{aligned} \tag{1}$$

where $f(x) = \log_{2016}(2030^x - 238^x)$. By substitution, we have that

$$x = f^3(x), \tag{2}$$

where $f^3(x) := f(f(f(x)))$. The same property holds also for y and z, so if α is a solution to the equation (2) if and only if (α, α, α) is a solution to the system (1). Thus, in order to solve the system of equation, we have to solve the equation (2). Let $A \subseteq \mathbb{R}$ and let

$$\text{Fix}(f) = \{x \in A \mid f(x) = x\}$$

be the set of the fixed points of f. We use the following lemma.

Lemma: Let $f : A \to A$ be an increasing function on $A \subseteq \mathbb{R}$ and let $\alpha \in A$. Then,

$$\text{Fix}(f) = \{\alpha\} \iff \text{Fix}(f^n) = \{\alpha\} \qquad \forall n \geq 2.$$

Proof. Assume that $\text{Fix}(f^n) = \{\alpha\}$ for all $n \geq 2$. Since $f^n(f(\alpha)) = f(f^n(\alpha)) = f(\alpha)$, then $f(\alpha) \in \text{Fix}(f^n)$. So, $f(\alpha) = \alpha$, i.e. $\alpha \in \text{Fix}(f)$. If $\beta \in \text{Fix}(f)$, then $\beta = f(\beta) = f(f(\beta)) = \ldots = f^n(\beta)$, which gives $\beta \in \text{Fix}(f^n)$, i.e. $\beta = \alpha$.

Conversely, let $\text{Fix}(f) = \{\alpha\}$ and let $\beta \in A$, $\beta \neq \alpha$. We will prove that $\beta \notin \text{Fix}(f^n)$. Since $\text{Fix}(f) = \{\alpha\}$, then $f(\beta) \neq \beta$. Assume without loss of generality that $f(\beta) > \beta$. Since f is increasing, we have $f(f(\beta)) \geq f(\beta) > \beta$, i.e. $f^2(\beta) > \beta$. Iterating the process, we get $f^n(\beta) > \beta$ for all $n \geq 2$, so $\beta \notin \text{Fix}(f^n)$.

Now, we have $f(x) = \log_{2016}(2030^x - 238^x)$ and

$$f(x) = x \implies 2030^x - 238^x = 2016^x \implies x = 2.$$

Observe that f is increasing on $[0, +\infty)$ and $f([0, +\infty)) = [0, +\infty)$. So, by the Lemma, we get $f^3(x) = x \implies x = 2$. From what we said at the beginning, it follows that $(2, 2, 2)$ is the unique solution to the given system of equations.

Also solved by Adam John Frederickson, Utah Valley University, UT, USA; G. C. Greubel, Newport News, VA, USA; Konstantinos Nakis, Athens, Greece; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S633. Let $ABCD$ be a convex quadrilateral with $CD = CB$ and $\angle BCD = 180^\circ - 2(\angle BAD)$. The orthogonal projection of A on BD is E and the orthogonal projections of the point E on AD and AB are F and K, respectively. Let O be the midpoint of the segment AE and let X be the intersection of AC and FK. Prove that $OX = AO \cdot \cos(\angle BAD)$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Theo Koupelis, Clark College, WA, USA

Let the perpendicular bisector from C to DB intersect the circumcircle of $\triangle BCD$ at P. Then $DPBC$ is cyclic and $\angle DPB = 2(\angle BAD)$. But $PB = PD$ because $\triangle BCD$ is isosceles, and thus P is the circumcenter of $\triangle DAB$. Also, PC is a diameter of the circle $(DPBC)$, and thus $\angle PDC = \angle PBC = 90^\circ$. Thus, CD andCB are external tangents to the circle (DAB). Therefore, AC is the A-symmedian of $\triangle DAB$. By construction, the quadrilateral $AFEK$ is cyclic because $\angle AFE = \angle AKE = 90^\circ$. Then O is the circumcenter of the circle $(AFEK)$. Also, $\angle BDF = \angle EDA = \angle FEA = \angle FKA$, and thus $DFKB$ is a cyclic quadrilateral, and FK is antiparallel to DB. It is well-known that a symmedian of a triangle from a certain vertex bisects any antiparallel to the side opposite to that vertex, and thus X is the midpoint of the chord FK. Therefore, $OX \perp FK$. Thus, $OX = OF \cdot \cos(\angle FOX) = OA \cdot \cos(\angle KAF) = OA \cdot \cos(\angle BAD)$.

S634. Prove that there are no integers a, b, c such that $a^3 - b^2 - c^2 + abc = 5$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We shall indeed prove for each k the following equation has no integer root.

$$y^2 - kyz + z^2 = k^3 - 5$$

If k is even then $\left(y - \frac{kz}{2}\right)^2 - \left(\frac{k^2}{4} - 1\right) z^2 = k^3 - 5$. Letting $d = \frac{k}{2}, u = y - \frac{kz}{2}$ then $u^2 - \left(d^2 - 1\right) z^2 = 8d^3 - 5$. If d is odd taking mod 8 we reach to the contradiction. If d is even, taking mod 4 we arrive at the contradiction. If k is odd then taking modulo 2 yields that y, z are even and therefore, $4 \mid y^2 + yz + z^2$ hence, $k^3 - 5 \equiv 9 \pmod 4$. That is, $k \equiv 1 \pmod 4$. Hence, $\left(y - \frac{kz}{2}\right)^2 - \frac{(k^2-4)z^2}{4} = k^3 - 5$ letting $(z, u) = (2v, y - vk)$ yielding $u^2 - \left(k^2 - 4\right) v^2 = k^3 - 5$ and $k \equiv 1 \pmod 4$.

If $k \equiv 1 \pmod{12}$ then $k + 2$ is divisible by 3 and $k^3 \equiv 1 \pmod 3$ yielding $v^2 \equiv 2 \pmod 3$, absurd. If $k \equiv 9 \pmod{12}$ then $k - 2$ is not divisible by 6 then if all the primes dividing $k - 2$ are of the form $12K + 1$ then $k - 2 \equiv \pm 1 \pmod{12}$ absurd. Hence, there is a prime $p \equiv 5$ or $7 \pmod{12}$ and $p \mid k - 2$. thus, $u^2 \equiv 3 \pmod p$ absurd.

If $k \equiv 5 \pmod{12}$ then $3 \mid k - 2$ taking $\pmod 3$ yielding $3 \mid u$. Writing $u = 3w, k = 12r + 5$ it follows that

$$3w^2 - (4r + 1)(12r + 7)v^2 = 2^6 3^2 r^3 + 2^4 3^2 5 r^2 + 2^2 3 5^2 r + 40$$

Taking mod 3 yields $-(r + 1)v^2 \equiv 1 \pmod 3$. Thus, $r \equiv 1 \pmod 3$ implying $k \equiv 17 \pmod{36}$.

Writing $k = 36s + 17$ and then $k - 2 = 3(12s + 5)$ if all primes dividing $12s + 5 = \frac{k-2}{3}$ are of the form $12K \pm 1$ then $\frac{k-2}{3} \equiv \pm 1 \pmod{12}$, absurd. Hence there is a prime $p \equiv 5$ or 7($\pmod{12}$ such that $p \mid k - 2$. Taking the last equation mod$p$ yielding

$$u^2 \equiv 3 (\mathrm{mod} p)$$

Absurd.

Also solved by Konstantinos Nakis, Athens, Greece.

S635. Find all positive rational numbers x such that

$$x^3 - \lfloor x \rfloor^3 - \{x\}^3 = \frac{162}{5},$$

where $\lfloor x \rfloor$ and $\{x\}$ are the greatest integer less than or equal to x and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Let $[x] = n$ and $\{x\} = f$. Then $(n+f)^3 - n^3 - f^3 = \frac{162}{5}$, implying $5nf(n+f) = 54$.

The equation $(5n)f^2 + (5n^2)f - 54 = 0$ is a quadratic and, since f is rational, its discriminant must be a perfect square. Hence $25n^4 + 1080n = (5k)^2$ for some positive integer$k$. Then $n = 5m$ for some positive integer m and so

$$625m^4 + 216m = k^2, \text{ but } (25m^2)^2 < 625m^4 + 216m < (25m^2+2)^2 \text{ for all } m \geq 3,$$

implying $625m^4 + 216m = (25m^2+1)^2$. However, $50m^2 - 216m + 1 = 0$ does not have integer solutions. Hence $m = 1$ or $m = 2$. Only $m = 1$ works, implying $k = 29$. Thus, $n = 5$ and $f = \frac{-125 \pm 5 \cdot 29}{50}$, yielding $x = 5 + \frac{2}{5} = \frac{27}{5}$.

Also solved by Srijan Sundar, Oxford, UK; Alexander Lee, Chadwick International School, South Korea; Soham Bhadra, India; Daniel Pascuas, Barcelona, Spain; Monil Patel, University of Calgary, Canada; Sundaresh Harige, India; Konstantinos Nakis, Athens, Greece; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S636. Prove that for any prime p the sum of the digits of $7^p + 13^p + 2023^p$ is not a prime.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Anderson Torres, Brazil

Let $F(n) = 7^n + 13^n + 2023^n$. Looking at $F(n) \pmod{10}$ we have

$$\begin{aligned}
F(n+2) &\equiv 7^n \cdot 7^2 + 13^n \cdot 13^2 + 2023^n \cdot 2023^2 \\
F(n+2) &\equiv 7^n \cdot 7^2 + 13^n \cdot 13^2 + 2023^n \cdot 2023^2 \\
F(n+2) &\equiv -(7^n + 13^n + 2023^n) \\
F(n+2) &\equiv -F(n)
\end{aligned}$$

Calculating for $n = 0, n = 1$ we obtain

$$\begin{aligned}
F(0) &\equiv 3 \pmod{10} \\
F(1) &\equiv 3 \pmod{10} \\
F(2) &\equiv 7 \pmod{10} \\
F(3) &\equiv 7 \pmod{10}
\end{aligned}$$

Therefore the last digit of $F(n)$ is at least 3, implying the sum of its digits is at least 3, with equality only when $n = 0$ - because if $n > 0$ then obviously $F(n)$ has at least $log_{10}(2023^n) > 3n > 1$ digits, implying $F(n)$ has at least 2 digits, therefore its digital sum is bigger than 3.

On the other hand, looking at $F(n) \pmod 3$, we have $F(n) = 1^n + 1^n + 1^n \equiv 0$.

It implies the sum of digits of $F(n)$ is multiple of 3.

Therefore, the sum of digits of $F(n)$ can possibly be prime only if it's equal to 3. And we know it can happen only if $n = 0$.

Therefore the digital sum of $F(n)$ is always a composite number. In particular, $F(p)$ is composite when p is prime.

Also solved by Srijan Sundar, Oxford, UK; Alexander Lee, Chadwick International School, South Korea; Soham Bhadra, India; Batakogias Panagiotis, High School of Velestino, Greece; Monil Patel, University of Calgary, Canada; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sundaresh Harige, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

Undergraduate problems

U631. Evaluate

$$\int_2^3 \frac{(x^2+2)\sqrt{x^4-x^2+4}}{x^3}\,dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Matthew Too, Brockport, NY, USA

Let $u = x - \frac{2}{x}$, $du = \left(1 + \frac{2}{x^2}\right)dx$. Then,

$$\int_2^3 \frac{(x^2+2)\sqrt{x^4-x^2+4}}{x^3}\,dx = \int_2^3 \left(1+\frac{2}{x^2}\right)\sqrt{3+\left(x-\frac{2}{x}\right)^2}\,dx = \int_1^{\frac{7}{3}} \sqrt{3+u^2}\,du.$$

We then use the substitution $u = \sqrt{3}\tan\theta$, $du = \sqrt{3}\sec^2\theta\,d\theta$ and the secant reduction formula to get

$$\int_1^{\frac{7}{3}} \sqrt{3+u^2}\,du = 3\int_{u=1}^{u=\frac{7}{3}} \sec^3\theta\,d\theta = \frac{3}{2}\sec\theta\tan\theta\Big|_{u=1}^{u=\frac{7}{3}} + \frac{3}{2}\int_{u=1}^{u=\frac{7}{3}} \sec\theta\,d\theta$$

$$= \left[\frac{3}{2}\sec\theta\tan\theta + \frac{3}{2}\ln|\tan\theta+\sec\theta|\right]_{u=1}^{u=\frac{7}{3}} = \left[\frac{1}{2}u\sqrt{u^2+3} + \frac{3}{2}\ln|u+\sqrt{u^2+3}| - \frac{3}{4}\ln 3\right]_1^{\frac{7}{3}}$$

$$= \frac{7\sqrt{19}}{9} + \frac{3}{2}\ln\left(\frac{7+2\sqrt{19}}{3}\right) - 1 - \frac{3}{2}\ln(3) = \frac{7\sqrt{19}-9}{9} + \frac{3}{2}\ln\left(\frac{7+2\sqrt{19}}{9}\right)$$

as desired.

Also solved by Daniel Pascuas, Barcelona, Spain; Monil Patel, University of Calgary, Canada; Yunyong Zhang, China Unicom, China; G. C. Greubel, Newport News, VA, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Sundaresh Harige, India; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U632. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Evaluate

$$S = \sum_{n=1}^{\infty} \frac{H_{n+2}}{n(n+1)}$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Henry Ricardo, Westchester Area Math Circle

We have, recognizing three telescoping series,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{H_{n+2}}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{H_{n+2}}{n} - \frac{H_{n+2}}{n+1} \right) \\
&= \sum_{n=1}^{N} \left(\frac{H_n + \frac{1}{n+1} + \frac{1}{n+2}}{n} - \frac{H_{n+1} + \frac{1}{n+2}}{n+1} \right) \\
&= \sum_{n=1}^{N} \left(\frac{H_n}{n} - \frac{H_{n+1}}{n+1} \right) + \sum_{n=1}^{N} \left(\frac{1}{n(n+1)} + \frac{1}{n(n+2)} - \frac{1}{(n+1)(n+2)} \right) \\
&= \sum_{n=1}^{N} \left(\frac{H_n}{n} - \frac{H_{n+1}}{n+1} \right) + \frac{1}{2} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2} \right) + \sum_{n=1}^{N} \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right) \\
&= H_1 - \frac{H_{N+1}}{N+1} + \frac{1}{2} \left(\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) + \frac{1}{2} - \frac{1}{(N+1)(N+2)},
\end{aligned}$$

which tends to $1 + \frac{3}{4} + \frac{1}{2} = \frac{9}{4}$ as $N \to \infty$.

Also solved by Srijan Sundar, Oxford, UK; Arkady Alt, San Jose, CA, USA; Soham Bhadra, India; Daniel Pascuas, Barcelona, Spain; Monil Patel, University of Calgary, Canada; Yunyong Zhang, China Unicom, China; G. C. Greubel, Newport News, VA, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Sundaresh Harige, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA; Matthew Too, Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U633. Evaluate

$$\lim_{n\to\infty} \frac{\ln \sqrt[3]{n} \cdot \ln(n+3)}{\sum\limits_{1\le i<j\le n} \frac{1}{ij}}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Note that $\ln \sqrt[3]{n} = \frac{\ln n}{3}$, and $\sum\limits_{1\le i<j\le n} \frac{1}{ij} = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{j-1} \frac{1}{ij} = \frac{1}{2}\left((H_n)^2 - H_n^{(2)}\right)$, where $H_n^{(2)} = \sum i = 1^n \frac{1}{i^2}$.

Since $\lim\limits_{n\to\infty} \frac{\ln n}{H_n} = 1$ and $\lim\limits_{n\to\infty} \frac{\ln(n+3)}{H_n} = 1$, it follows that

$$\lim_{n\to\infty} \frac{\ln \sqrt[3]{n} \cdot \ln(n+3)}{\sum\limits_{1\le i<j\le n} \frac{1}{ij}} = \lim_{n\to\infty} \frac{\ln \sqrt[3]{n} \cdot \ln(n+3)}{\frac{1}{2}\left((H_n)^2 - H_n^{(2)}\right)} = \frac{2}{3}.$$

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Daniel Pascuas, Barcelona, Spain; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Clark College, WA, USA; Matthew Too, Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U634. Let $A_1, A_2, \ldots, A_n$ be points lying on a circle with radius 1. Prove that there is a point P on this circle such that

$$PA_1 + PA_2 + \cdots + PA_n \geq \frac{4n}{\pi}.$$

Proposed by Karol Janowicz and Waldemar Pompe, Warsaw, Poland

Solution by Matthew Too, Brockport, NY, USA

Consider the function

$$f(\theta) = \sum_{i=1}^{n} \|P - A_i\|$$

where $P = (\cos\theta, \sin\theta)$ and $A_i = (\cos\alpha_i, \sin\alpha_i)$. Since

$$\begin{aligned}\|P - A_i\| &= \sqrt{(\cos\theta - \cos\alpha_i)^2 + (\sin\theta - \sin\alpha_i)^2} = \sqrt{2 - 2(\cos\theta\cos\alpha_i + \sin\theta\sin\alpha_i)} \\ &= \sqrt{2 - 2\cos(\theta - \alpha_i)},\end{aligned}$$

then the average of $f(\theta)$ on a circle with radius 1 is

$$\begin{aligned}f_{avg} &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sum_{i=1}^{n}\sqrt{2 - 2\cos(\theta - \alpha_i)}\,d\theta \\ &= \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi}\sqrt{2 - 2\cos(\theta - \alpha_i)}\,d\theta = \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta \\ &= \frac{n}{\pi}\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)d\theta = \frac{2n}{\pi}\left[\cos\left(\frac{\theta}{2}\right)\right]_{2\pi}^{0} = \frac{4n}{\pi}.\end{aligned}$$

Thus, according to the Mean Value Theorem for Integrals, there exists some $\theta \in (0, 2\pi)$ such that $f(\theta) = \frac{4n}{\pi}$. Furthermore, since f is not a constant function and $f(\theta)$ is less than $\frac{4n}{\pi}$ close to $\theta = \alpha_i$, then it must be greater than $\frac{4n}{\pi}$ elsewhere. Hence, such a point P satisfying the inequality exists. This completes the proof.

Also solved by Daniel Pascuas, Barcelona, Spain; Besfort Shala, University of Bristol, UK; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

U635. Evaluate

$$\int_0^1 \frac{\sin x \sin(\pi x)}{\cos \frac{2x-1}{2}} dx$$

Proposed by Vasile Lupulescu, University of Târgu Jiu, România

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy
$y = (2x-1)/2$ yields

$$\int_{-1/2}^{1/2} \frac{\sin \frac{2y+1}{2} \sin(\pi y + \frac{\pi}{2})}{\cos y} dy = \int_{-1/2}^{1/2} \frac{\sin \frac{2y+1}{2} \cos(\pi y)}{\cos y} dy =$$

$$= \int_{-1/2}^{1/2} \frac{(\sin y \cos \frac{1}{2} + \cos y \sin \frac{1}{2}) \cos(\pi y)}{\cos y} dy =$$

$$= \cos \frac{1}{2} \int_{-1/2}^{1/2} \tan y \cos(\pi y) dy + \sin \frac{1}{2} \int_{-1/2}^{1/2} \frac{\cos y \cos(\pi y)}{\cos y} dy =$$

$$= \sin \frac{1}{2} \int_{-1/2}^{1/2} \cos(\pi y) dy = \frac{2}{\pi} \sin \frac{1}{2}$$

Also solved by Arkady Alt, San Jose, CA, USA; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Monil Patel, University of Calgary, Canada; Yunyong Zhang, China Unicom, China; Sundaresh Harige, India; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Jodie Burdick, SUNY Brockport, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA; Matthew Too, Brockport, NY, USA.

U636. Evaluate

$$\iiint_D e^{\sqrt{x^2+y^2}/2}\frac{zy^2}{(x^2+y^2)^{\frac{3}{2}}}\frac{\left(\frac{1}{2}(x^2+y^2+z^2)-1\right)^3}{\sqrt{4-x^2-y^2-z^2}}\frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}},$$

where $D=\{(z-1)^2+y^2+x^2\le 1\}$.

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Solution by Matthew Too, Brockport, NY, USA

We will convert to spherical coordinates where $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$, $\rho^2=x^2+y^2+z^2$, and $dx\,dy\,dz=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$. Since the boundary of D is

$$(z-1)^2+y^2+x^2=1\Longrightarrow x^2+y^2+z^2=2z\Longrightarrow \rho=2\cos\phi,$$

and θ has no impact on the relationship between ρ and ϕ, then the region D is equivalent to

$$D=\{(\rho,\phi,\theta)\,|\,\theta\in[0,2\pi],\rho\in[0,2],\phi\in[0,\arccos(\rho/2)]\}.$$

The integral rewritten in terms of spherical coordinates is

$$\int_0^{2\pi}\int_0^2\int_0^{\arccos(\rho/2)} e^{\frac{1}{2}\rho\sin\phi}\cos\phi\sin^2\theta\cdot\frac{\rho(\rho^2-2)^3}{8\sqrt{4-\rho^2}}\,d\phi\,d\rho\,d\theta=$$

$$\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^2\frac{\rho(\rho^2-2)^3}{8\sqrt{4-\rho^2}}\left(\int_0^{\arccos(\rho/2)}e^{\frac{1}{2}\rho\sin\phi}\cos\phi\,d\phi\right)d\rho\right)=$$

$$\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^2\frac{(\rho^2-2)^3}{4\sqrt{4-\rho^2}}\left[e^{\frac{1}{2}\rho\sin\phi}\right]_0^{\arccos(\rho/2)}d\rho\right)=$$

$$\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^2\frac{(\rho^2-2)^3}{4\sqrt{4-\rho^2}}\left(e^{\frac{1}{2}\rho\sqrt{4-\rho^2}}-1\right)d\rho\right).$$

Using the substitution $u=\frac{1}{2}\rho\sqrt{4-\rho^2}$, $(\rho^2-2)^2=4(1-u^2)$, $du=\frac{-(\rho^2-2)}{\sqrt{4-\rho^2}}\,d\rho$, we evaluate the rightmost integral to get

$$\int_0^2\frac{(\rho^2-2)^3}{4\sqrt{4-\rho^2}}\left(e^{\frac{1}{2}\rho\sqrt{4-\rho^2}}-1\right)d\rho=\int_0^0(u^2-1)(e^u-1)\,du=0.$$

Thus,

$$\iiint_D e^{\sqrt{x^2+y^2}/2}\frac{zy^2}{(x^2+y^2)^{\frac{3}{2}}}\frac{\left(\frac{1}{2}(x^2+y^2+z^2)-1\right)^3}{\sqrt{4-x^2-y^2-z^2}}\frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}=0.$$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, WA, USA.

Olympiad problems

O631. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{x}{4y^2+yz+4z^2}+\frac{y}{4z^2+xz+4x^2}+\frac{z}{4x^2+xy+4y^2} \geq \frac{1}{45}+\frac{14(xy+yz+zx)}{135}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Denoting $p=x+y+z$, $q=xy+yz+zx$ the inequality becomes

$$\sum_{cyc}\frac{x}{4y^2+yz+4z^2} \geq \frac{1}{15p}+\frac{14q}{5p^3}.$$

Multiplying above inequalities with $4x^2+4y^2+4z^2+xy+yz+zx$, we get

$$x+y+z+\sum_{cyc}\frac{x^2(4x+y+z)}{4y^2+yz+4z^2} \geq \left(\frac{1}{15p}+\frac{14q}{5p^3}\right)(4p^2-7q) \tag{1}$$

By Caucgy-Schwarz Inequality, we have

$$\begin{aligned}\sum_{cyc}\frac{x^2(4x+y+z)}{4y^2+yz+4z^2} &= \sum_{cyc}\frac{x^2(4x+y+z)^2}{(4x+y+z)(4y^2+yz+4z^2)} \\ &\geq \frac{\left(\sum(4x^2+xy+zx)\right)^2}{\sum(4x+y+z)(4y^2+yz+4z^2)} \\ &= \frac{(4p^2-6q)^2}{8(x^3+y^3+z^3)+12xyz+21\sum x(y^2+z^2)} \\ &= \frac{(4p^2-6q)^2}{8p^3-3pq-27xyz}.\end{aligned}$$

From (1) we need to show that

$$p+\frac{(4p^2-6q)^2}{8p^3-3pq-27xyz} \geq \left(\frac{1}{15p}+\frac{14q}{5p^3}\right)(4p^2-7q). \tag{2}$$

Using Shur's Inequality i.e. $9xyz \geq 4pq-p^3$ we have two cases:

1. If $3 \leq t = \frac{p^2}{q} \leq 4$ it suffices to prove that

$$p+\frac{(4p^2-6q)^2}{8p^3-3pq-12pq+3p^3} \geq \left(\frac{1}{15p}+\frac{14q}{5p^3}\right)(4p^2-7q)$$

or

$$t+\frac{(4t-6)^2}{11t-15} \geq \left(\frac{1}{15}+\frac{14}{5t}\right)(4t-7)$$

or

$$\frac{(t-3)^2(361t-490)}{15t(11t-15)} \geq 0$$

obviously true.

2. If $t = \frac{p^2}{q} > 4$ $(xyz \geq 0)$ it suffices to prove that

$$p + \frac{(4p^2 - 6q)^2}{8p^3 - 3pq} \geq \left(\frac{1}{15p} + \frac{14q}{5p^3}\right)(4p^2 - 7q)$$

or

$$t + \frac{(4t-6)^2}{8t-3} \geq \left(\frac{1}{15} + \frac{14}{5t}\right)(4t-7)$$

or

$$\frac{328t^3 - 2041t^2 + 3375t - 882}{15t(8t-3)} \geq 0$$

which is true for $t > 4$.

Also solved by Arkady Alt, San Jose, CA, USA; Soham Bhadra, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Konstantinos Nakis, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, România.

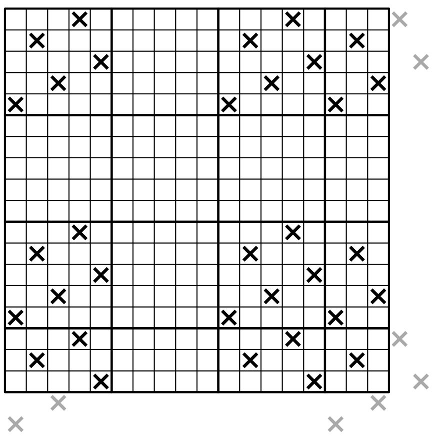
O632. Find the largest integer $n \geq 8$ with the following property: It is possible to mark 64 cells of an $n \times n$ board such that each 2×3 rectangle and each 3×2 rectangle contains at least one marked cell.

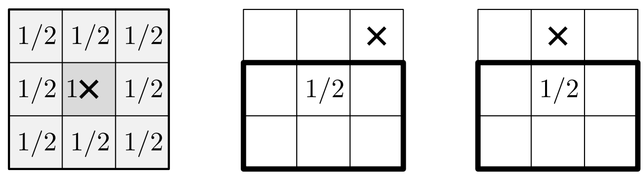
Proposed by Josef Tkadlec, Czech Republic

Solution by the author

Answer: $n = 18$.

For $n = 18$, an admissible subset of 64 cells is shown in the left figure. (The pattern has density 1/5, so one of its 5 translates must yield at most $\left\lfloor \frac{1}{5} \cdot 18^2 \right\rfloor = \left\lfloor \frac{1}{5} \cdot 324 \right\rfloor = 64$ marked cells.)





Now suppose m cells of an $s \times s$ board are marked such that each 2×3 rectangle and each 3×2 rectangle contains at least one marked cell. The key observation is that, in any 3×3 subsquare, if the center cell is not marked then at least 2 of the remaining 8 cells must be marked: Indeed, if at most one non-center cell is marked then there is either a 2×3 or a 3×2 sub-rectangle not containing that marked cell (see the top right figure). Using this observation, we now prove a lemma.

Lemma: $5m \geq s(s-2)$.

Proof: For each marked cell, assign +1 coin to itself and +1/2 coin to each of its (up to 8) neighboring cells. In this way, in total we distribute at most $m(1 + 8 \cdot 1/2) = 5m$ coins. By the observation, each of the interior $(s-2)^2$ cells must receive at least 1 coin. Moreover, each of the $4(s-2)$ perimeter (non-corner) cells must receive at least 1/2 coin. Thus $5m \geq (s-2)^2 + 2(s-2) = s(s-2)$ as required.

Using the lemma, the conclusion is immediate: For $m = 64$ and $s \geq 19$ we have $5m = 320 < 323 = 19 \cdot 17$, so $s < 19$.

Remark: On a 17×17 board, one can mark the $8^2 = 64$ cells with both coordinates even, thereby getting a weaker lower bound of 17. Asymptotically, that construction marks $\frac{1}{4}n^2 + \mathcal{O}(n)$ cells, whereas the construction in the solution marks only $\frac{1}{5}n^2 + \mathcal{O}(n)$ cells. The lemma shows that the constant 1/5 here is optimal.

O633. Let $ABCDEF$ and $A'B'C'D'E'F'$ be regular hexagons with the same orientation. Let $X = AA' \cap BB'$, $Y = DD' \cap EE'$, $Z = CC' \cap FF'$. Prove that points X, Y, Z are collinear.

Proposed by Waldemar Pompe

Solution by the author

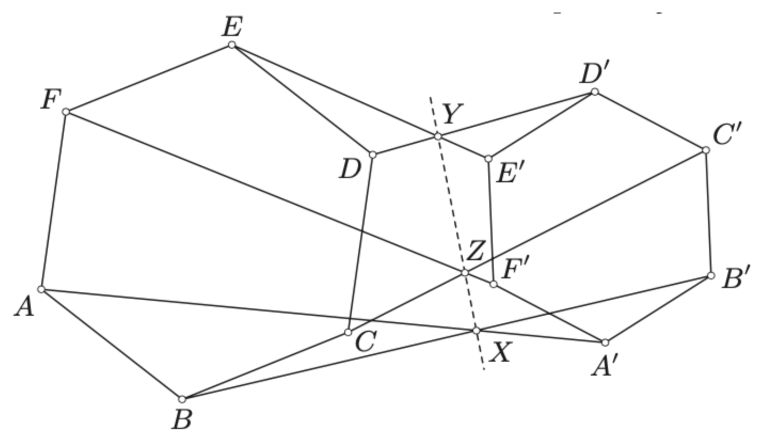


Fig. 1

The proof is based on the following observation about spiral similarites.

Let S and T be given points, and let f be a spiral similarity with center S. Moreover, let ω be a variable cirlce passing through points S and T. Circles ω and $f(\omega)$ intersect at S and X. Then points X lie on a fixed line.

For the proof, let O, O' be the centers of ω, $f(\omega)$, respectively. Then all triangles OSO' are spiral similar, with fixed angle $\angle OSO'$ and fixed ratio $O'S/OS$. Moreover, X is the reflection of point S in line OO', which implies that all triangles OSX are spiral similar. Thus X is obtained from O by some fixed spiral similarity with center S, so since points O lie in one line, so do all X.

We pass to the solution of the problem.

Let S be the center of the spiral similarity f mapping hexagon $ABCDEF$ to $A'B'C'D'E'F'$. Consider the circumcircles o_1, o_2, o_3 of triangles SAB, SDE, SCF, respectively. Obviously, the centers of o_1, o_2, and o_3 are collinear. By the well-known construction of the center of a spiral similarity, X is the second intersection point of circles o_1 and $f(o_1)$. Similarly, Y is the second intersection point of circles o_2 and $f(o_2)$, and Z is the second intersection point of circles o_3 and $f(o_3)$. The observation from the beginning gives the result.

O634. Let $200 < a_1 < \cdots < a_n$ be positive integers such that for each positive integer d, there are at most $d-1$ consecutive terms with difference d. prove that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \le \frac{1}{2}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Theo Koupelis, Clark College, WA, USA

Let $S := \sum_{k=1}^{n} \frac{1}{a_k}$. The maximum value of S occurs when we choose the smallest possible values a_k based on the given conditions. Thus, $a_1 = 201$, $a_2 = 203$ and $a_3 = 206$ (because there are 0 consecutive terms with difference 1, and 1 consecutive term with difference 2), $a_4 = 209$ and $a_5 = 213$ (because there are 2 consecutive terms with difference 3), $a_6 = 217$, and $a_7 = 221$ (because there are 3 consecutive terms with difference 4), etc. Thus,

$$S \le \left(\frac{1}{201}\right) + \left(\frac{1}{203} + \frac{1}{206}\right) + \left(\frac{1}{209} + \frac{1}{213} + \frac{1}{217}\right) + \left(\frac{1}{221} + \frac{1}{226} + \frac{1}{231} + \frac{1}{236}\right)$$
$$+ \left(\frac{1}{241} + \frac{1}{247} + \frac{1}{253} + \frac{1}{259} + \frac{1}{265}\right) + \cdots$$

By construction, we see that the difference in value between the denominators of consecutive first terms of the groups of numbers in parentheses is a term of an arithmetic sequence whose general term is given by $a_k = k(k+1)$, for $k = 1, 2, \ldots$. Therefore, the number in the denominator of each first term in each group is given by $b_n = 201 + \sum_{k=0}^{n} k(k+1) = 201 + (n^3 + 3n^2 + 2n)/3$, for $n = 0, 1, 2, \ldots$. With $\frac{1}{b_n}$ being the first term in each group of numbers in parentheses, the second term is given by $\frac{1}{b_n + n + 2} < \frac{1}{b_n + \frac{n}{3}}$, for $n \ge 1$. But

$$\frac{1}{b_n} + \frac{1}{b_n + n + 2} < \frac{2}{b_n + \frac{n}{3}} \Longleftrightarrow b_n > \frac{n(n+2)}{n+6},$$

which is obvious, because $b_n > n$ for all $n \ge 0$. Thus, setting $c^3 = 602$ we get

$$\begin{aligned} S &< \frac{1}{201} + \sum_{n=1}^{\infty} \frac{n+1}{201 + \frac{1}{3}(n^3 + 3n^2 + 2n) + \frac{1}{3}n} < \frac{1}{201} + \int_0^{\infty} \frac{3(x+1)}{(x+1)^3 + c^3}\,\mathrm{d}x \\ &= \frac{1}{201} + \frac{1}{2c}\left[\ln\frac{x^2 - x(c-2) + c^2 - c + 1}{c^3(x+c+1)^2} + 2\sqrt{3}\arctan\frac{2(x+1)-c}{\sqrt{3}c}\right]\Bigg|_0^{\infty} \\ &< \frac{1}{201} + 0.427132518\ldots < \frac{16}{37} < \frac{1}{2}. \end{aligned}$$

O635. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{128(ab+bc+ca)^2}{(a+b)(b+c)(c+a)}+\frac{81}{abc}\geq 225.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Let's note that we have equality when $a=b=c=1$. We note $P=(a+b)(b+c)(c+a)$, $q=ab+bc+ca$, $r=abc$. Using the Cauchy-Schwarz Inequality, we are looking for a number $k>0$ such that

$$\left(\frac{256q^2}{2P}+\frac{81}{r}\right)\left(2Pk^2+r\right)\geq(16qk+9)^2.$$

To have equality in this, we should have $\frac{16q}{2Pk}=\frac{9}{r}$ when $a=b=c=1$, so we get $k=\frac{1}{3}$. Hence, it remains to show that

$$\left(\frac{16q}{3}+9\right)^2\geq 225\left(\frac{2(a+b)(b+c)(c+a)}{9}+r\right)$$

or

$$(16q+27)^2\geq 225\left[2(a+b+c)(ab+bc+ca)-2abc+9r\right]$$

or

$$(16q+27)^2\geq 225\left(6q+7r\right).$$

Since $(ab+bc+ca)^2\geq 3abc(a+b+c)=9abc$, it suffices to prove that

$$(16q+27)^2\geq 225\left(6q+\frac{7q^2}{9}\right)\Longleftrightarrow 81(q-3)^2\geq 0$$

obviously true.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, România; Arkady Alt, San Jose, CA, USA.

O636. Prove that there are no nonzero polynomials $P(x)$ with real coefficients such that

$$P(-a+b+c)+P(a-b+c)+P(a+b-c)=0,$$

for all real numbers a,b,c which satisfy the condition $a^4+b^4+c^4=2$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Adam John Frederickson, Utah Valley University, UT, USA
This fails to hold even with the restriction $c=0$. Let a,b satisfy $a^4+b^4=2$. Then all four points

$$(a,b,0),\ (a,-b,0),\ (-a,b,0),\ (-a,-b,0)$$

satisfy $a^4+b^4+c^4=2$, and so we have the system of equations

$$\begin{cases} P(-a+b) & + & P(a-b) & + & P(a+b) & = & 0 \\ P(-a-b) & + & P(a+b) & + & P(a-b) & = & 0 \\ P(a+b) & + & P(-a-b) & + & P(-a+b) & = & 0 \\ P(a-b) & + & P(-a+b) & + & P(-a-b) & = & 0 \end{cases} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} P(a+b) \\ P(a-b) \\ P(-a+b) \\ P(-a-b) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

is $-3\neq 0$, so we must have

$$P(a+b)=P(a-b)=P(-a+b)=P(a-b)=0.$$

For instance, $P(a+b)=0$ when $a^4+b^4=2$. In other words, taking $a+b$ to be x, $P(x)$ is identically 0 whenever there exists an a such that $a^4+(x-a)^4=2$. No such polynomial exists unless $P\equiv 0$.

Also solved by Anderson Torres, Brazil; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Clark College, WA, USA.