Junior Problems

J643. Find all positive integers n such that

$$\sqrt{\binom{12n}{n+1}+1}-\sqrt{\binom{12n}{n-1}+1}=40.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J644. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{\sqrt{a+2a^4}}+\frac{1}{\sqrt{b+2b^4}}+\frac{1}{\sqrt{c+2c^4}}\geq\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Proposed by Mircea Becheanu, Canada

J645. Let $ABCD$ be a cyclic quadrilateral and let O the intersection of the diagonals AC and BD. Let P be a point on line BD such that $\measuredangle PAD = \measuredangle CAD$ and $BP^2 = 4OC(AO + AP)$. Show that O is the midpoint of the segment BP.

Proposed by Mihaela Berindeanu, Bucharest, România

J646. Let a, b be positive integers such that $\gcd(a, b) = 1$. Prove that $\gcd(a^2 + b^2, a^3 + b^3) \in \{1, 2\}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J647. Let a, b, c, p, q, r be positive real numbers such that $q + r \geq 2p$ and $a + b + c = p + q + r$. Prove that

$$\frac{a}{\sqrt{pa+qb+rc}}+\frac{b}{\sqrt{pb+qc+ra}}+\frac{c}{\sqrt{pc+qa+rb}}\geq\sqrt{3}$$

Proposed by Mircea Becheanu, Canada

J648. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$a+b+c+3abc\geq\frac{4}{a+b+c}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

Senior Problems

S643. Let $n \geq 2$ and $P(x) = x^{2n+1} + a_1 x^{2n} + \cdots + a_{2n+1}$ be a polynomial with real coefficients satisfying

$$a_2 + a_4 + \cdots + a_{2n} = \frac{3(3^{2n} - 1)}{2}.$$

Let $x_1, x_2, \ldots, x_{2n+1}$ be the roots of $P(x)$. Given that

$$(x_1 - 1)(x_2 + 1)(x_3 - 1)(x_4 + 1) \cdots (x_{2n+1} - 1) = 3^n \text{ and}$$
$$(x_1 + 1)(x_2 - 1)(x_3 + 1)(x_4 - 1) \cdots (x_{2n+1} + 1) = 3^{n+1},$$

evaluate $a_1 + a_3 + \cdots + a_{2n+1}$. Give an example of such a polynomial $P(x)$ in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S644. Let $a \geq b \geq c \geq d \geq e \geq 0$ such that $ab + bc + cd + de + ea = 5$. Prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 + 5(a + b + c + d + e) \geq 30.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

S645. Let ABC be a triangle with sidelengths $BC = a, CA = b, AB = c$, inradius r, circumradius R, and area K. Let A', B', C' be the tangency points of the incircle of the triangle with sides BC, CA, AB, respectively.

(i) Prove that one can construct a triangle Δ with sidelengths $a \cdot AA', b \cdot BB', c \cdot CC'$.

(ii) Let K' denotes the area of Δ. Prove that

$$9r^2 \leq \frac{K'}{K} \leq \frac{9}{4}R^2.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

S646. Let a, b, c be positive real numbers such that $ab + bc + ca = 2(a + b + c)$. Prove that

$$\frac{a}{b^2 + 4} + \frac{b}{c^2 + 4} + \frac{c}{a^2 + 4} \geq \frac{3}{4}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S647. Let ABC be an acute triangle with circumcircle Γ. The tangents at B and C to Γ intersect at X. Line AX intersects the circle Γ a second time at Y. Let M be the midpoint of side BC. Line YM and the perpendicular from A to BC intersect at D. Prove that $\triangle ABC \equiv \triangle DBC$.

Proposed by Mihaela Berindeanu, Bucharest, România

S648. Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 4$. Prove that

$$\frac{a}{a^2 + 4} + \frac{b}{b^2 + 4} + \frac{c}{c^2 + 4} + \frac{d}{d^2 + 4} \leq \frac{1}{5} + \frac{ab + ac + ad + bc + bd + cd}{10}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U643. Consider the sequence $(x_n)_{n\geq 2}$ defined by

$$x_n = \frac{\sqrt[n]{e}-1}{\sqrt[n^2]{e}-1} - n.$$

Evaluate $\lim\limits_{n\to\infty} n\left(x_n - \frac{1}{2}\right)$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ştefan Marinescu, Hunedoara, România

U644. Let $f\colon [-1,1] \to \mathbb{R}$ be a smooth function of order three such that $f(-1) = f(1) = f''(-1) = 0$. Prove that

$$\left(\int_{-1}^{1} f(x)dx\right)^2 \leq \frac{104}{315}\int_{-1}^{1}\left(f'''(x)\right)^2 dx$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

U645. Find all non-zero polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P\left((Q(x))^3\right) = x^2 P(x)(Q(x))^2, \ \forall x \in \mathbb{R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U646. Let $f : [0,1] \longrightarrow [0,1)$ be an integrable function. Prove that

$$\lim_{n\to\infty}\int_0^1 f(x)^n dx = 0.$$

Proposed by Mihai Piticari and Sorin Rădulescu, România

U647. Let $(p_k)_{k\geq 1}$ be the sequence of primes and let $q_n = \prod\limits_{k\leq n} p_k$. Evaluate

$$\lim_{n\to\infty}\frac{\sum\limits_{p|q_n}(\log p)^\alpha}{\omega(q_n)^{1-\alpha}(\log q_n)^\alpha},$$

where $\alpha \in (0,1)$ and $\omega(n)$ denotes the number of distinct primes of the positive integer n.

Proposed by Alessandro Ventulo, Milan, Italy

U648. Let T_n be a sequence defined by $T_1 = a$,

$$T_{n+1} = \frac{T_n^2}{2\sqrt{1+T_n^2}}, \ n \geq 1.$$

Evaluate $\prod\limits_{n=1}^{\infty}\frac{T_n^2+2}{2T_n^2+2}$.

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

Olympiad Problems

O643. Let a, b, c, λ be positive real numbers such that

$$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda} \le \frac{1}{\lambda}.$$

Prove that

$$a+b+c+\frac{15abc}{ab+bc+ca} \ge 16\lambda.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

O644. Prove that $k = 4$ is the least positive constant k such that

$$\left(\frac{ka_1+a_2+\cdots+a_9}{k+8}\right)^2 \ge \frac{a_1a_2+a_2a_3+\cdots+a_9a_1}{9}$$

whenever $a_1 \ge a_2 \ge \cdots \ge a_9 \ge 0$.

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

O645. Find all completely multiplicative functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(a^2+b^2+c^2) = f(ab+bc+ca-2)$$

for all positive integers a, b, c.

Proposed by Titu Andreescu, USA and Vlad Matei, România

O646. Let x, y, z be non-zero real numbers such that $x + y + z = xyz$. Prove that

$$\left|x+y+z-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right| \ge 2\sqrt{3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O647. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{a^2+3}+\frac{b}{b^2+3}+\frac{c}{c^2+3} \le \frac{1}{4}+\frac{ab+bc+ca}{6}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

O648. Let $A(x), B(x)$ be polynomials with integer coefficients for which there are polynomials $P_1(x), Q_1(x)$ with integer oefficients such that

$$P_1(x)A(x) + Q_1(x)B(x) = 1.$$

Prove that for each positive integer n there are integer polynomials $P_n(x), Q_n(x)$ such that

$$P_n(x)A(x)^n + Q_n(x)B(x)^n = 1.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran