A New Method to Solve and Refine Algebraic Inequalities

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Abstract. In this paper[1], we present a new method to solve and refine algebraic symmetric and homogeneous inequalities in three variables. Using this method we solve and refine some inequalities from Mathematical Reflections journal.

1. Introduction.

Let x, y, z positive numbers. We denote

$$s = x + y + z,\ p = xyz,\ \alpha = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9,\ t = \frac{(x + y + z)^3}{xyz} \geq 27$$

We obtain the following formulas:
1). $\sum xy = \frac{\alpha p}{s}$
2). $\sum x^2 = \frac{s^3 - 2p\alpha}{s}$
3). $\sum x^3 = s^3 - 3\alpha p + 3p$
4). $\sum x^4 = \frac{\left(s^3 - 2\alpha p\right)^2}{s^2} - \frac{2\alpha^2 p^2}{s^2} + 4sp$
5). $\sum (x - y)^4 = \frac{1}{s^2}\left(2s^6 - 12\alpha p s^3 + 18\alpha^2 p^2\right)$
6). $\sum \frac{x}{2x+y+z} = \frac{s^3 + 2p\alpha + 3p}{2s^3 + \alpha p + p}$
7). $\sum \frac{x}{y+z} = \frac{s^3 - 2\alpha p + 3p}{(\alpha - 1)p}$
8). $\prod \left(x^2 - xy + y^2\right) = -8p^2 - 3s^3 p + \alpha^2 p^2 + 10\alpha p^2 - \frac{3\alpha^3 p^3}{s^3}$
9). $\sum xy\,(x + y) = (\alpha - 3)\,p$
10). $\sum \frac{1}{(x+y)^2} = \frac{\left(s^3 + p\alpha\right)^2 - 4s^3 p(\alpha - 1)}{(\alpha - 1)^2 p^2 s^2}$
11). $\prod \left(2x^2 - xy + 2y^2\right) = -27p^2 - 20ps^3 + 8\alpha^2 p^2 + 54\alpha p^2 - \frac{20\alpha^3 p^3}{s^3}$
Proof.
1). It follows from $\sum x \cdot \sum \frac{1}{x} = \alpha$
2). $\sum x^2 = \left(\sum x\right)^2 - 2\sum xy = \frac{s^3 - 2\alpha p}{s}$
3). $\sum x^3 = \sum x \sum x^2 - \sum xy \sum x + 3xyz = s^3 - 3\alpha p + 3p$
4). $\sum x^4 = \left(\sum x^2\right)^2 - 2\left[\left(\sum xy\right)^2 - 2xyz\sum x\right] = \frac{\left(s^3 - 2\alpha p\right)^2}{s^2} - \frac{2\alpha^2 p^2}{s^2} + 4sp$
5). $\sum (x - y)^4 = 2\left(\sum x\right)^4 - 12\sum xy\left(\sum x\right)^2 + 18\,(xy)^2 = \frac{1}{s^2}\left(2s^6 - 12\alpha s^3 + 18\alpha^2 p^2\right)$
6). $\sum \frac{x}{2x+y+z} = \frac{\sum x(s+x)(s+z)}{\prod (s+x)} = \frac{\sum x\left(s^2 + s(s-x) + yz\right)}{2s^3 + s\sum yz + xyz} = \frac{s^3 + 2\alpha p + 3p}{2s^3 + \alpha p + p}$
7). $\sum \frac{x}{y+z} = \frac{\sum x \sum x^2 + 3xyz}{\prod (y+z)} = \frac{s^3 - 2\alpha p + 3p}{(\alpha - 1)p}$
8). $\prod \left(x^2 - xy + y^2\right) = -8\,(xyz)^2 - 3xyz\left(\sum x\right)^3 + \left(\sum xy\right)^2\left(\sum x\right)^2 +$

$$+10xyz\sum xy\sum x - 3\left(\sum xy\right)^3 = -8p^2 - 3s^3 p + \alpha^2 p^2 + 10\alpha p^2 - \frac{3\alpha^3 p^3}{s^3}$$

9). $\sum xy(x + y) = \sum xys - 3xyz = p\alpha - 3p$
10). $\sum \frac{1}{(x+y)^2} = \left[\frac{\sum (y+z)(z+x)}{\prod (x+y)}\right]^2 - 2\frac{\sum (x+y)}{\prod (x+y)} = \frac{\left(3\sum xy + \sum x^2\right)^2 - 4\sum x \prod (x+y)}{\prod (x+y)^2} = \frac{\left(s^3 + p\alpha\right)^2 - 4(\alpha - 1)s^3 p}{(\alpha - 1)^2 p^2 s^2}$

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In the following we denote $t_{1,2}=\frac{\alpha^2+18\alpha-27\pm\sqrt{(\alpha-1)(\alpha-9)^3}}{8}\ (t_1\le t_2)\,,\ t_3=4\alpha-9,\ t_4=\frac{\alpha^3}{4\alpha-9}, t=\frac{s^3}{p}$. Then it is true the theorem:

Theorem. Let $t=\frac{s^3}{p}$. Then:

$$t_3\le t_1\le t\le t_2\le t_4$$

Proof. From Shur inequality we have

$$s^3+9p\ge 4\sum x\sum xy \quad \text{or} \quad s^3+9p\ge 4\alpha p \quad \text{or} \quad \frac{s^3}{p}\ge 4\alpha-9 \quad \text{from which} \quad t\le t_3$$

Also from Shur inequality we have $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3+\frac{9}{xyz}\ge 4\sum\frac{1}{yz}$
from which it follows that $t\le t_4$.
Also it's well known that if x,y,z are the positive roots of equation $x^3-\sigma_1x^2+\sigma_2x-\sigma_3=0$
Then

$$(x-y)^2(y-z)^2(z-x)^2\ge 0\Longleftrightarrow 18\sigma_1\sigma_2\sigma_3-4\sigma_1^3\sigma_3+\sigma_1^2\sigma_2^2-4\sigma_2^3-27\sigma_3^2\ge 0$$

If we replace $\sigma_1=\sum x$, $\sigma_2=\sum xy, \sigma_3=xyz$ we obtain that:

$$18\alpha-4t+\alpha^2-\frac{4\alpha^3}{t}-27\ge 0 \quad \text{or} \quad t_1\le t\le t_2$$

It is easy to prove that $t_3\le t_1$ and $t_2\le t_4$, $\forall\alpha\ge 9$
Also t_1,t_2 is the best bound depend only of α for which $t_1\le t\le t_2$

2. Main Result

1. Let a,b,c positive numbers. Then

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\ge\frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca}+2$$

An Zhenping, Mathematical Reflections 2/2020

Solution. We have $f\left(t\right)=\sum\frac{a+b}{c}-\frac{4\sum a^2}{\sum ab}=\alpha+5-\frac{4t}{\alpha}\ge f\left(\frac{\alpha^3}{4\alpha-9}\right)=2+\frac{3(\alpha-9)}{4\alpha-9}\ge 2$
We obtain the following refinement: Let a,b,c positive numbers. Then

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\ge\frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca}+2+\frac{3\left(\alpha-9\right)}{4\alpha-9}\ge 2 \quad \text{Where} \quad \alpha=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

2. Let a,b,c positive numbers. Then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+4\left(\frac{a}{2a+b+c}+\frac{b}{2b+c+a}+\frac{c}{2c+a+b}\right)\ge\frac{9}{2}$$

Titu Zvonaru, S478, Mathematical Reflections

Solution. We have

$$f\left(t\right)=\sum\frac{a}{b+c}+4\sum\frac{a}{2a+b+c}=\frac{t-2\alpha+3}{\alpha-1}+\frac{4t+8\alpha+12}{2t+\alpha+1},$$

$\forall t\in[t_3,t_4]$ with positive root

$$t_0=\frac{1}{2}\left(2\sqrt{\left(\alpha-1\right)\left(3\alpha+5\right)}-\alpha-1\right)\le t_3=4\alpha-9$$

This inequality it's true if $\alpha > 3, 11...$ according WOLPHRAM ALPHA (WA) So for $\alpha \geq 9$. So f is increasing on $[t_3, t_4]$ or

$$f(t) \geq f(4\alpha - 9) = \frac{9}{2} + \frac{(\alpha - 9)(3\alpha - 11)}{2(\alpha - 1)(9\alpha - 17)}$$

We obtain the refinement.

Let a, b, c positive numbers. Then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + 4\left(\frac{a}{2a+b+c} + \frac{b}{2b+c+a} + \frac{c}{2c+a+b}\right) \geq$$

$$\geq \frac{9}{2} + \frac{(\alpha - 9)(3\alpha - 11)}{2(\alpha - 1)(9\alpha - 17)}$$

where $\alpha = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

3. Let a, b, c positive numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$a + b + c + \frac{3}{ab + bc + ca} \geq 4$$

An Zhenping, Mathematical Reflections 6/2021

Solution. Inequality from statement may be written as

$$(a+b+c)(ab+bc+ca) + 3 \geq 4(ab+bc+ca)$$

or after deconditionated

$$\sum a \sum ab \geq \left(4\sum ab - \sum a^2\right)\sqrt{\frac{\sum a^2}{3}}$$

If $4\sum ab - \sum a^2 \leq 0$ it's obviously true

If $4\sum ab - \sum a^2 > 0$ or

$$4\frac{\alpha p}{s} - \frac{s^3 - 2p\alpha}{s} > 0 \Rightarrow 6\alpha - t > 0$$

we obtain $t < 6\alpha$.

In this case we square (1) and obtain

$$3\left(\sum a \sum ab\right)^2 \geq \left(4ab - \sum a^2\right)^2 \sum a^2 \quad \text{or}$$

$$3\alpha^2 p^2 \geq \frac{\left(6\alpha p - s^3\right)^2}{s^3}\left(s^3 - 2p\alpha\right) \quad \text{or}$$

$$3\alpha^2 t \geq (6\alpha - t)^2 (t - 2\alpha) \quad \text{or} \quad (8\alpha - t)(t - 3\alpha)^2 \geq 0$$

which is true since $t < 6\alpha$ so $8\alpha - t > 2\alpha > 0$

4. Let a, b, c positive real numbers. Then

$$(a-b)^4 + (b-c)^4 + (c-a)^2 \leq 6\left(a^4 + b^4 + c^4 - abc(a+b+c)\right)$$

Nicuşor Zlota, S473, Mathematical Reflections

Solution. If $\alpha = 9$ it's obviously. Let $\alpha > 9$ we have:

$$f(t) = \frac{\sum (a-b)^4}{\sum a^4 - abc \sum a} = \frac{2s^6 - 12\alpha ps^3 + 18\alpha^2 p^2}{(s^3 - 2\alpha p)^2 - 2\alpha^2 p^2 + 4s^3 p - s^3 p} = \frac{2t^2 - 12\alpha t + 18\alpha^2}{(t-2\alpha)^2 - 2\alpha^2 + 3t}$$

$$\text{with} \quad f'(t) = \frac{2(t-3\alpha)\left(3t + 2\alpha t - 8\alpha^2 + 9\alpha\right)}{\left(2\alpha^2 - 4\alpha t + t^2 + 3t\right)^2}$$

Since $t \geq t_3$ we have

$$(3+2\alpha)t - 8\alpha^2 + 9\alpha \geq (4\alpha - 9)(3+2\alpha) - 8\alpha^2 + 9\alpha = 3(\alpha - 9) \geq 0$$

So f is increasing on $[t_3, t_4]$ or $f(t) \leq f(\alpha_4) = \frac{2(\alpha-9)(\alpha-3)^2}{\alpha^3 - 7\alpha^2 + 17\alpha - 18}$

$$\text{We have} \quad \frac{2(\alpha-9)(\alpha-3)^2}{\alpha^3 - 7\alpha^2 + 17\alpha - 8} < 2$$

so we obtain the following improvements: Let a, b, c positive real numbers. Then

$$(a-b)^4 + (b-c)^4 + (c-a)^4 \leq \frac{2(\alpha-9)(\alpha-3)^2}{\alpha^3 - 7\alpha^2 + 17\alpha - 18}\left(a^4 + b^4 + c^4 - abc(a+b+c)\right) \leq$$

$$\leq 2\left(a^4 + b^4 + c^4 - abc(a+b+c)\right)$$

5. Let a, b, c positive numbers such that $a + b + c = 3$. Then

$$\left(a^2 - ab + b^2\right)\left(b^2 - bc + c^2\right)\left(c^2 - ca + a^2\right) + 11abc \leq 12$$

An Zhenping, O476, Mathematical Reflections

Solution. We have after deconditionate

$$\prod \left(a^2 - ab + b^2\right) + 11abc\left(\frac{\sum a}{3}\right)^3 - 12\left(\frac{\sum a}{3}\right)^6 \overset{(8)}{=}$$

$$= -8p^2 - 3ps^3 + p^2\alpha^2 + 10\alpha p^2 - \frac{3\alpha^3}{s^3}p^3 + 11\frac{ps^3}{3} - \frac{12s^6}{27^2} =$$

$$= p^2\left(-8 - 3t + \alpha^2 + 10\alpha - \frac{3\alpha^3}{t} + \frac{11t}{27} - \frac{12t^2}{27^2}\right) = p^2 f(t)$$

$$f(t) = \alpha^2 + 10\alpha - 8 - \frac{70t}{27} - 3\frac{\alpha^3}{t} - \frac{4}{243}t^2, \ \forall t \in [t_3, t_4]$$

$$\text{with} \quad f'(t) = -\frac{70}{27} + \frac{3\alpha^3}{t^2} - \frac{8}{243}t^2 = \frac{-t^2\left(8t^2 + 630\right) + 729\alpha^3}{243t^2}$$

$$\text{But} \quad -t^2\left(8t^2 + 630\right) + 729\alpha^3 < -t_3^2\left(8t_3^2 + 630\right) + 729\alpha^3$$

$$\text{since} \quad -t_3^2\left(8t_3^2 + 630\right) + 729\alpha^3 < 0$$

if $\alpha > 3,91661$ we obtain since $\alpha \geq 9 > 3,9166$ that f is decreasing on $[t_3, t_4]$ or

$$f(t) \leq f(t_3) = -\frac{(\alpha-9)\left(13\alpha^2 + 936\alpha - 3402\right)}{243(4\alpha - 9)}, \ \forall \alpha \geq 9$$

We obtain the following refinement: Let a, b, c positive numbers such that $a + b + c = 3$. Then

$$\left(a^2-ab+b^2\right)\left(b^2-bc+c^2\right)\left(c^2-ca+a^2\right)+11abc-12\leq$$

$$\leq\frac{-a^2b^2c^2\left(\alpha-9\right)\left(13\alpha^2+936\alpha-3402\right)}{243\left(4\alpha-9\right)}\leq 0\quad\text{where}\quad\alpha=\frac{3}{a}+\frac{3}{b}+\frac{3}{c}$$

6. Let a,b,c the sides, lenghts of the sides of a triangle and F it's area. Then

$$a^2+b^2+c^2\leq 4F\sqrt{\frac{6r}{R}}+3\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

Marius Stănean, Mathematical Reflections 6/2020

Solution. From Ravi's substitution we have: $\frac{r}{R}=\frac{4}{\alpha-1}$ and

$$\sum(a-b)^2=2\sum x^2-2\sum xy=\frac{2s^2-4\alpha p}{s}-\frac{2p\alpha}{s}=\frac{2s^3-4\alpha p}{s}$$

Inequality may be written as

$$\frac{2s^3-2\alpha p}{s}\leq 4\sqrt{ps}\sqrt{\frac{24}{\alpha-1}}+\frac{3\left(2s^3-6\alpha p\right)}{s}\quad\left(F=\sqrt{xyz\left(x+y+z\right)}=\sqrt{ps}\right)\quad\text{or}$$

$$\frac{p}{s}\left[2t-2\alpha-8\sqrt{t}\sqrt{\frac{6}{\alpha-1}}-3\left(2t-6\alpha\right)\right]\leq 0$$

We consider the function $f:[t_3,t_4]\to R\ f\left(t\right)=4t+8\sqrt{t}\sqrt{\frac{6}{\alpha-1}}-16\alpha$
which is increasing on $[t_3,t_4]$ so $f\left(t\right)\geq f\left(t_3\right)=8\sqrt{\frac{6(4\alpha-9)}{\alpha-1}}-36\geq 0,\ \forall\alpha\geq 9$
We obtain the refinement

$$a^2+b^2+c^2\leq 4F\sqrt{\frac{6r}{R}}+3\left[(a-b)^2+(b-c)^2+(c-a)^2\right]-8r\sqrt{\frac{24\alpha-54}{\alpha-1}}+36r$$

$$\text{with}\quad\alpha=\left(a+b+c\right)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)$$

$$\left(\text{We have}\quad\frac{p}{s}=\frac{xyz}{x+y+z}=\frac{\prod\left(s-a\right)}{s}=\frac{F^2}{s^2}=r^2\right)$$

7. Let a,b,c positive real numbers. Then

$$\frac{a^3+b^3+c^3}{abc}+\frac{8abc}{\left(a+b\right)\left(b+c\right)\left(c+a\right)}\geq 4$$

Alessandro Ventullo, S435, Mathematical Reflections 1/2018

Solution. We have $\frac{\sum a^3}{abc}+\frac{8abc}{\prod(a+b)}=t-3\alpha+3+\frac{8}{\alpha-1}\geq 4+\frac{(\alpha-2)(\alpha-9)}{\alpha-1}\geq 4$
Refinement: Let a,b,c positive numbers, then

$$\frac{a^3+b^3+c^3}{abc}+\frac{8abc}{\left(a+b\right)\left(b+c\right)\left(c+a\right)}\geq 4+\frac{\left(\alpha-2\right)\left(\alpha-9\right)}{\alpha-1}$$

where $\alpha=\left(a+b+c\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

8. Let a,b,c positive numbers. Prove that

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\geq\frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca}+2$$

An Zhenping, S514, Mathematical Reflections 2/2020

Solution. We have $f(t)=\sum\frac{a+b}{c}-\frac{4\sum a^2}{\sum ab}=\alpha+5-\frac{4t}{\alpha},\ \forall t\in[t_3,t_4]$
which is decreasing, so

$$f(t)\geq f(t_4)=f\left(\frac{\alpha^3}{4\alpha-9}\right)=2+\frac{3(\alpha-9)}{4\alpha-9}$$

So we obtain the improvement: Let a,b,c positive numbers. Prove that

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\geq\frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca}+2+\frac{3(\alpha-9)}{4\alpha-9}$$

9. Let a,b,c positive real numbers. Then

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3(a+b+c)}{2}}$$

Mircea Becheanu, Mathematical Reflections 2/2020

Solution. We denote $\sqrt{b+c}=x,\ \sqrt{c+a}=y,\ \sqrt{a+b}=z$ so

$$\left(x^2+y^2+z^2\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-2(x+y+z)\geq\sqrt{3\left(x^2+y^2+z^2\right)}\quad\text{or}$$

$$ps\left[(t-2\alpha)\alpha-2t-\sqrt{3t(t-2\alpha)}\right]\geq 0$$

Let $f:[t_3,t_4]\to R,\ f(t)=(t-2\alpha)\alpha-2t-\sqrt{3t(t-2\alpha)}$ with $f'(t)=\frac{\sqrt{3}(\alpha-t)}{\sqrt{t(t-2\alpha)}}+\alpha-2$ with roots of $f'(t)=0,\ t'_{1,2}=\alpha\left(1\pm\frac{\alpha-2}{\sqrt{\alpha^2-4\alpha+1}}\right)$ $(t'_1\leq t'_2)$ since $t'_2<t_3,\ \forall\alpha>5,044$ we have not critic points.
So f' have the same sign on $[t_3,t_4]$. But $f'(t_3)>0,\ \forall\alpha>5,044$ so if $\alpha\geq 9$ so $f(t)\geq 0,\ \forall\alpha\in[t_3,t_4]$ *or*

$$f(t)\geq f(t_1)\geq f(t_3)$$

Since $f(t_3)>0,\ \forall\alpha\geq 9$ it follows the statement we obtain the refinement

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3(a+b+c)}{2}}+$$

$$+\left[2\alpha^2-17\alpha+18-\sqrt{3}\sqrt{8\alpha^2-54\alpha+8}\right]\frac{\sqrt{\prod(a+b)}}{\left(\sum(a+b)\right)^2}$$

$$\text{where}\quad\alpha=\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)\left(\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}+\frac{1}{\sqrt{a+b}}\right)$$

Also let a,b,c positive numbers such that

$$\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)\left(\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}\right)=10$$

Then

$$\text{Then}\quad\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3(a+b+c)}{2}}+\left(\frac{200-75\sqrt{3}}{4}\right)\sqrt{\frac{\prod(a+b)}{\sum\sqrt{a+b}}}$$

Solution. We have $f(t_1)=\frac{200-75\sqrt{3}}{4}$ and since if $\alpha=10,\ f(t)\geq f(t_1)$ we obtain the statement.

10. Let x,y,z be nonnegative real numbers. Prove that

$$\frac{x^3+y^3+z^3+3xyz}{\sum\limits_{cyclic} xy(x+y)}+\frac{5}{4}\geq (xy+yz+zx)\left[\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right]$$

Marius Stănean, Mathematical Reflections 4/2021

Solution. We have $\frac{\sum x^3+3xyz}{\sum\limits_{cyclic} xy(x+y)}-\sum xy\sum\frac{1}{(x+y)^2}+\frac{5}{4}\overset{(9)(10)}{=}\frac{s^3-3\alpha p+6p}{p(\alpha-3)}-$

$$-\frac{p\alpha}{9}\cdot\frac{\left(s^3+\alpha p\right)^2-4s^3(\alpha-1)p}{p^2s^2(\alpha-1)^2}+\frac{5}{4}=\frac{t-3\alpha+6}{\alpha-3}-$$

$$-\frac{\alpha\left[(t+\alpha)^2-4(\alpha-1)t\right]}{(\alpha-1)^2t}+\frac{5}{4}=f(t),\ t\in[t_1,t_2]$$

$$\text{with}\quad f'(t)=\frac{(\alpha+1)t^2+\alpha^4-3\alpha^3}{(\alpha-3)(\alpha-1)^2t^2}>0,\ \forall\alpha\geq 9,\ t\in[t_1,t_2]$$

so

$$\text{so}\quad f(t)\geq f(t_1)=\frac{(\alpha-9)\left(\alpha-5-\sqrt{(\alpha-1)(\alpha-9)}\right)}{4(\alpha-1)(\alpha-3)}\geq 0,\ \forall\alpha\geq 9$$

We obtain the refinement

$$\frac{\sum x^3+3xyz}{\sum\limits_{cyclic} xy(x+y)}+\frac{5}{4}\geq\sum xy\sum\frac{1}{(x+y)^2}+\frac{(\alpha-9)(\alpha-5)-\sqrt{(\alpha-1)(\alpha-9)}}{4(\alpha-1)(\alpha-3)}$$

where $\alpha=\sum x\sum\frac{1}{x}$

11. Let x,y,z positive real numbers such that $x+y+z=1$. Prove that:

$$\frac{1}{x^3+y^3+z^3}+\frac{24}{xy+yz+zx}\geq 81$$

Nyugen Viet Hung, Mathematical Reflections 1/2021

Solution. After deconditionate we obtain:

$$\frac{1}{\sum x^3}+\frac{24}{\sum yz\sum x}\geq\frac{81}{\left(\sum x\right)^3}\overset{3)}{\Rightarrow}\frac{1}{s^3-3p\alpha+3p}+\frac{24}{p\alpha}\geq\frac{81}{s^3}\quad\text{or}\quad\frac{1}{p}f(t)\geq 0$$

$$\text{where}\quad f(t)=\frac{1}{t-3\alpha+3}+\frac{24}{\alpha}-\frac{81}{t}\geq 0,\ \forall t\in[t_3,t_4]\quad\text{with}$$

$$f'(t)=\frac{(8t-27\alpha+27)(10t-27\alpha+27)}{t^2(t-3\alpha+3)^2},\ t\in[t_3,t_4]$$

$$\text{sine}\quad t\geq 4\alpha-9\Rightarrow 8t-27\alpha+27\geq 32\alpha-72-27\alpha+27-5\alpha-45\geq 0\ \ (\alpha\geq 9)$$

$$\text{Also}\quad 10t-27\alpha+27\geq 40\alpha-90-27\alpha+27=13\alpha-63\geq 54>0$$

so $f'(t)\geq 0$ or f is increasing so

$$f(t)\geq f(4\alpha-9)=\frac{(\alpha-9)(19\alpha-144)}{(\alpha-6)\alpha(4\alpha-9)}\geq 0$$

So inequality from statement accept the following refinement: Let $x, y, z > 0$, $x+y+z=1$, $\alpha = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ then

$$\frac{1}{x^3+y^3+z^3} + \frac{24}{xy+yz+zx} \geq 81 + \frac{(\alpha-9)(19\alpha-14)}{xyz(\alpha-6)\alpha(4\alpha-9)} \geq 81$$

12. Let x, y, z be nonnegative numbers. Prove that

$$x^3+y^3+z^3 \geq \sqrt{\frac{1}{3}\prod_{cyclic}(2x^2-xy+2y^2)} \geq \sum xy(x+y) - 3xyz$$

Marius Stănean, Mathematical Reflections 4/2020

Solution. The left side of inequality from statement may be written as according (3) and (11)

$$3\left(s^3-3\alpha p+3p\right)^2 \geq -27p^2 - 20ps^3 + 8p^2\alpha^2 + 54p^2\alpha - 20\frac{p^3\alpha^3}{s^3}$$

or $p^2 f(t) \geq 0$, $\forall t \in [t_3, t_4]$ where

$$f(t) = 3(t-3\alpha+3)^2 + 27 + 20t - 8\alpha^2 - 54\alpha + \frac{20\alpha^3}{t} \geq 0,\ \forall t \in [t_3, t_4]$$

with

$$\text{with} \quad f'(t) = -\frac{20\alpha^3}{t^2} - 18\alpha + 6t + 38 \geq \frac{-20\alpha^3}{t^2} + 6(4\alpha-9) - 18\alpha + 38 =$$

$$= 6\alpha - 16 - \frac{20\alpha^3}{t^2} \geq 6\alpha - 16 - \frac{20\alpha^3}{(4\alpha-9)^2} \geq 0,\ \forall \alpha > 5,8808$$

(we use $t \geq t_3 = 4\alpha - 9$).
It follows that f is increasing, so

$$f(t) \geq f(t_3) = (\alpha-9)^2\left(\alpha - \frac{9}{4}\right) \geq 0$$

so we obtain the refiement. $\forall x, y, z \geq 0$ it's true.

$$x^3+y^3+z^3 \geq \sqrt{\frac{1}{3}\prod_{cyclic}(2x^2-xy+2y^2) + \frac{1}{3}(\alpha-9)^2\left(\alpha-\frac{9}{4}\right)x^2y^2z^2}$$

where $\alpha = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$
The right side of inequality from statement it's equivalent with

$$\prod_{cyclic}\left(2x^2-xy+2y^2\right) \geq 3\left[\sum xy(x+y) - 3xyz\right]^2$$

or from (11)

$$-27p^2 - 20ps^3 + 8\alpha^2p^2 + 54\alpha p^2 - \frac{20\alpha^3p^3}{s^3} \geq 3p^2(\alpha-6)^2 \quad \text{or} \quad p^2 g(t) \geq 0$$

$$-27 - 20t + 8\alpha^2 + 54\alpha - \frac{20\alpha^3}{t} - 3(\alpha-6)^2 \geq 0,\ t \in [t_1, t_2]$$

or in an equivalent form

$$4t^2-\left(\alpha^2+18\alpha-27\right)t+4\alpha^3\le 0,\ t\in[t_1,t_2]$$

which is true, since the roots of

$$4t^2-\left(\alpha^2+18\alpha-27\right)t+4\alpha^3=0 \text{ are } t_1 \text{ and } t_2$$

If we consider the function $g:[t_1,t_2]\to R$

$$g(t)=-27-20t+8\alpha^2+54\alpha-\frac{20\alpha^3}{t}-3(\alpha-6)^2$$

with $g'(t)=\frac{20\left(\alpha^3-t^2\right)}{t^2}$ with root $t'=\alpha^{\frac{3}{2}}\in[t_1,t_2]$ we observe that t' is a maximum point for f. It follows that $g(t)\le g(t')=(\sqrt{\alpha}-3)(\sqrt{\alpha}+1)$. We obtain that for each positive numbers x,y,z holds the inequality

$$\frac{1}{3}\prod_{cyclic}\left(2x^2-xy+2y^2\right)\le\left(\sum_{xyxlix}xy(x+y)-3xyz\right)^2+\frac{x^2y^2z^2}{3}(\sqrt{\alpha}-3)(\sqrt{\alpha}+1)$$

where $\alpha=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

13. Prove that the following inequality holds for all positive real numbers.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{9(a+b+c)}{ab+bc+ca}\ge 8\left(\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\right)$$

Nyugen Viet Hung, Mathematical Reflections 1/2019

Solution.

$$\text{We have}\quad\sum\frac{a}{a^2+bc}=\frac{\sum a\left(b^2+ca\right)\left(c^2+ab\right)}{\prod\left(a^2+bc\right)}=$$

$$=\frac{-abc(a+b+c)^2+(ab+bc+ca)^2(a+b+c)-2abc(ab+bc+ca)}{8a^2b^2c^2+abc\left(\sum a\right)^3-6abc\sum ab\sum a+\left(\sum ab\right)^3}=$$

$$=\frac{-ps^2+\frac{p^2\alpha^2}{s^2}\cdot s-2p\frac{p\alpha}{s}}{8p^2+ps^3-6p\frac{p\alpha}{s}s+\frac{p^3\alpha^3}{s^3}}=\frac{\left(-ps^3+p^2\alpha^2-2p^2\alpha\right)s^2}{8p^2s^3+ps^6-6\alpha p^2s^3+p^3\alpha^3}$$

The inequality from statement may be written as:

$$\frac{\alpha}{s}+\frac{9s^2}{p\alpha}\ge\frac{8s^2\left(-ps^3+p^2\alpha^2-2p^2\alpha\right)}{8p^2s^3+ps^6-6p^2\alpha s^3+p^3\alpha^3}\quad\text{or}$$

$$\alpha+\frac{9t}{\alpha}-\frac{8t\left(-t+\alpha^2-2\alpha\right)}{8t+t^2-6\alpha t+\alpha^3}\ge 0\quad\text{or}$$

$$f(t)=9t^3+\left(72-46\alpha+\alpha^2\right)t^2+\left(24\alpha^2-5\alpha^3\right)t+\alpha^5\ge 0,\forall t\in[t_3,t_4]$$

We have

$$f'(t)=27t^2+\left(2\alpha^2+144-92\alpha\right)t-5\alpha^3+24\alpha^2,\forall t\in[t_3,t_4]$$

$$\text{with}\quad t'_{1,2}=\frac{1}{27}\left(-\alpha^2+46\alpha-72\pm\sqrt{\alpha^4+43\alpha^3+1612\alpha^2-6624\alpha+5184}\right),\left(t'_1\le t'_2\right)$$

Since $t_2' < 4\alpha - 9$, $\forall \alpha > 3,666$ so if $\alpha \geq 9$ it follows that $f'(t) \geq 0$, $\forall t \in [t_3, t_4]$ so f is increasing on $[t_3, t_4]$ or $f(t) \geq f(t_3)$. But $f(t_3) \geq 0$, $\forall \alpha \geq 9$ (WA) from which it follows the statement.

14. Let a, b, c be the side lenghts of triangle with inradius r and circumradius R. Prove that

$$\frac{R}{r} + 1 + \sqrt{5} \geq \left(3 + \sqrt{5}\right) \frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

Marius Stănean, Mathematical Reflections 4/2021,

Solution. We have using Ravi substitution that

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} = \frac{\sum (y+z)^2}{\sum (x+y)(x+z)} = \frac{2\sum x^2 + 2\sum yz}{\sum x^2 + 3\sum yz} =$$

$$= \frac{2s^3 - 2\alpha p}{s^3 + \alpha p} = \frac{2t - 2\alpha}{t + \alpha} \text{ and } \frac{R}{r} = \frac{\alpha - 1}{4}$$

Inequality from statement may be written as

$$\frac{\alpha - 1}{4} + 1 + \sqrt{5} \geq \left(3 + \sqrt{5}\right) \frac{(2t - 2\alpha)}{t + \alpha}$$

Let $f : [t_1, t_2] \to R$

$$f(t) = \frac{\left(3 + \sqrt{5}\right)(2t - 2\alpha)}{t + \alpha} - \frac{\alpha + 3}{4} - \sqrt{5}$$

which is increasing on $[t_1, t_2]$ so $f(t) \leq f(t_2)$.

We will prove that $f(t_2) \leq 0$, which is true according W.A. if $\alpha \geq 9$ with equality if $\alpha = 9$ or $\alpha = \frac{25+12\sqrt{5}}{5} \simeq 10,36$.

If $\alpha = 9$ we obtain $x = y = z$ or $a = b = c$.

If $\alpha = \frac{25+12\sqrt{5}}{5}$ we obtain $y = z = \frac{x}{\sqrt{5}}$ so $a = \frac{2}{\sqrt{5}}$, $b = \left(1 + \frac{1}{\sqrt{5}}\right) k$, $c = \left(1 + \frac{1}{\sqrt{5}}\right) k$, $k > 0$

15. Let x, y, z positive numbers such that

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \alpha \geq 9$$

Find the minimum and maximum of

$$\left(x^3 + y^3 + z^2\right)\left(\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3}\right)$$

Marius Drăgan

Solution. We have

$$x^3 \sum \frac{1}{x^3} = \frac{\sum x^3 \sum y^3 z^3}{(xyz)^3} = \frac{1}{p^3}\left(9p^3 + 3p^2 s^3 - 18p^2 \frac{p\alpha}{s} s - 3p \frac{p\alpha}{s} s^4 + \frac{p^3\alpha^3}{s^3} s^3 + 9p \frac{p^2\alpha^2}{s^2} s^2 - 3\frac{p^4\alpha^4}{s^4} s + \right.$$

$$\left. +3p \frac{p^3\alpha^3}{s^3}\right) = 9 + \frac{3s^3}{p} - 18\alpha - \frac{3\alpha s^3}{p} + \alpha^3 + 9\alpha^2 - 3\alpha^4 \frac{p}{s^3} + \frac{3\alpha^3 p}{s^3} =$$

$$= 9 + 3t - 18\alpha - 3\alpha t + \alpha^3 + 9\alpha^2 - \frac{3\alpha^4}{t} + \frac{3\alpha^3}{t} =$$

$$= \alpha^3 + 9\alpha^2 - 18\alpha + 9 + (3 - 3\alpha)t + \frac{3\alpha^3 - 3\alpha^4}{t} = f(t), \ t \in [t_1, t_2]$$

$$\text{with}\quad f'(t)=3-3\alpha-\frac{3\alpha^3-3\alpha^4}{t^2}=(3-3\alpha)\left(1-\frac{\alpha^3}{t^2}\right)=\frac{(3-3\alpha)\left(t^2-\alpha^3\right)}{t^2}$$

t	t_1		$\alpha^{\frac{3}{2}}$		t_2
$f'(t)$	$+++$	$++$	0	$--$	$---$
$f(t)$	$f(t_1)$	$\nearrow$	$f\left(\alpha^{\frac{3}{2}}\right)$	$\searrow$	$f(t_2)$

According WA, $t_1\le\alpha^{\frac{3}{2}}\le t_2$, $\forall\alpha\ge 9$. So $\min\limits_{t\in[t_1,t_2]} f(t)=\min\{f(t_1),f(t_2)\}$, $\max\limits_{t\in[t_1,t_2]} f(t)=f\left(\alpha^{\frac{3}{2}}\right)$.
We have:

$$f(t_1)=\frac{1}{4}\left(\alpha^3-15\alpha^2+63\alpha-45\right),f(t_2)=\frac{1}{4}\left(\alpha^3-15\alpha^2+63\alpha-45\right),f\left(\alpha^{\frac{3}{2}}\right)=\left(\sqrt{\alpha^3}-3\alpha+3\right)^2$$

so

$$\min\left\{\left(x^3+y^3+z^3\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)/x,y,z>0,\ (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\alpha\ge 9\right\}=$$

$$=\frac{1}{4}\left(\alpha^3-15\alpha^2+63\alpha-45\right)\quad\text{and}$$

$$\max\left\{\left(x^3+y^3+z^3\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)/x,y,z>0,\ (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\alpha\ge 9\right\}=\left(\alpha^{\frac{3}{2}}-3\alpha+3\right)^2$$

16. Let a,b,c positive numbers. Prove that

$$\frac{b+c}{a^2}+\frac{c+a}{b^2}+\frac{a+b}{c^2}-\frac{9}{a+b+c}\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Adrian Andreescu, Mathematical Reflections 2/2020

Solution. We have in an equivalent form

$$s\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\sum\frac{1}{a}-\frac{9}{s}-\sum\frac{1}{a}\ge 0\quad\text{or}$$

$$s\left[\left(\sum\frac{1}{a}\right)^2-2\sum\frac{1}{ab}\right]-2\sum\frac{1}{a}-\frac{9}{s}\ge 0\quad\text{or}$$

$$s\left(\frac{\alpha^2}{s^2}-\frac{2s}{p}\right)-\frac{2\alpha}{s}-\frac{9}{s}\ge 0\ \text{or}\ \frac{\alpha^2-2\alpha-9}{s}-\frac{2s^2}{p}\ge 0\quad\text{where}$$

$$f(t)=\alpha^2-2\alpha-9-2t\ge 0,\ \forall t\in[t_3,t_4]\ \text{ or }\ \frac{1}{s}f(t)\ge 0,\ \forall t\in[t_3,t_4]$$

But $f(t)\ge f(t_4)$, $\forall\alpha\ge 9$

$$f(t_4)=\alpha^2-2\alpha-9-\frac{2\alpha^3}{4\alpha-9}=\frac{(\alpha-9)\left(2\alpha^2+\alpha-9\right)}{4\alpha-9}\ge 0,\ \forall\alpha\ge 9$$

We obtain the following refinement

$$\sum\frac{b+c}{a^2}-\frac{9}{\sum a}\ge\sum\frac{1}{a}+\frac{1}{\sum a}\cdot\frac{(\alpha-9)\left(2\alpha^2+\alpha-9\right)}{4\alpha-9}\ge\sum\frac{1}{a}$$

where $\alpha = \sum a \sum \frac{1}{a}$

17. Prove that for every positive real numbers holds

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}+\frac{21}{4}$$

Nyugen Viet Hung, Mathematical Reflections 1/2020

Solution. We have $f(t) = \sum a \sum \frac{1}{a} - \frac{27\sum a^3}{\left(\sum a\right)^3} = \alpha - \frac{27\left(s^3-3p\alpha+3p\right)}{s^3} =$

$$= \alpha - 27 + \frac{81\alpha - 81}{t}, \ \forall t \in [t_1, t_2]$$

Since f is decreasing on $[t_1, t_2]$. We have:

$$f(t) \geq f(t_2) = \alpha - 27 + \frac{8(81\alpha-81)}{\alpha^2+16\alpha-27+\sqrt{(\alpha-1)(\alpha-9)^3}} = g(\alpha)$$

So $f(t) \geq \inf\limits_{\alpha\geq 9} g(\alpha) \overset{WA}{=} \frac{21}{4}$ at $\alpha = \frac{27}{2}$. The equality come true if

$$\left\{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{27}{2}, \frac{9\sum a^3}{\left(\sum a\right)^3} = \frac{11}{4}, \ a,b,c>0\right\}$$

Solution are of form $(k,k,4k)$, $k \neq 0$.
We obtain the following refinement: Prove that for every positive real numbers holds

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}+\alpha-$$

$$-\frac{87}{4}+\frac{8(81\alpha-81)}{\alpha^2+18\alpha-27+\sqrt{(\alpha-1)(\alpha-9)^3}} \geq \frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}+\frac{21}{4}$$

where $\alpha = (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

18. Let x, y, z positive numbers such that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \alpha \geq 9$$

Find the minimum and the maximum of $\left(x^4+y^4+z^4\right)\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right)$

Solution. We have: $\sum x^4 \cdot \sum \frac{1}{x^4} = \frac{1}{(xyz)^4}\sum x^4 \sum y^4 z^4 =$

$$= \frac{1}{p^4}\left(8p^3s^3+16p^3\frac{p\alpha}{s}s+2p^2s^6-4p^2\frac{p\alpha}{s}s^4-28p^2\frac{p^2\alpha^2}{s^2}s^2+8p^2\frac{p^3\alpha^3}{s^3}-\right.$$

$$\left.-4p\frac{p^2\alpha^2s^5}{s^2}+\frac{p^4\alpha^4}{s^4}s^4+16p\frac{p^3\alpha^3}{s^3}s^3-\frac{4p^5\alpha^5}{s^5}s^2-4p\frac{p^4\alpha^4}{s^4}s+\frac{2p^6\alpha^6}{s^6}\right) =$$

$$= 8t+16\alpha+2t^2-4\alpha t-28\alpha^3+\frac{8\alpha^3}{t}-4\alpha^2 t+\alpha^4+16\alpha^3-$$

$$-\frac{4\alpha^5}{t}-\frac{4\alpha^4}{t}+\frac{2\alpha^6}{t^2} = f(t), \ \forall t \in [t_1, t_2]$$

$$\text{We have} \quad f'(t) = \frac{4\left(t^2-\alpha^3\right)\left[t^2-\left(\alpha^2+\alpha-2\right)t+\alpha^3\right]}{t^3}$$

with roots $t_6=\alpha^{\frac{2}{3}}$, $t_{4,5}=\frac{\alpha^2+\alpha-2\pm\sqrt{\alpha^4-2\alpha^3-3\alpha^2-4\alpha+4}}{2}$ $(t_4\le t_5)$

According WA we have $t_4\le t_1$, $t_2\le t_5$, $t_1\le t_6$, $t_6\le t_2$, $\forall\alpha\ge 9$. So $t_4\le t_6\le t_2\le t_5$, $\forall\alpha\ge 9$ so if $t\in[t_1,t_6]$ f is increasing and if $t\in[t_6,t_2]$ f is decreasing so

$$\min_{t\in[t_1,t_2]} f(t)=\min\left\{f(t_1),f(t_2)\right\},\quad \max_{t\in[t_1,t_2]} f(t)=f(t_6)$$

We use WA. So

$$f(t_1)=\alpha^4+16\alpha^3-\frac{1}{2}\left(\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)\alpha^2-$$

$$-27\alpha^2-\frac{1}{2}\left(\alpha^2+18\alpha-\sqrt{(\alpha-1)(\alpha-9)^3}-27\right)\alpha+\frac{\left(\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)^2}{32}=$$

$$=\frac{32\alpha^5+32\alpha^4-64\alpha^3}{\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}}+34\alpha-\sqrt{(\alpha-1)(\alpha-9)^3}-27+\frac{128\alpha^6}{\left(\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)^2}=$$

$$=\alpha^4+16\alpha^3-\frac{\alpha^4}{2}-9\alpha^3+\frac{27}{2}\alpha^2-27\alpha^2-\frac{\alpha^3}{2}-9\alpha^2+\frac{27\alpha}{2}+\frac{\alpha^4}{16}+\frac{\alpha^3}{4}+\frac{135\alpha^2}{8}-$$

$$-\frac{243\alpha}{4}+\frac{729}{16}++\left(\frac{\alpha^2}{2}+\frac{\alpha}{2}\right)\sqrt{(\alpha-1)(\alpha-9)^3}+\left(-\frac{\alpha^2}{16}-\frac{9\alpha}{8}+\frac{27}{16}\right)\cdot$$

$$\cdot\sqrt{(\alpha-1)(\alpha-9)^3}-\frac{7}{16}\alpha^4-\frac{37}{4}\alpha^3+\frac{179}{8}\alpha^2-\frac{117}{4}\alpha+\frac{297}{16}$$

$$+\left(-\frac{7}{16}\alpha^2+\frac{10}{16}\alpha-\frac{11}{16}\right)\sqrt{(\alpha-1)(\alpha-9)^3}-\sqrt{(\alpha-1)(\alpha-9)^3}+$$

$$+34\alpha-27=\frac{\alpha^4}{8}-\frac{5\alpha^3}{2}+\frac{67\alpha^2}{4}-\frac{85\alpha}{2}+\frac{297}{8}$$

In same way we obtain $f(t_2)=f(t_1)$, so

$$\min_{x,y,z>0}\left(x^4+y^4+z^4\right)\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right)=\frac{\alpha^4}{8}-\frac{5\alpha^3}{2}+\frac{67\alpha^2}{4}-\frac{85\alpha}{2}+\frac{297}{8}$$

If $\alpha=11$ we obtain:

$$\min_{x,y,z>0}\left(x^4+y^4+z^4\right)\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right)=99$$

which represent S520 from Mathematical Reflections 4/2020, author Marius Stănean.

Also

$$\max\left(x^4+y^4+z^4\right)\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right)=\alpha\left(\sqrt{\alpha}-2\right)^2\left(\alpha-2\sqrt{\alpha}-2\right)^2$$

19. Find the greatest constant k such the inequality

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\lambda\ge\frac{2}{3}(1+\lambda)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

holds for all positive real numbers.

Solution.

$$\frac{s^3-2p\alpha}{p\alpha}+\lambda\geq\frac{2}{3}\left(1+\lambda\right)\frac{s^3-2p\alpha+3p}{p\left(\alpha-1\right)}$$

$$\frac{t-2\alpha}{\alpha}+\lambda\geq\left(\frac{2}{3}+\frac{2}{3}\lambda\right)\frac{t-2\alpha+3}{\alpha-1}$$

$$\frac{t-2\alpha}{\alpha}-\frac{2}{3}\cdot\frac{t-2\alpha+3}{\alpha-1}\geq\lambda\left(\frac{2t-4\alpha+6}{3\alpha-3}-1\right)\quad\text{or}$$

$$\frac{\left(\alpha-3\right)t-2\alpha^2}{\alpha\left(2t-7\alpha+9\right)}\geq\lambda\quad\text{since}$$

$$2t-7\alpha+9\geq2\left(4\alpha-9\right)-7\alpha+9=\alpha-9\geq0$$

Let $f:[t_1,t_2]\to R$, $f\left(t\right)=\frac{(\alpha-3)t-2\alpha^2}{\alpha(2t-7\alpha+9)}$ with $f'\left(t\right)=\frac{1}{\alpha}\cdot\frac{\left(-5\alpha^2+30\alpha-27\right)}{(-7\alpha+t+9)^2}$, $\forall t\in[t_1,t_2]$ since $-5\alpha^2+30\alpha-27<0$ if $\alpha>\frac{15+3\sqrt{10}}{5}\simeq4,89$ so if $\alpha\geq9$ it follows that f is decreasing on $[t_1,t_2]$ so $f\left(t\right)\geq f\left(t_2\right)$ so $\lambda\leq f\left(t_2\right),\forall\alpha\geq9$ so the greatest constant is

$$\lambda_0=\inf_{\alpha\geq9}f\left(t_2\right)=\inf_{\alpha\geq9}\frac{\left(\alpha-3\right)t_2-2\alpha^2}{\alpha\left(2t_2-7\alpha+9\right)}=\frac{1}{2}$$

20. Prove that in any triangle ABC

$$\frac{\left(a+b\right)\left(b+c\right)\left(c+a\right)}{4abc}\leq1+\frac{R}{2r}$$

Marius Stănean, Mathematical Reflections 6/2020

Solution Inequality may be written as

$$\frac{\left(a+b\right)\left(b+c\right)\left(c+a\right)}{4abc}\leq1+\frac{abc}{8\prod\left(s-a\right)}$$

Using the Ravi subtitution $a=y+z$, $b=z+x$, $c=x+y$ we obtain

$$\frac{\prod\left(x+y+2z\right)}{4\prod\left(y+z\right)}\leq1+\frac{\prod\left(x+z\right)}{8xyz}$$

$$\text{But}\quad\prod\left(x+y+2z\right)=2\left(\sum x\right)^3+\sum xy\sum x+xyz$$

$$\text{so}\quad\frac{2s^3+p\alpha+p}{4\left(p\alpha-p\right)}\leq1+\frac{p\alpha-p}{8p}\ \text{or}$$

$$\frac{2t+\alpha+1}{4\alpha-4}\leq1+\frac{\alpha-1}{8}\quad\text{or}\quad f\left(t\right)=\frac{2t+\alpha+1}{4\alpha-4}-1-\frac{\alpha-1}{8}\leq0,\ \forall t\in[t_3,t_4]$$

$$\text{But}\quad f\left(t\right)\leq f\left(t_4\right)=-\frac{\left(\alpha-9\right)\left(7\alpha-9\right)}{8\left(\alpha-1\right)\left(4\alpha-9\right)}\leq0,\ \forall\alpha\geq9$$

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