The Best Constant for an Inequality from Mathematical Reflections

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In Mathematical Reflections 1 (2018), Alessandro Ventullo proposed the following inequality:

S435. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + \frac{8}{(a+b)(b+c)(c+a)} \geq 4.$$

Here is one possible solution:

Starting from the inequality

$$x^3 + y^3 \geq xy(x+y) \iff (x+y)(x-y)^2 \geq 0$$

for all $x, y > 0$, we can write

$$\frac{a^3+b^3}{2} \geq \frac{ab(a+b)}{2} = \frac{a+b}{2c},$$

and similarly

$$\frac{b^3+c^3}{2} \geq \frac{b+c}{2a}, \quad \frac{c^3+a^3}{2} \geq \frac{c+a}{2b}.$$

Summing up these last 3 inequalities and applying the AM-GM Inequality, we get

$$\begin{aligned} a^3 + b^3 + c^3 + \frac{8}{(a+b)(b+c)(c+a)} &\geq \frac{a+b}{2c} + \frac{b+c}{2a} + \frac{c+a}{2b} + \frac{8}{(a+b)(b+c)(c+a)} \\ &\geq 4\sqrt[4]{\frac{a+b}{2c} \cdot \frac{b+c}{2a} \cdot \frac{c+a}{2b} \cdot \frac{8}{(a+b)(b+c)(c+a)}} \\ &= 4. \end{aligned}$$

□

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In the same year, Titu Zvonaru proposed in *Revista de Matematica din Timisoara* 2 (2018) a stronger form of this inequality.

OBJ.139 Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + \frac{64}{(a+b)(b+c)(c+a)} \geq 11.$$

Solution: The expression $a^3 + b^3 + c^3$ suggests appealing to Schur's Inequality i.e.

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

or

$$a^3 + b^3 + c^3 + 5abc \geq (a+b)(b+c)(c+a).$$

Using this inequality and AM-GM Inequality, we have

$$\begin{aligned}a^3+b^3+c^3+\frac{64}{(a+b)(b+c)(c+a)} &= a^3+b^3+c^3+5abc+\frac{64}{(a+b)(b+c)(c+a)}-5\\ &\geq (a+b)(b+c)(c+a)+\frac{64}{(a+b)(b+c)(c+a)}-5\\ &\geq 2\sqrt{(a+b)(b+c)(c+a)\cdot\frac{64}{(a+b)(b+c)(c+a)}}-5\\ &= 11.\end{aligned}$$

□

In both inequalities, the equality holds when $a=b=c=1$.

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The problem we set ourselves is what is the best constant k so that:

$$a^3+b^3+c^3+\frac{8k}{(a+b)(b+c)(c+a)} \geq 3+k, \tag{1}$$

for all $a,b,c>0$ such that $abc=1$.

In (1) we take $a=b=t>0$, $c=\dfrac{1}{t^2}$ so we have

$$2t^3+\frac{1}{t^6}+\frac{4kt^3}{(t^3+1)^2} \geq 3+k,$$

or

$$(t-1)^2\left(\frac{1}{t^6}+2t^3+\frac{4}{t^3}+5-k\right) \geq 0.$$

This is true for all $t>0$ if and only if

$$k \leq \inf_{t>0}\left(\frac{1}{t^6}+2t^3+\frac{4}{t^3}+5\right).$$

Consider the function $f:(0,\infty)\mapsto\mathbb{R}$ defined as

$$f(x)=\frac{1}{x^2}+2x+\frac{4}{x}+5.$$

Since

$$f'(x)=\frac{2(x^3-2x-1)}{x^3}=\frac{2(x+1)(x^2-x-1)}{x^3},$$

we conclude that f is a decreasing function on $\left(0,\dfrac{1+\sqrt{5}}{2}\right)$ and a increasing function on $\left(\dfrac{1+\sqrt{5}}{2},\infty\right)$. Therefore

$$f(x) \geq f\left(\frac{1+\sqrt{5}}{2}\right)=\frac{11+5\sqrt{5}}{2}$$

for all $x>0$ which means $k \leq \dfrac{11+5\sqrt{5}}{2}$.

Next, we will show that the inequality holds for $k=\dfrac{11+5\sqrt{5}}{2}\approx 11.0901699$ which would mean that this value of k is the best. We have the following statement:

Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$a^3+b^3+c^3+\frac{4(11+5\sqrt{5})}{(a+b)(b+c)(c+a)}\geq\frac{17+5\sqrt{5}}{2}.$$

When does equality hold?

Solution. To prove the inequality, we will use the SOS-Schur technique and the method of undetermined coefficients (see [2]). Without loss of generality suppose that $a\geq b\geq c$. In homogeneous form, the inequality can be written as follows

$$\frac{a^3+b^3+c^3-3abc}{abc}\geq\frac{(11+5\sqrt{5})\left[(a+b)(b+c)(c+a)-8abc\right]}{2(a+b)(b+c)(c+a)},$$

$$\frac{(a+b+c)(X+Y)}{abc}\geq\frac{(11+5\sqrt{5})\left[2cX+(a+b)Y\right]}{2(a+b)(b+c)(c+a)},$$

where we denote $X=(a-b)^2\geq 0$, $Y=(a-c)(b-c)\geq 0$. To prove this inequality, because $a+b\geq 2c$, it is enough to show that

$$2(a+b+c)(a+c)(b+c)\geq(11+5\sqrt{5})abc.$$

We try using the Weighted AM-GM Inequality. Let $\alpha,\beta>0$ such that:

$$a+b+c=\alpha\cdot\left(\frac{a}{\alpha}\right)+\alpha\cdot\left(\frac{b}{\alpha}\right)+c\geq(2\alpha+1)\left(\frac{a}{\alpha}\right)^{\frac{\alpha}{2\alpha+1}}\left(\frac{b}{\alpha}\right)^{\frac{\alpha}{2\alpha+1}}c^{\frac{1}{2\alpha+1}},$$

$$a+c=\beta\cdot\left(\frac{a}{\beta}\right)+c\geq(\beta+1)\left(\frac{a}{\beta}\right)^{\frac{\beta}{\beta+1}}c^{\frac{1}{\beta+1}},$$

$$b+c=\beta\cdot\left(\frac{b}{\beta}\right)+c\geq(\beta+1)\left(\frac{b}{\beta}\right)^{\frac{\beta}{\beta+1}}c^{\frac{1}{\beta+1}}.$$

We set the condition

$$\begin{aligned}\tfrac{\alpha}{2\alpha+1}+\tfrac{\beta}{\beta+1}&=1\\ \tfrac{1}{2\alpha+1}+\tfrac{2}{\beta+1}&=1.\end{aligned}$$

that passes on $\alpha\beta=\alpha+1$. Let's choose $\alpha=\beta$ so $\alpha^2-\alpha-1=0$ equation that has a positive root $\alpha=\frac{1+\sqrt{5}}{2}$. Therefore

$$\begin{aligned}2(a+b+c)(a+c)(b+c)\geq&\frac{2(2\alpha+1)(\alpha+1)^2}{\alpha^{\frac{2\alpha}{2\alpha+1}+\frac{2\alpha}{\alpha+1}}}abc\\ =&\frac{2(2\alpha+1)(\alpha+1)^2}{\alpha^2}abc\\ =&\frac{2(2\alpha+1)(\alpha+1)^2}{\alpha+1}abc\\ =&2(5\alpha+3)abc=(11+5\sqrt{5})abc\end{aligned}$$

that is exactly what we desired. The equality holds when $a=b=c$ and $a=b=\left(\frac{1+\sqrt{5}}{2}\right)c$. □

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An extension to 4 variables of this inequality would look like this:

$$a^4+b^4+c^4+d^4+\frac{16k}{(a+b)(b+c)(c+d)(d+a)}\geq 4+k, \tag{2}$$

for all $a,b,c,d>0$ such that $abcd=1$.

Remarks. 1) For $k = 4$ the inequality results easily. By the Power Mean Inequalities, we have

$$\left(\frac{a^4+b^4+c^4+d^4}{4}\right)^{\frac{1}{4}} \geq \frac{a+b+c+d}{4}.$$

Also, by the AM-GM Inequality

$$(a+b)(b+c)(c+d)(d+a) \leq \left(\frac{a+b+b+c+c+d+d+a}{4}\right)^4 = \frac{(a+b+c+d)^4}{16}.$$

Finally, using this and by the AM-GM Inequality

$$\begin{aligned} a^4+b^4+c^4+d^4+\frac{16}{(a+b)(b+c)(c+d)(d+a)} \geq & \frac{(a+b+c+d)^4}{4^3}+\frac{4^5}{(a+b+c+d)^4} \\ \geq & 2\sqrt{4^2} = 8. \end{aligned}$$

The equality holds when $a = b = c = d$.

2) The inequality also occurs for $k = 7$. First we prove the following inequality:
(Tran Le Bach-Vasile Cîrtoaje) If $a, b, c, d > 0$ then we have

$$a^4+b^4+c^4+d^4+8abcd \geq \sum_{cyc} abc(a+b+c).$$

Solution. Without loss of generality suppose that $d = \min\{a, b, c, d\}$and let $a = d + x$, $b = d + y$, $c = d + z$, where $x, y, z \geq 0$. After a few calculations, not very easy, we get the following inequality

$$\left(3\sum_{cyc} x^2 - 2\sum_{cyc} xy\right) d^2 + 2\left(2\sum_{cyc} x^3 - \sum_{cyc} xy(x+y)\right) d + \sum_{cyc} x^4 - xyz(x+y+z) \geq 0,$$

clearly true because

$$\sum_{cyc} x^2 \geq \sum_{cyc} xy,$$

$$2\sum_{cyc} x^3 - \sum_{cyc} xy(x+y) = \sum_{cyc}(x+y)(x-y)^2 \geq 0,$$

$$\sum_{cyc} x^4 - xyz(x+y+z) = \frac{1}{2}\sum_{cyc}(x^2-y^2)^2 + \frac{1}{2}\sum_{cyc} z^2(x-y)^2 \geq 0.$$

□

Homogenizing the inequality (2) for $k = 7$, we get

$$\frac{a^4+b^4+c^4+d^4}{abcd} + \frac{112abcd}{(a+b)(b+c)(c+d)(d+a)} \geq 11. \tag{3}$$

Now, we can normalize the inequalities with $a + b + c + d = 4$. Using

$$(a+b)(b+c)(c+d)(d+a) \leq \left(\frac{a+b+b+c+c+d+d+a}{4}\right)^4 = 16$$

and

$$\frac{a^4+b^4+c^4+d^4}{abcd} \geq \frac{(a+b+c+d)(abc+bcd+cda+dab)}{abcd} - 12 = 4\sum_{cyc}\frac{1}{a} - 12$$

to prove (3) it suffices to show that

$$4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)+7abcd\geq 23.$$

We prove this inequality using stronger Mixing Variable Method(see [3]). Note that the inequality is symmetric, so without loss of generality, we may assume $a\geq b\geq c\geq d$. Denote by

$$f(a,b,c,d)=4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)+7abcd-23.$$

We have

$$\begin{aligned}f(a,b,c,d)-&f\left(a,\frac{b+d}{2},c,\frac{b+d}{2}\right)\\&=4\left(\frac{1}{b}+\frac{1}{d}-\frac{4}{b+d}\right)+7ac\left[bd-\frac{(b+d)^2}{4}\right]\\&=\frac{(b-d)^2\left[16-7abcd(b+d)\right]}{4bd(b+d)}\geq 0,\end{aligned}$$

because $abcd\leq\left(\dfrac{a+b+c+d}{4}\right)^4=1$ and $b+d\leq\dfrac{a+b+c+d}{2}=2$. Hence, according to stronger Mixing Variable Method, we only need to consider the inequality in case $a=4-3x$, $b=c=d=x\leq 1$. In this case, the problem becomes

$$4\left(\frac{1}{4-3x}+\frac{3}{x}\right)+7x^3(4-3x)\geq 23,$$

or

$$\frac{(x-1)^2(63x^4-42x^3-35x^2-28x+48)}{x(4-3x)}\geq 0$$

which is true for $0<x\leq 1$.

3) We leave it to the readers to find better k constants, possibly the best k.

References

[1] Titu Andreescu, Marius Stănean, *116 Algebraic Inequalities from the AwesomeMath Year-Round Program*, 2018.

[2] Titu Andreescu, Marius Stănean, *118 Inequalities for Mathematics Competitions*, 2019.

[3] Titu Andreescu, Marius Stănean, *New, Newer, and Newest Inequalities*, 2021.