Junior problems

J637. Find all positive integers n such that

$$3 \cdot n! - 1 = \sqrt{(2n-1)! + 1}.$$

Proposed by Adrian Andreescu, Dallas, USA

Solution by Polyahedra, Polk State College, USA

Notice that for all $n \geq 1$,

$$\frac{(2n+1)!}{(n+1)!^2} - \frac{(2n-1)!}{n!^2} = \frac{(2n-1)!(3n^2-1)}{(n+1)!^2} > 0,$$

so for all $n \geq 5$,

$$\sqrt{\frac{(2n-1)!}{n!^2} + \frac{1}{n!^2}} > \sqrt{\frac{(2n-1)!}{n!^2}} > 3 > 3 - \frac{1}{n!}.$$

For $n \in \{1, 2, 3, 4\}$, only $n = 4$ satisfies the equation.

Also solved by G. C. Greubel, Newport News, VA, USA; Jennifer Lee, Chattahoochee High School, GA, USA; Michael Lincoln, Suny Brockport, USA; Sundaresh. H .R., Shivamogga, India; Theo Koupelis, Clark College, WA, USA; Yoonwoo Lee; NYSS Problem Solving Group; Michel Faleiros Martins, São Paulo, SP, Brazil; Soham Bhadra, Patha Bhavan, Kolkata, India.

J638. Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{\sqrt{c(a+2b+3c)}}+\frac{b+2c}{\sqrt{a(b+2c+3a)}}+\frac{c+2a}{\sqrt{b(c+2a+3b)}}\geq\frac{3\sqrt{6}}{2}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Michel Faleiros Martins, São Paulo, SP, Brazil

By AM-GM, we are seeking for a number $k > 0$ such that

$$\sum_{cyc}\frac{a+2b}{\sqrt{c(a+2b+3c)}}\geq\sum_{cyc}\frac{a+2b}{\frac{kc+\frac{a+2b+3c}{k}}{2}}$$

implies equality for $a = b = c$. Therefore, $k = \sqrt{6}$. Hence, by Titu's lemma applied after AM-GM, we have

$$\frac{1}{2\sqrt{6}}\sum_{cyc}\frac{a+2b}{\sqrt{c(a+2b+3c)}}\geq\sum_{cyc}\frac{(a+2b)^2}{(a+2b)(a+2b+9c)}\geq\frac{\left(\sum_{cyc}(a+2b)\right)^2}{\sum\limits_{cyc}(a+2b)(a+2b+9c)}$$

$$=\frac{\frac{3}{4}\left(12(a^2+b^2+c^2)+24(ab+bc+ca)\right)}{5(a^2+b^2+c^2)+31(ab+bc+ca)}\geq\frac{3}{4},$$

where in the last step we used that $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$, that is, $a^2+b^2+c^2\geq ab+bc+ca$. Thus, the proposed inequality is immediate from the above. The equality holds iff $a = b = c$.

Also solved by Nguyen Viet Hung, Hanoi University of Science, Vietnam; Polyahedra, Polk State College, FL, USA; NYSS Problem Solving Group; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J639. Solve the equation

$$176x - 4\lfloor x \rfloor^2 - 88\{x\}^2 = 2023,$$

where $\lfloor x \rfloor$ and $\{x\}$ are the integer part and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Adam John Frederickson, Utah Valley University, UT, USA

Let $n = \lfloor x \rfloor$ and $r = \{x\}$. Then $x = n + r$, and

$$\begin{aligned} 0 &= -176(n+r) + 4n^2 + 88r^2 + 2023 \\ &= 4(n-22)^2 + 88(1-r)^2 - 1. \end{aligned}$$

Then $n = 22$, and

$$88(1-r)^2 - 1 = 0 \quad \Rightarrow \quad r = 1 \pm \frac{1}{\sqrt{88}}.$$

Since $0 \le r < 1$, we must have

$$r = 1 - \frac{1}{\sqrt{88}} \Rightarrow x = n + r = 23 - \frac{1}{\sqrt{88}}.$$

Also solved by Polyahedra, Polk State College, FL, USA; NYSS Problem Solving Group; Michel Faleiros Martins, São Paulo, SP, Brazil; Soham Bhadra, Patha Bhavan, Kolkata, India; Kyle Song, Hopkins School in New Haven, CT, USA; Adrianna Godfrey, SUNY Brockport, USA; Hyunbin Yoo, South Korea; Jennifer Lee, Chattahoochee High School, GA, USA; Matthew Too, Brockport, NY, USA; Sundaresh. H .R., Shivamogga, India; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Daniel Pascuas, Barcelona, Spain; Le Hoang Bao, TienGiang, Vietnam; Arkady Alt, San Jose, CA, USA..

J640. Let x, y, z be positive real numbers such that $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. Prove that

$$\sqrt{xy+3}+\sqrt{yz+3}+\sqrt{zx+3} \leq \frac{3(x+y+z+1)}{2}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition we get

$$x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{x+y+z}.$$

It follows that

$$(x+y+z)^2 \geq 9,$$

or

$$x+y+z \geq 3.$$

Now we use the Cauchy-Schwarz inequalitie to obtain

$$\sum_{\text{cyc}} \sqrt{xy+3} \leq \sqrt{3(xy+yz+zx+9)} \leq \sqrt{(x+y+z)^2+27}.$$

It suffices to show that

$$(x+y+z)^2+27 \leq \frac{9}{4}(x+y+z+1)^2.$$

This is equivalent to

$$5(x+y+z)^2+18(x+y+z)-99 \geq 0,$$

or

$$(x+y+z-3)(5(x+y+z)+33) \geq 0$$

which is true because $x+y+z \geq 3$ and we are done.

Also solved by Polyahedra, Polk State College, FL, USA; Michel Faleiros Martins, São Paulo, SP, Brazil; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J641. Find all positive integers a and b such that

$$\frac{a^2}{b} - \frac{b^2}{a} = 2023.$$

Proposed by Mircea Becheanu, Canada

Solution by Polyahedra, Polk State College, USA

Let $d = (a, b)$, then $a = dx$ and $b = dy$ with $(x, y) = 1$. The equation becomes $d(x^3 - y^3) = 7 \cdot 17^2 xy$. Since $x^3 - y^3$ is relatively prime to both x and y, it must divides $7 \cdot 17^2$. If $x - y \geq 17$, then $x^3 - y^3 > 17^3 > 7 \cdot 17^2$. Therefore, $x - y \in \{1, 7\}$.

If $x - y = 7$, then $x^2 + xy + y^2 = 3y^2 + 21y + 49 = 17^2$ by considering remainders modulo 3. There is no integer solution y in this case. If $x - y = 1$, then $x^2 + xy + y^2 = 3y^2 + 3y + 1 \in \{7, 17^2, 7 \cdot 17^2\}$ by considering remainders modulo 3 again. Only $3y^2 + 3y + 1 = 7$ yields the integer solution $y = 1$. Therefore, $x = 2$, $d = 2 \cdot 17^2$, and it is easy to check that $a = 4 \cdot 17^2 = 1156$ and $b = 2 \cdot 17^2 = 578$ do satisfy the equation.

Also solved by Isabella Kim, Academy of the Holy Angels, NJ, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Michel Faleiros Martins, São Paulo, SP, Brazil; G. C. Greubel, Newport News, VA, USA; Jennifer Lee, Chattahoochee High School, GA, USA; Theo Koupelis, Clark College, WA, USA.

J642. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove that

$$3\sqrt[6]{8abc} \le \sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} \le 3\sqrt{2abc}.$$

Proposed by Marius Stănean, Zalău, România

First solution by Polyahedra, Polk State College, USA
By the AM-GM inequality, $3=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{3}{\sqrt[3]{abc}}$, so $abc \ge 1$. Also, since

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}=(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-2(a+b+c)=9abc-2(a+b+c),$$

by the AM-GM inequality,

$$\begin{aligned}\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} &\ge 3\sqrt[6]{\left(a+\frac{b}{c}\right)\left(b+\frac{c}{a}\right)\left(c+\frac{a}{b}\right)}\\ &= 3\sqrt[6]{abc+a^2+b^2+c^2+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}+1}\\ &= 3\sqrt[6]{8abc+(a-1)^2+(b-1)^2+(c-1)^2+2(abc-1)}\\ &\ge 3\sqrt[6]{8abc}.\end{aligned}$$

Next, since

$$a+b+c=abc\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right) \le \frac{abc}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=3abc,$$

by the Cauchy-Schwarz inequality,

$$\begin{aligned}\sqrt{\frac{1}{c}}\cdot\sqrt{ca+b}+\sqrt{\frac{1}{a}}\cdot\sqrt{ab+c}+\sqrt{\frac{1}{b}}\cdot\sqrt{bc+a} &\le \sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(ab+bc+ca+a+b+c)}\\ &\le 3\sqrt{2abc}.\end{aligned}$$

Second solution by the author

For the right side by Cauchy-Schwarz Inequality, we have

$$\left(\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}}\right)^2 \le (ca+b+ab+c+bc+a)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)$$

so

$$\begin{aligned}\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} &\le \sqrt{3(ab+bc+ca+a+b+c)}\\ &\le \sqrt{6(ab+bc+ca)} = 3\sqrt{2abc},\end{aligned}$$

because $(ab+bc+ca)^2 \ge 3abc(a+b+c) \Longrightarrow ab+bc+ca \ge a+b+c$.

For the left side by AM-GM Inequality, we have

$$\begin{aligned}\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} &\ge 3\sqrt[6]{\left(a+\frac{b}{c}\right)\left(b+\frac{c}{a}\right)\left(c+\frac{a}{b}\right)}\\ &= 3\sqrt[6]{a^2+b^2+c^2+abc+1+\frac{a^2b^2+b^2c^2+c^2a^2}{abc}}\\ &= 3\sqrt[6]{(\sum a)^2-2\sum ab+abc+1+\frac{(\sum ab)^2}{abc}-2\sum a}\\ &= 3\sqrt[6]{(a+b+c-1)^2+4abc}.\end{aligned}$$

We need to show that

$$(a+b+c-1)^2 \ge 4abc,$$

or after we homogenize the inequality

$$\left(a+b+c-\frac{3abc}{ab+bc+ca}\right)^2 \cdot \frac{3abc}{ab+bc+ca} \ge 4abc,$$

or

$$3\left(ab(a+b)+bc(b+c)+ca(c+a)\right)^2 \ge 4(ab+bc+ca)^3,$$

or

$$3\sum_{cyc}(a^4b^2+a^2b^4)+6abc\sum_{cyc}a^3+2\sum_{cyc}a^3b^3-6abc\sum_{cyc}(a^2b+ab^2)-6a^2b^2c^2 \ge 0,$$

that is

$$3[(4,2,0)]+3[(4,1,1)]+[(3,3,0)] \ge 6[(3,2,1)]+[(2,2,2)]$$

which follows by Muirhead's Inequality i.e

$$[(4,2,0)] \ge [(4,1,1)] \ge [(3,3,0)] \ge [(3,2,1)] \ge [(2,2,2)].$$

Also solved by Arkady Alt, San Jose, CA, USA; Henry Ricardo, Westchester Area Math Circle; Michel Faleiros Martins, São Paulo, SP, Brazil; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Senior problems

S637. Let $a, b > 0$ and $c < 0$ such that

$$2a^2 + \frac{1}{8a^2} = 7, 2b^2 + \frac{1}{8b^2} = 17, 2c^2 + \frac{1}{8c^2} = 31.$$

Prove that

$$(4ab+1)(4bc+1)(4ca+1) = (8abc+1)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Michel Faleiros Martins, São Paulo, SP, Brazil
Let $x = 4a^2 > 0$, $y = 4b^2 > 0$, and $z = 4c^2 > 0$. Thus, adjusting the sign of ab, bc, ca, and abc in terms of x, y, and z, we obtain

$$\begin{aligned}(4ab+1)(4bc+1)(4ca+1) - (8abc+1)^2 &= (\sqrt{xy}+1)(-\sqrt{yz}+1)(-\sqrt{zx}+1) - (-\sqrt{xyz}+1)^2 \\ &= \sqrt{xyz}\left(-\frac{x+1}{\sqrt{x}} - \frac{y+1}{\sqrt{y}} + \frac{z+1}{\sqrt{z}} + 2\right) \\ &= \sqrt{xyz}\left(-x^{\frac{1}{2}} - x^{-\frac{1}{2}} - y^{\frac{1}{2}} - y^{-\frac{1}{2}} + z^{\frac{1}{2}} + z^{-\frac{1}{2}} + 2\right) \\ &= \sqrt{xyz}\,(-4-6+8+2) \\ &= 0,\end{aligned}$$

where we used that $x^1 + x^{-1} = 14$ implies $\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 - 2 = 14$, that is, $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = \sqrt{14+2} = 4$. Similarly, $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = \sqrt{34+2} = 6$ and $z^{\frac{1}{2}} + z^{-\frac{1}{2}} = \sqrt{62+2} = 8$.

Also solved by G. C. Greubel, Newport News, VA, USA; Sundaresh. H .R., Shivamogga, India; Theo Koupelis, Clark College, WA, USA; Arkady Alt, San Jose, CA, USA.

S638. Let a,b,c be positive real numbers. Prove that

$$\frac{6a^2}{(2b+c)(2c+b)}+\frac{6b^2}{(c+a)(2a+c)}+\frac{6c^2}{(2a+b)(b+a)}\ge 1+\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Using Cauchy-Schwarz Inequality, we have

$$\sum_{cyc}\frac{a^2}{(2b+c)(2c+b)}=\sum_{cyc}\frac{a^4}{a^2(2b+c)(2c+b)}\ge\frac{(a^2+b^2+c^2)^2}{\sum a^2(2b+c)(2c+b)}$$
$$=\frac{(a^2+b^2+c^2)^2}{4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)}.$$

We need to show that

$$\frac{(a^2+b^2+c^2)^2}{4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)}-\frac{1}{3}\ge\frac{a^2+b^2+c^2-ab-bc-ca}{6(ab+bc+ca)},$$

or

$$\frac{3(a^4+b^4+c^4)+2(a^2b^2+b^2c^2+c^2a^2)-5abc(a+b+c)}{4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)}\ge\frac{a^2+b^2+c^2-ab-bc-ca}{2(ab+bc+ca)}.$$

Without loss of generality, we may assume that $c=\max\{a,b,c\}$.

$$\frac{3(a^2-b^2)^2+3(a^2-c^2)(b^2-c^2)+5(ac-bc)^2+5(ab-ac)(ab-bc)}{4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)}\ge\frac{(a-b)^2+(a-c)(b-c)}{2(ab+bc+ca)}.$$

Therefore it suffices to prove that

$$2(ab+bc+ca)\left[3(a+b)^2+5c^2\right]\ge 4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)$$

that is

$$6a^3b+6a^3c+6ab^3+6b^3c+8a^2b^2+13a^2bc+13ab^2c+5abc^2+10ac^3+10bc^3-4a^2c^2-4b^2c^2\ge 0,$$

clearly true, and

$$2(ab+bc+ca)\left[3(a+c)(b+c)+5ab\right]\ge 4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)$$

that is

$$12a^2b^2+2a^2c^2+2b^2c^2+17a^2bc+17ab^2c+13abc^2+6ac^3+6bc^3\ge 0$$

obviously true.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam.

S639. Let

$$f(n) = \binom{n^2-1}{3}, \quad n = 2,3,4,\ldots$$

Find all positive integers $k \ge 2$ such that $f(k+3) = f(k) + 2023$.

Proposed by Adrian Andreescu, Dallas, USA

Solution by Jean Heibig, ISAE-SUPAERO, Toulouse, France

As $f(n) = \frac{(n^2-1)(n^2-2)(n^2-3)}{6}$, then $f(k+3) - f(k) = \frac{2k+3}{2}\left(\alpha_k + \beta_k(6k+9) + (6k+9)^2\right)$, with

$$\begin{cases} \alpha_k & = (k^2-1)(k^2-2) + (k^2-2)(k^2-3) + (k^2-3)(k^2-1) \\ \beta_k & = (k^2-1) + (k^2-2) + (k^2-3) \end{cases}$$

This shows that $k \mapsto f(k+3) - f(k)$ is strictly increasing, so there is at most one solution. As $2k+3 \mid 2023 = 7 \times 17^2$, the solution k, if it exists, is such that

$$2k+3 \in \{7, 17, 119, 289, 2023\} \iff k \in \{2, 7, 58, 143, 1010\}$$

Trying these values gives for $k = 2$:

$$f(k+3) = 4 \times 22 \times 23 = 2024 = 1 + 2023 = f(k) + 2023$$

Conclusion: $k = 2$ is the unique solution to the equation.

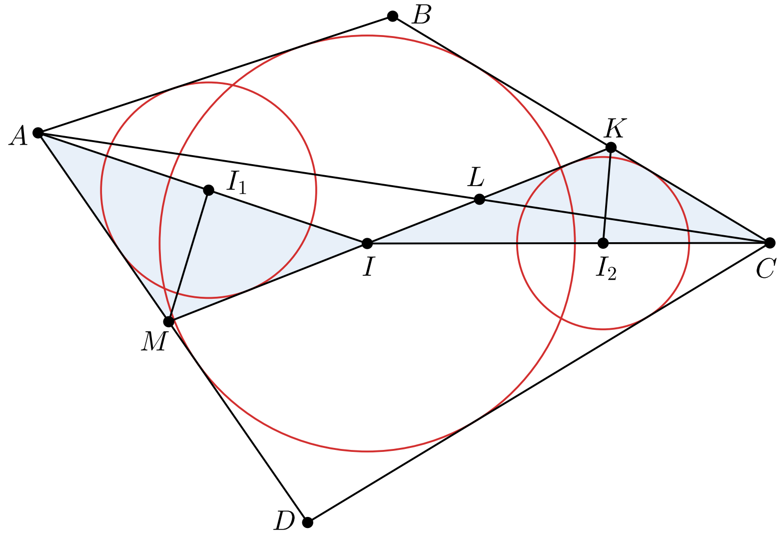
Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Adam John Frederickson, Utah Valley University, UT, USA; G. C. Greubel, Newport News, VA, USA; Kaleigh Perkins, SUNY Brockport, NY, USA; Sundaresh. H .R., Shivamogga, India; Theo Koupelis, Clark College, WA, USA.

S640. Let $ABCD$ be a convex quadrilateral with incenter I and inradius r. A line passing through I intersects segments BC, CA, AD at points K, L, M, respectively. Let r_1 be the radius of a circle tangent to segments AB, AM, MK and let r_2 be the radius of a circle tangent to segments CD, CK, KM. Set $a = AL/LC$. Prove that

$$a\left(\frac{1}{r_1} - \frac{1}{r}\right) = \frac{1}{r_2} - \frac{1}{r}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Li Zhou, Polk State College, USA



Let I_1, I_2 be the centers of the circles with radii r_1, r_2, respectively. Then

$$\frac{AL \cdot MI}{LC \cdot IK} = \frac{[AMI]}{[CKI]} = \frac{AM}{CK},$$

so

$$a = \frac{AL}{LC} = \frac{AM}{IM} \cdot \frac{IK}{CK} = \frac{AI_1}{I_1I} \cdot \frac{II_2}{I_2C} = \frac{r_1}{r - r_1} \cdot \frac{r - r_2}{r_2},$$

from which the claim follows. Notice that AI_2, CI_1, IL are concurrent by Ceva's theorem.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Theo Koupelis, Clark College, WA, USA.

S641. Solve in integers the equation

$$2x^3 - 3x^2y^2 + 2y^3 = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Since the equation is equivalent to $(2x+2y-1)(x+y+1)^2 = 3(x+y+xy)^2$, we must have $2x+2y-1 = 3m^2$ and $x+y+xy = m(x+y+1)$ for some odd integer m. Then

$$x + y = \frac{3m^2+1}{2} \text{ and } xy = \frac{(m-1)(3m^2+1)}{2} + m,$$

so x, y are the roots of the quadratic equation

$$t^2 - \frac{3m^2+1}{2}t + \frac{(m-1)(3m^2+1)}{2} + m = 0,$$

the discriminant of which is $\frac{3}{4}(m-1)^2(3m^2-2m+3)$. Suppose that $m \neq 1$. Then $m = 3n$ and $9n^2-2n+1 = k^2$, where n and k are integers. Therefore, $8 = (3k+9n-1)(3k-9n+1)$, which forces $18n-2 \in \{\pm 7, \pm 2\}$, an impossibility for any odd n. Hence, $m = 1$, so $x = y = 1$. Finally, $x = y = 1$ obviously satisfies the equation.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Jean Heibig, ISAE-SUPAERO, Toulouse, France.

S642. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\le\frac{1}{92}\left(68+\frac{1}{abc}\right).$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Denote

$$f(a,b,c)=\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}-\frac{1}{92}\left(68+\frac{1}{abc}\right).$$

Without loss of generality, we may assume that $a\le b\le c$. We evaluate

$$f(a,b,c)-f\left(\frac{a+b}{2},\frac{a+b}{2},c\right)=\frac{(a-b)^2(a^2+4ab+b^2-6)}{(a^2+3)(b^2+3)((a+b)^2+12)}-\frac{(a-b)^2}{92abc(a+b)^2}$$
$$\le 0$$

because $c\ge\frac{a+b+c}{3}=1$ and $a+b=3-c\le 2$ so

$$a^2+4ab+b^2=(a+b)^2+2ab\le(a+b)^2+\frac{(a+b)^2}{2}\le 6.$$

Therefore it suffice to prove that

$$f(t,t,3-2t)\le 0,\qquad 0\le t\le 1$$

or

$$\frac{(t-1)^2(136t^5-340t^4+198t^3+12t^2-9t-9)}{92t^2(3-2t)(t^2+3)(t^2-3t+3)}\le 0$$

which is true for $t\in[0,1]$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

Undergraduate problems

U637. Let H_n be the n-th harmonic number $H_n = \sum_{k=1}^{n} 1/k$. Evaluate the following limit

$$\lim_{n\to\infty} n\left(\frac{n-H_n}{n}\right)^n.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by the author

The solution follows by taking logarithms and using the second order approximation to $\ln(1+x)$ when x tends to zero. Let L denote the proposed limit. then

$$\begin{aligned}\ln L &= \lim_{n\to\infty}\left(\ln n + n\ln\left(\frac{n-H_n}{n}\right)\right)\\ &= \lim_{n\to\infty}\left(\ln n + n\ln\left(1-\frac{H_n}{n}\right)\right)\\ &= \lim_{n\to\infty}(\ln n - H_n) = -\gamma\end{aligned}$$

where $\gamma = \lim_{n\to\infty}(H_n - \ln n)$ is the Euler-Mascheroni constant. Therefore, $L = e^{-\gamma}$.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Daniel Pascuas, Barcelona, Spain; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; G. C. Greubel, Newport News, VA, USA; Matthew Too, Brockport, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Sundaresh. H .R., Shivamogga, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U638. For $x \in \left(0, \frac{\pi}{2}\right)$, calculate:

$$\int \frac{1-\sin x}{3\sin x + 5(1+\cos x)e^x} dx$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by the author

Is considered $f : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R} \quad f(x) = \tan\frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

Formulas used:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad \text{and} \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\int \frac{1-\sin x}{3\sin x + 5(1+\cos x)e^x} dx = \int \frac{\frac{1-\sin x}{1+\cos x}}{\frac{3\sin x}{1+\cos x} + 5e^x} dx \quad \text{where} \quad \begin{cases} \frac{1-\sin x}{1+\cos x} = \frac{1+\tan^2\frac{x}{2} - 2\tan\frac{x}{2}}{2} = f'(x) - f(x) \\ \frac{\sin x}{1+\cos x} = \tan\frac{x}{2} = f(x) \end{cases}$$

So, it must be calculated

$$\int \frac{f'(x)-f(x)}{3f(x)+5e^x} dx = \frac{1}{3}\int \frac{3f'(x)-3f(x)}{3f(x)+5e^x} dx = \frac{1}{3}\int \frac{3f'(x)+5e^x}{3f(x)+5e^x} dx - \frac{1}{3}\int \frac{3f(x)+5e^x}{3f(x)+5e^x} dx =$$

$$= \frac{1}{3}\int \frac{(3f(x)+5e^x))'dx}{3f(x)+5e^x} - \frac{1}{3}\int 1dx = \frac{1}{3}\ln|3f(x)+5e^x| - \frac{1}{3}x + C$$

$$f(x) > 0)\forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \int \frac{1-\sin x}{3\sin x + 5(1+\cos x)e^x} dx = \frac{1}{3}\ln\left(3\tan\frac{x}{2} + 5e^x\right) - \frac{1}{3}x + C$$

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Adam John Frederickson, Utah Valley University, UT, USA; G. C. Greubel, Newport News, VA, USA; Matthew Too, Brockport, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sundaresh. H .R., Shivamogga, India; Theo Koupelis, Clark College, WA, USA; Ankush Kumar Parcha, NewDelhi, India.

U639. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \geq 1$, and let

$$F(a,b,c) = \frac{a+b+c}{3} - \sqrt{\frac{ab+bc+ca}{3}}.$$

Prove that

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

Proposed by Vasile Cârtoaje, Ploeşti and Vasile Mircea Popa, Sibiu, România

Solution by the authors

Since $F(a,b,c) \geq 0$ and $F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \geq 0$, it suffices to prove the homogeneous inequality

$$F(a,b,c) \geq (a^4bc)^{1/3} \cdot F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

for $a = \min\{a, b, c\}$ and without the condition $a^4bc \geq 1$. Due to homogeneity, we may set $a = 1$, hence $b, c \geq 1$. Thus, we need to show the inequality

$$F(1,b,c) \geq (bc)^{1/3} \cdot F\left(1, \frac{1}{b}, \frac{1}{c}\right),$$

which is equivalent to

$$\frac{1+b+c}{3} - \sqrt{\frac{b+c+bc}{3}} \geq (bc)^{1/3}\left(\frac{b+c+bc}{3bc} - \sqrt{\frac{1+b+c}{3bc}}\right).$$

Denote

$$s = \frac{b+c}{2}, \qquad p = \sqrt{bc},$$

with $s \geq p \geq 1$. For fixed p, the desired inequality is equivalent to $F(s) \geq 0$, where

$$F(s) = 2s + 1 - \sqrt{3(2s+p^2)} - p^{-4/3}\left[2s + p^2 - p\sqrt{3(2s+1)}\right].$$

We have

$$F'(s) = A - p^{-4/3}B,$$

where

$$A = 2 - \sqrt{\frac{3}{2s+p^2}}, \qquad B = 2 - p\sqrt{\frac{3}{2s+1}}.$$

We will show that $F'(s) \geq 0$. Since $A \geq 2 - 1 > 0$, it suffices to consider the case $B > 0$, when

$$F'(s) \geq A - B = p\sqrt{\frac{3}{2s+1}} - \sqrt{\frac{3}{2s+p^2}} \geq p\sqrt{\frac{3}{2s+1}} - \sqrt{\frac{3}{2s+1}} \geq 0.$$

From $F'(s) \geq 0$, it follows that $F(s)$ is increasing, hence $F(s) \geq F(p)$. So, we need to show that $F(p) \geq 0$, i.e.

$$2p + 1 - \sqrt{3(p^2+2p)} - p^{-1/3}\left[2 + p - \sqrt{3(2p+1)}\right] \geq 0,$$

$$\frac{(p-1)^2}{2p+1+\sqrt{3(p^2+2p)}} - \frac{p^{-1/3}(p-1)^2}{2+p+\sqrt{3(2p+1)}} \geq 0.$$

It is true if

$$\frac{p^{1/3}}{2p+1+\sqrt{3(p^2+2p)}} - \frac{1}{2+p+\sqrt{3(2p+1)}} \geq 0.$$

Substituting $p = x^3$, where $x \geq 1$, we need to prove that

$$\frac{x}{2x^3 + 1 + \sqrt{3(x^6 + 2x^3)}} - \frac{1}{2 + x^3 + \sqrt{3(2x^3 + 1)}} \geq 0,$$

i.e.

$$x^4 - 2x^3 + 2x - 1 \geq \sqrt{3}x\left(\sqrt{x^4 + 2x} - \sqrt{2x^3 + 1}\right),$$

$$(x-1)^3(x+1) \geq \frac{\sqrt{3}x(x-1)^3(x+1)}{\sqrt{x^4 + 2x} + \sqrt{2x^3 + 1}}.$$

Thus, we need to show that

$$\sqrt{x^4 + 2x} + \sqrt{2x^3 + 1} \geq \sqrt{3}x.$$

Indeed,

$$\sqrt{x^4 + 2x} + \sqrt{2x^3 + 1} - \sqrt{3}x > \sqrt{2x^3 + 1} - \sqrt{3}x = \frac{(x-1)^2(2x+1)}{\sqrt{2x^3 + 1} + \sqrt{3}x} \geq 0.$$

The equality occurs for $a = b = c \geq 1$.

Remark: The inequality is true in the particular case $a, b, c \geq 1$, which involves $a^4bc \geq 1$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U640. Let $(x_n)_{n\geq 2}$ be the sequence defined by

$$x_n = \frac{\sqrt[n]{e}-1}{\sqrt[n^2]{e}-1} - n.$$

Prove that $\lim\limits_{n\to\infty} x_n = \dfrac{1}{2}$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ştefan Marinescu, Hunedoara, România

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By the Taylor expansion for e^x, we have

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left(\frac{\sqrt[n]{e}-1}{\sqrt[n^2]{e}-1} - n\right)\\
&= \lim_{n\to\infty}\left(\frac{\frac{1}{n}+\frac{\frac{1}{n^2}}{2}+O\left(\frac{1}{n^3}\right)}{\frac{1}{n^2}+O\left(\frac{1}{n^4}\right)} - n\right)\\
&= \lim_{n\to\infty}\left(\frac{n+\frac{1}{2}+O\left(\frac{1}{n}\right)}{1+O\left(\frac{1}{n^2}\right)} - n\right)
\end{aligned}$$

from where $\lim\limits_{n\to\infty} x_n = \dfrac{1}{2}$ and the problem is done.

Also solved by Arkady Alt, San Jose, CA, USA; Michel Faleiros Martins, São Paulo, SP, Brazil; Adam John Frederickson, Utah Valley University, UT, USA; G. C. Greubel, Newport News, VA, USA; Daniel Pascuas, Barcelona, Spain; Le Hoang Bao, TienGiang, Vietnam; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Henry Ricardo, Westchester Area Math Circle; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Sundaresh. H .R., Shivamogga, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U641. Let $P(x,y,z)$ be a polynomial with rational coefficients. Prove that there is a polynomial $Q(x,y,z)$ with rational coefficients such that

$$P(x,y,z)Q(x,y,z) = R\left(x^2y, y^2z, z^2x\right),$$

for some polynomial $R(x,y,z)$ with rational coefficients.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Li Zhou, Polk State College, USA

Write $P(x,y,z) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where each a_i is a polynomial in y,z with rational coefficients. Now $P(x,y,z) = a_n(x+c_1)\cdots(x+c_n)$, where each c_i is in the algebraic closure of the field of rational functions in y,z. Let $\sigma_1,\ldots,\sigma_n$ be the elementary symmetric polynomials of $c_1,\ldots,c_n$, then each $\sigma_i = a_{n-i}/a_n$. Now let

$$F(x,y,z) = a_n^8 \prod_{i=1}^{n}\left(x^2 - c_i x + c_i^2\right)\left(x^6 - c_i^3 x^3 + c_i^6\right),$$

then the coefficients of $F(x,y,z)/a_n^8$ are symmetric functions of $c_1,\ldots,c_n$, thus are polynomials of $\sigma_1,\ldots,\sigma_n$ with rational coefficients. Furthermore,

$$P(x,y,z)F(x,y,z) = a_n^9\left(x^9 + c_1^9\right)\cdots\left(x^9 + c_n^9\right),$$

so the coefficients of $P(x,y,z)F(x,y,z)/a_n^9$ are also polynomials of $\sigma_1,\ldots,\sigma_n$ with rational coefficients. Therefore, $F(x,y,z)$ and $P(x,y,z)F(x,y,z)$ are polynomials in x and x^9, respectively, whose coefficients are polynomials in $a_0(y,z),\ldots,a_n(y,z)$ with rational coefficients.

Similarly and successively, we have a polynomial $G(x,y,z)$ in y whose coefficients are polynomials in x^9, z with rational coefficients and a polynomial $H(x,y,z)$ in z whose coefficients are polynomials in x^9, y^9 with rational coefficients, such that $P(x,y,z)F(x,y,z)G(x,y,z)H(x,y,z)$ is a polynomial in x^9, y^9, z^9 with rational coefficients. Let $u = y^2z$, $v = z^2x$, and $w = x^2y$. Then $x^9 = vw^4/u^2$, $y^9 = wu^4/v^2$, and $z^9 = uv^4/w^2$. Finally, to clear the denominators, let $Q(x,y,z)$ be a suitable power of $uvw = (xyz)^3$ times $F(x,y,z)G(x,y,z)H(x,y,z)$, then $P(x,y,z)Q(x,y,z)$ is a polynomial $R(u,v,w)$ with rational coefficients.

U642. Evaluate

$$\lim_{n\to\infty} n\sin\big((2\pi n)^p + 2^p\pi^p n^{p-1}p\big)^{\frac{1}{p}}$$

Proposed by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Italy

Solution by the author

$$\big((2\pi n)^p + p2^p\pi^p n^{p-1}\big)^{1/p} = 2\pi n\left(1+\frac{p2^p\pi^p n^{p-1}}{(2\pi n)^p}\right)^{1/p} = 2\pi n\left(1+\frac{p}{n}\right)^{1/p} =$$
$$= 2\pi n\left(1+\frac{1}{n}+\frac{1}{2p}\left(\frac{1}{p}-1\right)\frac{p^2}{n^2}+O(\frac{1}{n^3})\right) = 2\pi(n+1)+\left(\frac{1}{p}-1\right)\frac{\pi p}{n}+$$
$$+O(\frac{1}{n^2})$$

thus using $\sin x = x + O(x^3)$, $x\to 0$

$$n\sin\left(\big((2\pi n)^p + p2^p\pi^p n^{p-1}\big)^{1/p}\right) = n\sin\left(\left(\frac{1}{p}-1\right)\frac{\pi p}{n}+O(\frac{1}{n^2})\right) =$$
$$= \left(\frac{1}{p}-1\right)\pi p + O(\frac{1}{n}) \to \ \pi(1-p)$$

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Daniel Pascuas, Barcelona, Spain; Le Hoang Bao, TienGiang, Vietnam; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O637. Let a, b, c be positive real numbers such that $a+b+c=ab+bc+ca$. Prove that

$$\sqrt[3]{a^3+7}+\sqrt[3]{b^3+7}+\sqrt{c^3+7} \le 2(a+b+c).$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

First, we prove the following inequality for $t>0$:

$$\sqrt[3]{t^3+7} \le \frac{9t-7}{8}+\frac{7}{2(t+1)}$$

or

$$\left[\frac{9t-7}{8}+\frac{7}{2(t+1)}\right]^3-t^3-7 \ge 0$$

or

$$\frac{(7(t-1)^2(31t^4-88t^3+318t^2+464t+811)}{512(t+1)^3} \ge 0$$

which is true because $31t^4+318t^2 \ge 2\sqrt{31\cdot 318}t^2 \ge 88t^2$.

Therefore we need to show that

$$\frac{9(a+b+c)-21}{8}+\sum_{cyc}\frac{7}{2(a+1)} \le 2(a+b+c)$$

or

$$\sum_{cyc}\frac{4}{a+1} \le a+b+c+3.$$

Homogenizing, the inequality becomes successively,

$$\sum_{cyc}\frac{4(a+b+c)}{a(a+b+c)+ab+bc+ca} \le \frac{(a+b+c)^3}{(ab+bc+ca)^2}+\frac{3(a+b+c)}{ab+bc+ca},$$

$$\sum_{cyc}\frac{4(ab+bc+ca)}{a(a+b+c)+ab+bc+ca} \le \frac{(a+b+c)^2}{ab+bc+ca}+3,$$

$$12-\sum_{cyc}\frac{4a(a+b+c)}{a(a+b+c)+ab+bc+ca} \le \frac{(a+b+c)^2}{ab+bc+ca}+3,$$

$$\frac{a+b+c}{ab+bc+ca}+\sum_{cyc}\frac{4a}{a(a+b+c)+ab+bc+ca} \ge \frac{9}{a+b+c}.$$

But from Cauchy-Schwarz Inequality, we have

$$\sum_{cyc}\frac{a}{a(a+b+c)+ab+bc+ca} = \sum_{cyc}\frac{a^2}{a^2(a+b+c)+a(ab+bc+ca)} \ge$$

$$\ge \frac{(a+b+c)^2}{(a^2+b^2+c^2)(a+b+c)+(a+b+c)(ab+bc+ca)}$$

$$= \frac{a+b+c}{a^2+b^2+c^2+ab+bc+ca}.$$

Hence, it suffices to prove that

$$\frac{a+b+c}{ab+bc+ca}+\frac{4(a+b+c)}{a^2+b^2+c^2+ab+bc+ca}\ge\frac{9}{a+b+c},$$

$$\frac{1}{ab+bc+ca}+\frac{4}{a^2+b^2+c^2+ab+bc+ca}\ge\frac{9}{(a+b+c)^2}.$$

This is true because from Cauchy-Schwarz Inequality, we have

$$\frac{1}{ab+bc+ca}+\frac{4}{a^2+b^2+c^2+ab+bc+ca}\ge\frac{(1+2)^2}{ab+bc+ca+a^2+b^2+c^2+ab+bc+ca}=\frac{9}{(a+b+c)^2}.$$

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Jean Heibig, ISAE-SUPAERO, Toulouse, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA.

O638. Let $1 \leq a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there is a sequence $b_1, b_2, \ldots$ of positive integers with $b_i > a_i$ such that the only multiplicative function $f : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ satisfying the condition $f(b_i + b_j) = f(b_i) + f(b_j)$ is $f(n) = n$ for all n.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Li Zhou, Polk State College, USA

For each $k \in \mathbb{N}$, there is a prime $p_k > a_{2k} \geq k+1$. Let $b_{2k-1} = p_k$ and $b_{2k} = kp_k$. Clearly, $b_i > a_i$ for all $i \geq 1$. Suppose that f is such a multiplicative function satisfying the condition. For each $k \geq 1$,

$$f(p_k)f(1+k) = f\left(p_k(1+k)\right) = f(b_{2k-1} + b_{2k}) = f(b_{2k-1}) + f(b_{2k}) = f(p_k) + f(k)f(p_k),$$

so $f(1+k) = 1 + f(k)$ since $f(p_k) \neq 0$. By induction, $f(n+1) = 1 + nf(1)$ for all $n \geq 1$. Finally, for any prime p, $f(p) = f(1)f(p)$, thus $f(1) = 1$, completing the proof.

Also solved by Dion Aliu, Kosovo.

O639. Let a, b, c, λ be positive real numbers such that

$$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda}\le\frac{1}{\lambda}$$

Prove that

$$abc \ge 8\lambda^3 \text{ and } a+b+c+\frac{3abc}{ab+bc+ca}\ge 8\lambda.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Michel Faleiros Martins, São Paulo, SP, Brazil

Since all inequalities are homogeneous, we can take $\lambda = 1$. After clearing denominators, the condition yields

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\le 1 \quad\Leftrightarrow\quad abc \ge a+b+c+2.$$

By AM-GM inequality, we get

$$abc = a+b+c+2 \ge 3(abc)^{\frac{1}{3}}+2 \quad\Leftrightarrow\quad ((abc)^{\frac{1}{3}}-2)((abc)^{\frac{1}{3}}+1)^2\ge 0 \quad\Leftrightarrow\quad abc\ge 8.$$

To prove the other one, let $a+b+c=p$, $ab+bc+ca=q$, and $abc=r$. Then,

$$p+\frac{3r}{q}\ge 8 \quad\Leftrightarrow\quad pq+3r\ge 8q.$$

Thus, it suffices to show that $pq+3(p+2)=p(q+3)+6\ge 8q$. But, $p^2\ge 3q$, so that it is enough to prove that $\sqrt{3q}(q+3)\ge 8q-6$. From the well-known $q^2\ge 3pr$ and $p^3\ge 27r$, we obtain $p\ge 6$ and $q\ge 12$, using that $r=abc\ge 8$. After squaring both sides, we are left with

$$3q(3+q)^2\ge(8q-6)^2 \quad\Leftrightarrow\quad (q-12)(q-3)(3q-1)\ge 0,$$

which is obviously true.

In either case, the equality holds iff $a=b=c=2$, or equivalently, $a=b=c=2\lambda$ in the proposed inequalities.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Theo Koupelis, Clark College, WA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

O640. Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 > 1$ and $a_{n+1} = \frac{1+a_n^2}{2}$, for all $n \geq 0$. Prove that

$$\prod_{k=0}^{n} \frac{1+a_k}{a_k} \geq \left(\frac{(1+a_0)(n+1)}{(1+a_0)n+a_0} \right)^{n+1}.$$

Proposed by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Italy

Solution by Jean Heibig, ISAE-SUPAERO, Toulouse, France

$\forall n \in \mathbb{N}, a_{n+1} - a_n = (a_n-1)^2/2 \geq 0$, so (a_n) is an increasing sequence with $\forall n \in \mathbb{N}, a_n > 1$. As $\forall k \in [[1,n]], 0 < a_k < a_k + 1$, then

$$\begin{aligned} & \sum_{k=1}^{n} \frac{a_k}{a_k+1} \leq n \\ \iff & (1+a_0) \sum_{k=0}^{n} \frac{a_k}{a_k+1} \leq (1+a_0)n + a_0 \\ \iff & \frac{(1+a_0)(n+1)}{(1+a_0)n+a_0} \leq \frac{n+1}{\sum_{k=0}^{n} \frac{a_k}{a_k+1}} \end{aligned}$$

Finally, with $\forall k \in [[0,n]], (1+a_k)/a_k > 0$, using the GM-HM inequality, we get:

$$\frac{n+1}{\sum_{k=0}^{n} \frac{a_k}{a_k+1}} \leq \sqrt[n+1]{\prod_{k=0}^{n} \frac{1+a_k}{a_k}}$$

which yields the desired conclusion.

Remark: this formula works with any positive sequence.

Also solved by Arkady Alt, San Jose, CA, USA; Michel Faleiros Martins, São Paulo, SP, Brazil; Theo Koupelis, Clark College, WA, USA.

O641. Let $\{F_n\}_{n\geq 0}$, $F_0 = 0$, $F_1 = 1$ be the Fibonacci sequence. Find all triples (n, a, b) of nonnegative integers for which $F_n = p^a q^b$, where p, q are distinct primes of the form $4k + 3$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Li Zhou, Polk State College, USA

Notice that for (n, a, b), $(1, 0, 0)$ and $(2, 0, 0)$ give $F_1 = F_2 = 1$, $(4, 1, 0)$ and $(4, 0, 1)$ give $F_4 = 3^1$, and $(8, 1, 1)$ yields $F_8 = 3^1 7^1$. We show that these are the only such triples.

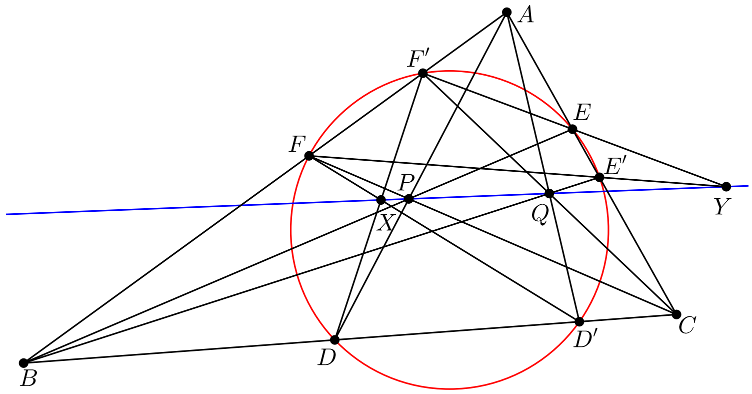
First, $F_n \equiv 0 \pmod 2$ if and only if $n \equiv 0 \pmod 3$. Therefore, assume that $n = 2^u m$ for some $u \geq 0$ and odd m not divisible by 3. If $m = 1$ and $u \geq 4$, then F_n is a multiple of $F_{16} = 3 \cdot 7 \cdot 47$. Thus, consider $m > 1$. It is known that all prime divisors of F_m are $\equiv 1 \pmod 4$. See Lemmermeyer, Franz (2000), *Reciprocity Laws: From Euler to Eisenstein*, Springer, p. 73. Since $F_m \mid F_n$, F_n has a prime divisor $\equiv 1 \pmod 4$. Therefore, n must be $1, 2, 4$, or 8.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Jean Heibig, ISAE-SUPAERO, Toulouse, France.

O642. Point P lies inside triangle ABC. Let $D = AP \cap BC$, $E = BP \cap CA$, $F = CP \cap AB$. The circumcircle of triangle DEF intersects sides BC, CA, AB for the second time at points D', E', F', respectively. Set $X = DF' \cap D'F$ and $Y = EF' \cap E'F$. prove that points X, P, Y are collinear.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Li Zhou, Polk State College, USA



By the power of a point, $AE \cdot AE' = AF \cdot AF'$, $BF \cdot BF' = BD \cdot BD'$, and $CD \cdot CD' = CE \cdot CE'$. Therefore,

$$\frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB},$$

so, by Ceva's theorem, AD', BE', CF' concur at a point Q. Applying Pappus's theorem to A, F', F on AB and C, D', D on CB, we get that X, P, Q are collinear. Likewise, Y and $Z = DE' \cap D'E$ are on PQ as well.