Junior Problems

J637. Find all positive integers n such that

$$3 \cdot n! - 1 = \sqrt{(2n-1)! + 1}.$$

Proposed by Adrian Andreescu, Dallas, USA

J638. Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{\sqrt{c(a+2b+3c)}} + \frac{b+2c}{\sqrt{a(b+2c+3a)}} + \frac{c+2a}{\sqrt{b(c+2a+3b)}} \geq \frac{3\sqrt{6}}{2}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

J639. Solve the equation

$$176x - 4\lfloor x \rfloor^2 - 88\{x\}^2 = 2023,$$

where $\lfloor x \rfloor$ and $\{x\}$ are the integer part and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J640. Let x, y, z be positive real numbers such that $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Prove that

$$\sqrt{xy+3} + \sqrt{yz+3} + \sqrt{zx+3} \leq \frac{3(x+y+z+1)}{2}.$$

Proposed by Marius Stănean, Zalău, România

J641. Find all positive integers a and b such that

$$\frac{a^2}{b} - \frac{b^2}{a} = 2023.$$

Proposed by Mircea Becheanu, Canada

J642. Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove that

$$3\sqrt[6]{8abc} \leq \sqrt{a + \frac{b}{c}} + \sqrt{b + \frac{c}{a}} + \sqrt{c + \frac{a}{b}} \leq 3\sqrt{2abc}.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S637. Let $a, b > 0$ and $c < 0$ such that

$$2a^2 + \frac{1}{8a^2} = 7, \;\; 2b^2 + \frac{1}{8b^2} = 17, \;\; 2c^2 + \frac{1}{8c^2} = 31.$$

Prove that

$$(4ab+1)(4bc+1)(4ca+1) = (8abc+1)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S638. Let a, b, c be positive real numbers. Prove that

$$\frac{6a^2}{(2b+c)(2c+b)} + \frac{6b^2}{(2c+a)(2a+c)} + \frac{6c^2}{(2a+b)(2b+a)} \geq 1 + \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, România

S639. Let $f(n) = \binom{n^2-1}{3}$, $n = 2, 3, 4, \ldots$. Find all positive integers $k \geq 2$ such that $f(k+3) = f(k) + 2023$.

Proposed by Adrian Andreescu, Dallas, USA

S640. Let $ABCD$ be a convex quadrilateral with incenter I and inradius r. A line passing through I intersects segments BC, CA, AD at points K, L, M, respectively. Let r_1 be the radius of a circle tangent to segments AB, AM, MK and let r_2 be the radius of a circle tangent to segmnets CD, CK, KM. Set $a = AL/LC$. Prove that

$$a\left(\frac{1}{r_1} - \frac{1}{r}\right) = \frac{1}{r_2} - \frac{1}{r}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

S641. Solve in integers the equation

$$2x^3 - 3x^2y^2 + 2y^3 = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S642. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2+3} + \frac{1}{b^2+3} + \frac{1}{c^2+3} \leq \frac{1}{92}\left(68 + \frac{1}{abc}\right).$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U637. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Evaluate

$$\lim_{n\to\infty} n\left(\frac{n-H_n}{n}\right)^n.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U638. Evaluate

$$\int \frac{1-\sin x}{3\sin x + 5(1+\cos x)e^x}\, dx$$

where $0 < x < \pi/2$.

Proposed by Mihaela Berindeanu, Bucharest, România

U639. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \geq 1$, and let

$$F(a,b,c) = \frac{a+b+c}{3} - \sqrt{\frac{ab+bc+ca}{3}}.$$

Prove that

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

Proposed by Vasile Cârtoaje, Ploeşti and Vasile Mircea Popa, Sibiu, România

U640. Let $(x_n)_{n\geq 2}$ be the sequence defined by

$$x_n = \frac{\sqrt[n]{e}-1}{\sqrt[n^2]{e}-1} - n.$$

Prove that $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ştefan Marinescu, Hunedoara, România

U641. Let $P(x,y,z)$ be a polynomial with rational coefficients. Prove that there is a polynomial $Q(x,y,z)$ with rational coefficents such that

$$P(x,y,z)Q(x,y,z) = R\left(x^2y, y^2z, z^2x\right),$$

for some polynomial $R(x,y,z)$ with rational coefficients.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U642. Evaluate

$$\lim_{n\to\infty} n \sin\left((2\pi n)^p + 2^p\pi^p n^{p-1} p\right)^{\frac{1}{p}}$$

Proposed by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Italy

Olympiad Problems

O637. Let a, b, c be positive real numbers such that $a+b+c = ab+bc+ca$. Prove that

$$\sqrt[3]{a^3+7}+\sqrt[3]{b^3+7}+\sqrt[3]{c^3+7} \leq 2(a+b+c).$$

Proposed by Marius Stănean, Zalău, România

O638. Let $1 \leq a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there is a sequence $b_1, b_2, \ldots$ of positive integers with $b_i > a_i$ such that the only multiplicative function $f : \mathbb{N} \longrightarrow \mathbb{Z} - \{0\}$ satisfying the condition $f(b_i + b_j) = f(b_i) + f(b_j)$ is $f(n) = n$ for all n.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O639. Let a, b, c, λ be positive real numbers such that

$$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda} \leq \frac{1}{\lambda}.$$

Prove that

$$abc \geq 8\lambda^3 \quad \text{and} \quad a+b+c+\frac{3abc}{ab+bc+ca} \geq 8\lambda.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O640. Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 > 1$ and $a_{n+1} = \frac{1+a_n^2}{2}$, for all $n \geq 0$. Prove that

$$\prod_{k=0}^{n} \frac{1+a_k}{a_k} \geq \left(\frac{(1+a_0)(n+1)}{(1+a_0)n+a_0}\right)^{n+1}.$$

Proposed by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Italy

O641. Let $(F_n)_{n\geq 0}$, $F_0 = 0,\ F_1 = 1$ be the *Fibonacci* sequence and p, q be distinct primes of the form $4k+3$. Find all triples (n, a, b) of nonnegative integers for which $F_n = p^a q^b$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O642. Point P lies inside triangle ABC. Let $D = AP \cap BC$, $E = BP \cap CA$, $F = CP \cap AB$. The circumcircle of triangle DEF intersects sides BC, CA, AB for the second time at points D', E', F', respectively. Set $X = DF' \cap D'F$, $Y = EF' \cap E'F$. Prove that points X, P, Y are collinear.

Proposed by Waldemar Pompe, Warsaw, Poland