An Asymptotic Analysis on the Distribution of Primes via a Continuous Extension of the Factorial Function

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Abstract

We discuss some introductory ideas from analytic number theory in a way that is accessible to the high-school student. In particular, we present an intuitive derivation of the Gamma function and subsequently derive Stirling's Approximation. We then apply these methods to demonstrate the truth of Bertrand's Postulate as proven by Ramanujan in 1919.

1 Introduction

It can be noticed that a spectacular variety of problems and theorems in number theory arise from creative manipulation of just the axioms of arithmetic; one does not need to look much farther than recent olympiad problems to see this. Problems and theorems relating to the distribution of primes—on the other hand—paint a very different picture. Besides Euclid's proof that there are infinitely many primes, it is relatively difficult to derive further properties regarding the distribution of primes from just the axioms of arithmetic. One seemingly-simple, but very difficult, problem about the distribution of primes is as follows:

Theorem 1 (Bertrand's Postulate)**.** *For every real number $x \geq 1$, there exists at least one prime number p such that $x \leq p \leq 2x$.*

In this paper, we survey the methods behind Ramanujan's brilliant two-page proof of Theorem 1. In particular, we build up key results of analytic number theory mostly from single-variable calculus, including an intuitive derivation of a real-valued factorial function, Stirling's approximation, and a key theorem to connect prime numbers with the factorial function. The paper will conclude with an annotation of Ramanujan's proof.

A Brief Note on Notation

We use $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ to denote the sets of natural numbers (integers greater than 0), integers, and real numbers respectively. We let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and $\{x\} \equiv x - \lfloor x \rfloor$. We further use $\log(x)$ to denote the base-e (natural) logarithm. Finally, the symbol $\sim$, used here as $f(x) \sim g(x)$, means "asymptotically equivalent to" which is shorthand for stating $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ [2].

2 Extending the Factorial

The integer-valued factorial function, denoted $n!$, is recursively defined as $n! \equiv n \cdot (n-1)!$ with base case $0! = 1$. When plotting the integer-valued factorial in Figure 1, it appears as if the factorial function could fit on a continuous curve. This gives the possibility that we can extend the standard integer-valued factorial function to a (continuous) real-valued one.

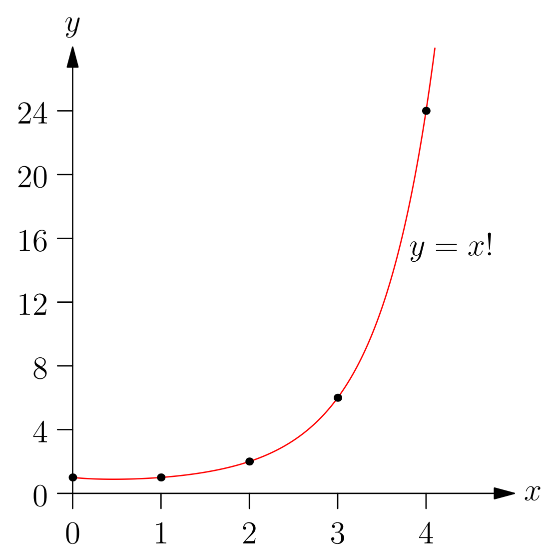

Figure 1: An extension of the factorial to $\mathbb{R}$

Naturally, calculus has the power to extend a discrete function to a continuous curve. Notice that the recursive definition of the factorial function is analogous to taking the derivative of a function, $f(x)$, raised to the k^{th} power with respect to x; i.e., if we denote $g(x,k) \equiv (f(x))^k$ then

$$\frac{d}{dx}(f(x))^k = k \cdot (f(x))^{k-1} \cdot \frac{d(f(x))}{dx} \tag{1}$$

$$= (k \cdot g(x, k-1)) \cdot f'(x) \tag{2}$$

The $k \cdot g(x, k-1)$ factor in (2) resembles the factorial's recursive definition. Hence, we can try multiplying $(f(x))^k$ by another function, say $h(x)$, to cancel the $f'(x)$ term. Then we get:

$$\frac{d(h(x)(f(x))^k)}{dx} = k \cdot (f(x))^{k-1} \cdot f'(x) \cdot h(x) + (f(x))^k h'(x) \tag{3}$$

$$= k \cdot (f(x))^{k-1} + (f(x))^k h'(x) \tag{4}$$

Observe then that we need $(f(x))^k h'(x)$ to disappear (or equal 0), which does not appear possible via the setup above (as it would imply $h'(x) = 0 \Longrightarrow h(x) =$ constant from which it is unclear how to proceed).

But this gives us an idea. Above, we still managed, via differentiation of a product, to obtain the quantity $k \cdot (f(x))^{k-1}$ which is of a similar form as the recursion we need for the factorial, albeit there was an extra term. Instead of differentiating, what if we integrated the same product? Differentiation of a product and integration of a product (or, integration by parts) both result in very similar expressions. Integrating $(f(x))^k$ yields:

$$\int (f(x))^k \, dx = x(f(x))^k - k \int x(f(x))^{k-1} \cdot f'(x) \, dx \tag{5}$$

$$= x(f(x))^k + k \int (f(x))^{k-1} \cdot (-x \cdot f'(x)) \, dx \tag{6}$$

In the integrand of (6), we want $-x \cdot f'(x)$ to equal 1; that way, we would get the familiar expression $k \int f(x)^{k-1} \, dx$ which then resembles the recursive definition of the factorial. We hence set $f(x) = -\log(x) \Longrightarrow f'(x) = -\frac{1}{x}$ so that the integration now reads:

$$\int (-\log(x))^k \, dx = x(-\log(x))^k + k \int (-\log(x))^{k-1} \, dx \tag{7}$$

We want only the $k \cdot \int (-\log(x))^{k-1} \, dx$ term and for the $x(-\log(x))^k$ term to reduce to 0. Observe $x(-\log(x))^k$ has a zero at $x = 1$. Furthermore, by L'Hôpital's rule,

$$\lim_{x \to 0^+} x \cdot (-\log(x))^k = \lim_{x \to 0^+} \frac{(\log(\frac{1}{x}))^k}{\frac{1}{x}} \tag{8}$$

$$= \lim_{u \to \infty} \frac{(\log(u))^k}{u} \tag{9}$$

$$= \lim_{u \to \infty} k \cdot \frac{(\log(u))^{k-1}}{u} \tag{10}$$

$$\vdots$$

$$= \lim_{u \to \infty} k! \cdot \frac{\log(u)}{u} \tag{11}$$

$$= \lim_{u \to \infty} k! \cdot \frac{1}{u} \tag{12}$$

$$= 0 \tag{13}$$

where we looked at the limit as $x \to 0^+$ as $-\log(x)$ is not well-defined for non-positive x. Thus, if we set our limits of integration to go from 0 to 1, the $x(-\log(x))^k$ reduces to 0:

$$\int_0^1 (-\log(x))^k \, dx = \Big[x(-\log(x))^k\Big]_0^1 + k \int_0^1 (-\log(x))^{k-1} \, dx \tag{14}$$

$$= k \int_0^1 (-\log(x))^{k-1} \, dx \tag{15}$$

Hence, the function:

$$\Pi(k) \equiv \int_0^1 (-\log(x))^k \, dx \tag{16}$$

has the property that $\Pi(k) = k \cdot \Pi(k-1)$. Furthermore, observe that $\Pi(0)$ is

$$\Pi(0) = \int_0^1 (-\log(x))^0 \, dx = \lim_{b \to 0^+} \int_b^1 \, dx = 1 \tag{17}$$

Thus, we have found a continuous function that is equal to $k!$ for $k \in \mathbb{N}$.

We rewrite $\Pi(x)$ to remove the logarithm by substituting $u = \log(\frac{1}{x}) = -\log(x) \implies x = e^{-u}$ so $\frac{dx}{du} = -e^{-u}$:

$$\int_0^1 (-\log(x))^k \, dx = \int_\infty^0 u^k \cdot (-e^{-u} \, du) \tag{18}$$

$$= \int_0^\infty u^k e^{-u} \, du \tag{19}$$

Thus, we can alternately define $\Pi(k) \equiv \int_0^\infty u^k e^{-u} \, du$.

Remark 1 (Relation to the Gamma function)*.* The above formulation is, according to [5], Gauss' formulation of the Gamma function. The Gamma function has integrand $u^{k-1}e^{-u}$ and consequently satisfies $\Gamma(k) = (k-1)!$ (as opposed to $\Pi(k) = k!$).

3 Stirling's Approximation

Now that we have extended the factorial to $\mathbb{R}$, we can form an approximation that converges to $\Pi(k)$ asymptotically. The following approximation, combined with number-theoretic properties of the factorial, may prove helpful in deducing the asymptotic properties of the distribution of primes.

The integrand $u^k e^{-u}$ is rather unwieldy due to the product involved. Rather, we can convert it into a single exponential: $u^k e^{-u} = e^{k\log(u)-u}$. Since we are interested in an asymptotic formula for large k, we are motivated to express the integrand in the form $e^{kh(x)}$ for some arbitrary function $h(x)$. Hence, we use the substitution $u = kx \implies \frac{du}{dx} = k$.

$$\begin{aligned}\Pi(k) &= \int_0^\infty (kx)^k e^{-kx} \cdot (k\,dx) \\ &= k^{k+1} \cdot \int_0^\infty x^k e^{-kx}\,dx \\ &= k^{k+1} \cdot \int_0^\infty e^{k(\log(x)-x)}\,dx \qquad (20)\end{aligned}$$

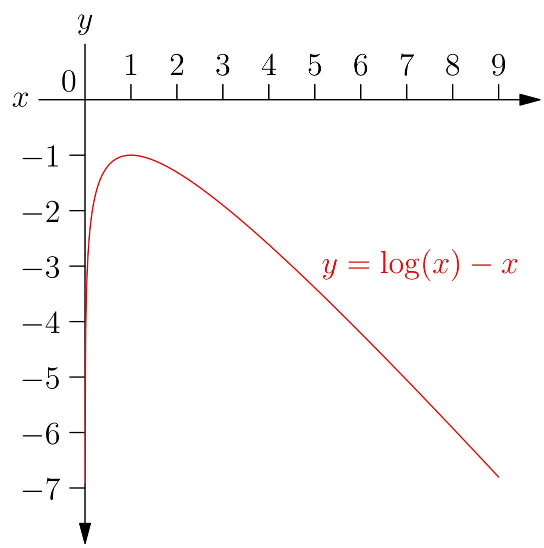

Figure 2: Plot of $\log(x) - x$

We are hence interested in $h(x) \equiv \log(x) - x$. Evidently, $h'(x) = x^{-1} - 1$ and $h''(x) = -x^{-2}$. Observe $h(x)$ is continuous on the interval $(0,\infty)$ and that $h'(x) = 0$ only at $x = 1$. Furthermore, as $h''(1) < 0$, it follows that $h(1) = -1$ is an absolute maximum.

Now observe that because $h(x)$ is strictly increasing prior to $x = 1$ and $h(x)$ is strictly decreasing after $x = 1$, most contributions to (20) come from values of x close to 1, especially as $k \to \infty$. Thus, we break the integral into:

$$S \equiv \int_0^{1-\delta} e^{k(\log(x)-x)}\,dx + \int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\,dx + \int_{1+\delta}^{\infty} e^{k(\log(x)-x)}\,dx \qquad (21)$$

$$= \int_0^\infty e^{k(\log(x)-x)}\,dx \qquad (22)$$

for $0 \leq \delta < 1$.

When we state "contributions" we rather mean that

$$S \approx \int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx, \tag{23}$$

with respect to k which follows from a brief graphical analysis of the integrand.

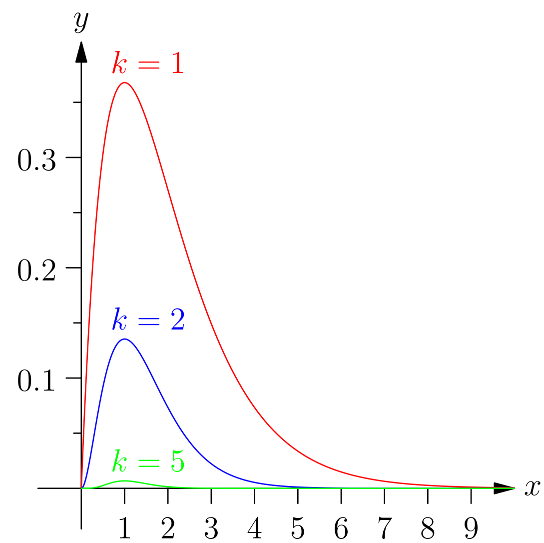

Figure 3: Plots of $y = e^{k(\log(x)-x)}$ for $k = 1, 2, 5$

Viewing the integral S as the area underneath the displayed curves, portions of the graphs between 0 and 1 and from 1 to infinity seem to contribute very little to the value of the integral as k increases. Thus, as $k \to \infty$, (23) seems to intuitively hold, because the "tails" of (23) converge too fast to 0 and are thus negligible ("the tails" are the integrals with bounds $0 \to 1-\delta$ and $1+\delta \to \infty$).

In order to rigorously proceed, we must be careful in the way we deal with limits; our **claim** in (23) is that as $k \to \infty$, S becomes dominated by $\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx$. Thus, we mean to look at the limit as $k \to \infty$ first, gain an approximation, then see what happens in smaller and smaller intervals $(1-\delta, 1+\delta)$.

Now, on the interval $(0, 1-\delta)$, let the least upper bound of $e^{\log(x)-x}$ be $\alpha \in \mathbb{R}$ (i.e., the least real number that $e^{\log(x)-x}$ is always less than on the interval $(0, 1-\delta)$ is α). Then,

$$\int_{0}^{1-\delta} e^{k(\log(x)-x)}\, dx < \int_{0}^{1-\delta} \alpha^k\, dx = B_\alpha \cdot \alpha^k \tag{24}$$

for some positive real number B_α which implicitly depends on δ since as $\delta \to 0$, $B_\alpha \to 1$. Further note that $\alpha < e^{-1}$ since $\log(x) - x$ attains a maximum value of -1; hence, $\alpha^k < e^{-k}$.

We can do something similar with the other tail on $(1+\delta, \infty)$. Let the least upper bound of $e^{\log(x)-x}$ be $\beta < e^{-1}$. Unlike the previous tail, this tail is bounded by ∞ but similar reasoning applies:

$$\int_{1+\delta}^{\infty} e^{k(\log(x)-x)}\,dx < \int_{1+\delta}^{\infty} \beta^k\,dx = \lim_{c\to\infty}\int_{1+\delta}^{c} \beta^k\,dx = \lim_{c\to\infty}(c-1-\delta)\beta^k \tag{25}$$

Now, we have to consider $\lim\limits_{k\to\infty}\lim\limits_{c\to\infty}(c-1-\delta)\beta^k$ which we immediately see as ∞ since we first consider the limit as $c \to \infty$, *then* the limit as $k \to \infty$. Hence, this naïve approach doesn't create any useful bounds as we simply obtain that this integral is bounded by positive infinity. We do, however, expect that the above integral can be bounded by β^k since the $c-1-\delta$ term is linear and β^k is exponential. So instead consider

$$\begin{aligned}\int_{1+\delta}^{\infty} e^{k(\log(x)-x)}\,dx < \beta^{k-1}\int_{1+\delta}^{\infty} e^{\log(x)-x}\,dx &= \beta^{k-1}\cdot\left(-xe^{-x}-e^{-x}\right)\Big|_{1+\delta}^{\infty}\\ &= \beta^{k-1}\cdot\left((1+\delta)e^{-(1+\delta)}+e^{-(1+\delta)}\right)\end{aligned} \tag{26}$$

Hence $\int_{1+\delta}^{\infty} e^{\log(x)-x}\,dx$ is finite (and constant with respect to k), implying that we can write

$$\int_{1+\delta}^{\infty} e^{k(\log(x)-x)}\,dx < B_\beta \cdot \beta^k \tag{27}$$

for some constant B_β that implicitly depends on δ (From (26), $B_\beta = \frac{(2+\delta)e^{-(1+\delta)}}{\beta}$. Furthermore, as $\delta \to 0$, $\beta \to e^{-1}$ since β is the least upper bound of $e^{\log(x)-x}$ for $x \in [1+\delta, \infty)$. Hence, $B_\beta \to 2$ as $\delta \to 0$).

Naturally, we note that $\alpha < e^{\log(x)-x}$ for all $x \in [1-\delta, 1]$ and $\beta < e^{\log(x)-x}$ for all $x \in [1, 1+\delta]$. Intuitively then, the tail integrals are much smaller than $\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\,dx$ for large k. It is possible to use the Mean Value Theorem and the aforementioned bounds to prove that as $k \to \infty$, the ratio of each tail to $\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\,dx$ approaches 0; however, this approach requires quite delicate and complicated analysis because we have to demonstrate properties regarding the mean value of $\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\,dx$, which constantly varies with respect to k, in relation to each tail. Rather, it suffices to simply find an expression that is asymptotically equivalent to $\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\,dx$, and then show that the tails are asymptotically negligible to this expression. We will proceed in the latter fashion.

To continue our efforts at approximating, we now turn to the 2^{nd} order Taylor Series expansion of $\log(x) - x$ at $x = 1$ since the Taylor Series expansion converges to $\log(x) - x$ as $\delta \to 0$. We choose to approximate by the 2^{nd} order since it matches the concavity of $\log(x) - x$ in the interval $(1 - \delta, 1 + \delta)$ as $x = 1$ is an extrema. By Taylor's theorem (Lagrange's form), for each $x \in [1 - \delta, 1 + \delta]$, there exists $z \in [1 - \delta, 1 + \delta]$ such that:

$$\log(x) - x = -1 - \frac{1}{2z^2}(x-1)^2 \tag{28}$$

Observe that the linear term disappears from the Taylor Series expansion since 1 is an absolute maximum of $\log(x) - x$ and that $-\frac{1}{z^2}$ is the second derivative of $\log(x) - x$ evaluated at z. Further note that as $\delta \to 0$, $z \to 1$.

To complete the approximation, we must analyze the effect of the Taylor Series approximation. We first show a nontrivial result, and then apply that to our approximation:

Lemma 1. $\displaystyle\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

Proof.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dy\, dx = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 \tag{29}$$

where the last step follows from the fact that, in the 3D space, we may split the integral completely into its x and y components. When we do this, the integral is composed of two identical parts, one in terms of x and the other in terms of y. Hence, (29) follows. Moreover, $e^{-x^2}e^{-y^2} = e^{-(x^2+y^2)}$ reminds us of the equation of a circle in the Cartesian plane: $x^2 + y^2 = r^2$. Thus, we express (29) in polar form:

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 &= \int_0^{\infty}\int_0^{2\pi} r \cdot e^{-r^2}\, d\theta\, dr && (30)\\
&= 2\pi \int_0^{\infty} r \cdot e^{-r^2}\, dr && (31)\\
&= 2\pi \cdot -\frac{1}{2} \cdot \left(e^{-\infty} - 1\right) && (32)\\
&= \pi && (33)
\end{aligned}$$

where the step in (30) follows from a conversion to a polar coordinate system and the extra factor of r is the conversion factor from switching to polar coordinates. Lemma 1 follows by taking the square-root of both sides of (30). □

Now, recall the definition of z in (28) and note that z is solely dependent on x and does not depend on k. Furthermore, let $z_{\min}$ and $z_{\max}$ be the minimum and maximum value of z respectively for $x \in [1-\delta, 1+\delta]$. Hence, from (28),

$$\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx = \int_{1-\delta}^{1+\delta} e^{k\left(-1-\frac{1}{2z^2}(x-1)^2\right)}\, dx \tag{34}$$

from which we obtain

$$e^{-k} \cdot \int_{1-\delta}^{1+\delta} e^{-\frac{k(x-1)^2}{2z_{\min}^2}}\, dx \leq \int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx \leq e^{-k} \cdot \int_{1-\delta}^{1+\delta} e^{-\frac{k(x-1)^2}{2z_{\max}^2}}\, dx \tag{35}$$

Note that the upper and lower bounds of (35) are quite similar to the form in Lemma 1. To evaluate the upper and lower bounds in (35) then, we morph the bounds and the integrand into a form similar to Lemma 1. Starting off with the lower bound, we need $\frac{k}{2z_{\min}^2}(x-1)^2$ to become t^2; hopefully, the upper and lower limits of integration will go to infinity as k approaches infinity. Thus, we consider the substitution: $t = \sqrt{\frac{k}{2z_{\min}^2}}(x-1) \implies dt = \sqrt{\frac{k}{2z_{\min}^2}}\, dx$. The lower limit of $1-\delta$ is then transformed to $-\delta \cdot \sqrt{\frac{k}{2z_{\min}^2}}$ and the upper limit is similarly mapped to $\delta \cdot \sqrt{\frac{k}{2z_{\min}^2}}$. Thus, the lower bound in (35) becomes:

$$e^{-k} \cdot \frac{\sqrt{2z_{\min}^2}}{k^{\frac{1}{2}}} \int_{-\delta\cdot\sqrt{\frac{k}{2z_{\min}^2}}}^{\delta\cdot\sqrt{\frac{k}{2z_{\min}^2}}} e^{-t^2}\, dt \leq \int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx \tag{36}$$

and by a similar manipulation of the upper bound:

$$\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx \leq e^{-k} \cdot \frac{\sqrt{2z_{\max}^2}}{k^{\frac{1}{2}}} \int_{-\delta\cdot\sqrt{\frac{k}{2z_{\max}^2}}}^{\delta\cdot\sqrt{\frac{k}{2z_{\max}^2}}} e^{-t^2}\, dt \tag{37}$$

Now, this is where the order of limits is very important. We first consider the limit of $k \to \infty$ and then $\delta \to 0$. As $k \to \infty$, the limits of integration converge to $-\infty$ and $+\infty$ respectively for both the lower and upper bounds in (36) and (37). Thus, the entire integral term converges to $\sqrt{\pi}$ as per Lemma 1 in (36) and (37). This is true for all values of $0 < \delta < 1$. So as $\delta \to 0$, the value of the integral does not change. Furthermore, as $\delta \to 0$, both $z_{\min}, z_{\max} \to 1$. Thus, both the upper and lower bounds are asymptotically equivalent to $e^{-k} \cdot \frac{\sqrt{2\pi}}{k^{\frac{1}{2}}}$. It follows by the squeeze theorem then, that

$$\int_{1-\delta}^{1+\delta} e^{k(\log(x)-x)}\, dx \sim e^{-k} \cdot \frac{\sqrt{2\pi}}{k^{\frac{1}{2}}} \tag{38}$$

Recall now that the "tails" were bounded in (24) and (27) by:

$$\int_0^{1-\delta} e^{k(\log(x)-x)}\,dx < B_\alpha \cdot \alpha^k \qquad \text{and} \qquad \int_{1+\delta}^{\infty} e^{k(\log(x)-x)}\,dx < B_\beta \cdot \beta^k \tag{39}$$

Once again looking at the limit as $k \to \infty$ first, because both $\alpha, \beta < e^{-1}$, it follows that the ratio of each tail to the asymptotic expression in (38) converges to 0 as $k \to \infty$. Hence, the contribution of each tail integral is indeed negligible. Therefore,

$$\int_0^{\infty} e^{k(\log(x)-x)}\,dx \sim e^{-k} \cdot \frac{\sqrt{2\pi}}{k^{\frac{1}{2}}} \tag{40}$$

Now, we go back to the original problem: approximating the factorial function, or equivalently, approximating $\Pi(k)$ in the limit of large k. For the benefit of the reader, we repeat one of our initial steps:

$$\Pi(k) = k^{k+1} \cdot \int_0^{\infty} e^{k(\log(x)-x)}\,dx \tag{41}$$

Putting together (40) and (41), we obtain

Theorem 2 (Stirling's Approximation). *$\Pi(k) \sim \left(\frac{k}{e}\right)^k \cdot \sqrt{2\pi k}$.*

Remark 2. The above method of obtaining Stirling's Approximation is an example of using Laplace's Method for Integral Approximation; the above proof is based on a more general method described in [3], albeit the above proof takes into account the infinite bounds of integration.

Remark 3. From the approximation above, we also have that $\log(\Pi(k))$ is asymptotically equivalent to $k\log(k) - k + \frac{1}{2}\log(2\pi k)$. Note that Stirling's approximation is rapidly convergent to $\Pi(x)$:

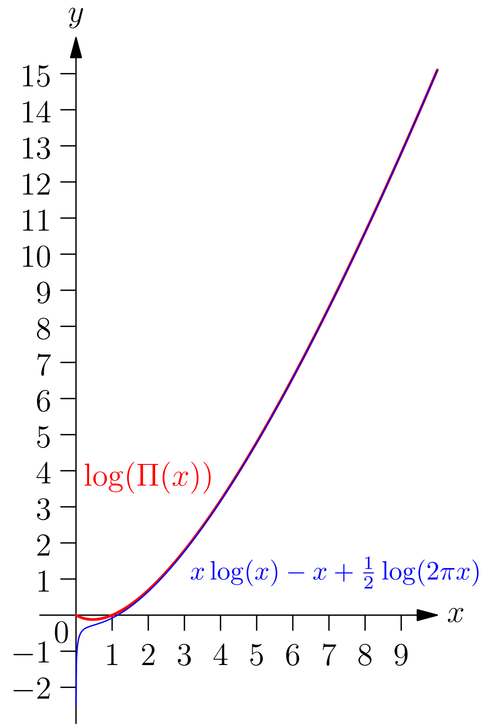

Figure 4: A comparison of $\log(\Pi(x))$ with Stirling's Approximation

Moreover, $\Pi(x)$ is a strictly increasing concave-up function on $[1, \infty)$ (the proof of this fact requires Real Analysis which is beyond the scope of this paper). The first derivative of $x\log(x) - x + \frac{1}{2}\log(2\pi x)$ is $\log(x) + \frac{1}{2x}$ and its second derivative is $\frac{2x-1}{2x^2}$ which are both positive for all $x \in [1, \infty)$. Therefore, both $\log(\Pi(x))$ and $x\log(x) - x + \frac{1}{2}\log(2\pi x)$ are strictly increasing and concave-up on $x \in [1, \infty)$. We can express $\log(\Pi(x))$ as $x\log(x) - x + \frac{1}{2}\log(2\pi x) + \text{remainder}$. Since Stirling's Approximation gives asymptotic equivalence upto a constant term of $\sqrt{2\pi}$, the remainder must have decreasing end-behavior (otherwise we would not get asymptotic equivalence to a constant term). It follows that the remainder is strictly decreasing for $x \geq 1$ since $\log(\Pi(x))$ and $x\log(x) - x + \frac{1}{2}\log(2\pi x)$ are strictly increasing, concave-up functions on $x \in [1, \infty)$. The remainder is thus negligible for $x \geq 2$ since the difference between $\log(\Pi(x))$ and $x\log(x) - x + \frac{1}{2}\log(2\pi x)$ is empirically less than or equal to 0.041 for $x \geq 2$.

Along with the number-theoretic properties of the factorial, we can now use this asymptotic approximation to rigorize certain properties of the distribution of primes.

4 Ramanujan's Proof of Bertrand's Postulate

The primes have a direct connection to all numbers, namely they are the building blocks of numbers. All numbers can be broken into prime factors. When taking a look at the factorial, this is true too. But, because the factorial is the product of consecutive integers, there are some nice properties in relation to its prime factors.

Rather than analyzing a product, it is often easier to comprehend a sum. One useful way to turn a product into a sum is by taking the logarithm of the product. So, we will deal with

$$\log(n!) = \sum_{k=1}^{n} \log(k) \tag{42}$$

It may be useful to look at the factorial as just a number and try to find the multiplicity of its prime divisors.

So, consider, for now, the number 2. How many times does 2 divide $n!$ for some positive integer n? We can count the number of even integers less than or equal to n, which is simply $\lfloor n/2 \rfloor$, since each even integer contributes a factor of 2. Observe that we still neglect some factors of 2 in $n!$ because of the multiples of 4 which contribute an additional factor of 2. We hence add the number of multiples of 4 less than or equal to n to the number of even integers less than or equal to n, giving us $\lfloor n/2 \rfloor + \lfloor n/4 \rfloor$. Observe even this does not capture all factors of 2 in $n!$ because of the multiples of 8 which contribute three factors of 2. Thus, continuing this process, we find that, as famously attributed to Legendre, the number of factors of 2 that divide $n!$ is

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \tag{43}$$

Generalizing (43) to a prime number p gives that the number of factors of p that divide $n!$ is

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \tag{44}$$

Now, we can try to express the factorial in terms of its prime factors. The multiplicity of a prime p in $n!$ is given by (44); hence, iterating over all primes $p \leq n$ gives:

$$\log(n!) = \sum_{p \leq n} \log(p) \cdot \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

$$= \sum_{p \leq n} \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \cdot \log(p) \tag{45}$$

where our notation above is to be read as "the sum over all primes p less than or equal to n." Note that both summations are *finite*: one summation ranges across the *finitely* many primes less or equal to n; the upper limit of ∞ in the second summation could be replaced with an upper bound of $k_{\max} = \lfloor \log_p(n) \rfloor$ (as for all $k > k_{\max}$, $\lfloor n/p^k \rfloor = 0$) and hence the second summation is also finite.

A simple property of summation is that we can switch the order of summation (it does not matter the order in which we sum the elements of a finite summation; the same is not necessarily true of an infinite summation, hence the clarification made in the prior paragraph). Thus, we are justified in writing

$$\begin{aligned} \log(n!) &= \sum_{k=1}^{\infty} \sum_{p \leq n} \left\lfloor \frac{n}{p^k} \right\rfloor \log(p) \\ &= \sum_{k=1}^{\infty} \sum_{p \leq n^{1/k}} \left\lfloor \frac{n}{p^k} \right\rfloor \log(p) \end{aligned} \tag{46}$$

where the second equality follows from the fact that if $p > n^{1/k} \implies p^k > n$, then $\left\lfloor \frac{n}{p^k} \right\rfloor = 0$.

It may be a bit nicer to remove the floor function, so we may have to change the order in which we do the summation again, this time of

$$\sum_{p \leq n^{1/k}} \left\lfloor \frac{n}{p^k} \right\rfloor \log(p) = \sum_{p \leq n^{1/k}} \sum_{m=1}^{\lfloor n/p^k \rfloor} \log(p) \tag{47}$$

To swap this time, observe what the current order is doing. It sums $\log(p)$, $\lfloor n/p^k \rfloor$ number of times, then repeats for each prime $p \leq n^{1/k}$. When we swap the order, observe we must sum $\log(p)$ for each prime, up to a certain prime in each iteration, and then repeat as many times as necessary. Now consider fixing m and suppose that for some prime q, $\lfloor n/q^k \rfloor = m$. From the order of summation, the term $\log(q)$ should be summed precisely m times. When we swap the order of summation, $\log(q)$ should be added exactly once on iterations $1, 2, \ldots, m$ of the outer sum. Hence, if the counter $m \leq \lfloor n/q^k \rfloor$, $\log(q)$ should be counted. Observe, by definition, $m \leq \lfloor n/q^k \rfloor \leq n/q^k$. Moreover, if $m \leq n/q^k = \lfloor n/q^k \rfloor + \{n/q^k\} < \lfloor n/q^k \rfloor + 1$, it follows that

$m - \lfloor n/q^k \rfloor < 1$. Because $m, \lfloor n/q^k \rfloor \in \mathbb{Z}$, $m - \lfloor n/q^k \rfloor < 1$ implies $m - \lfloor n/q^k \rfloor \leq 0$ so $m \leq \lfloor n/q^k \rfloor$. Thus, $m \leq \lfloor n/q^k \rfloor \iff m \leq n/q^k$. This condition rearranges to $q \leq \left(\frac{n}{m}\right)^{1/k}$, and we hence find:

$$\sum_{p \leq n^{1/k}} \left\lfloor \frac{n}{p^k} \right\rfloor \log(p) = \sum_{m=1}^{\infty} \sum_{p \leq \left(\frac{n}{m}\right)^{\frac{1}{k}}} \log(p) \tag{48}$$

Putting all of this together we find that

Theorem 3. $\log\lfloor n \rfloor ! = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{p \leq \left(\frac{n}{m}\right)^{\frac{1}{k}}} \log(p)$

where we have put the m-summation out at the front to keep $\frac{n}{m}$ constant over iterations of k.

Theorem 3 provides an elementary connection between the primes and the factorial. And this is exactly what Srinivasa Ramanujan saw and makes clever use of in his proof of Theorem 1. In particular, it may be possible to relate the asymptotic behavior of the primes to the asymptotic behavior of the factorial. For simplicity, we begin by breaking the sums in Theorem 3 into parts.

Definition 1. We define the first Chebyshev function to be $\vartheta(x) \equiv \sum_{p \leq x} \log(p)$.

Definition 2. We define the second Chebyshev function to be $\psi(x) \equiv \sum_{k=1}^{\infty} \sum_{p \leq x^{1/k}} \log(p) \equiv \sum_{k=1}^{\infty} \vartheta(x^{\frac{1}{k}})$.

This way, we find that $\log\lfloor n \rfloor ! = \sum_{m=1}^{\infty} \psi\left(\frac{n}{m}\right)$ by Theorem 3. Moreover, due to the infinitude of primes, $\vartheta(x)$ and $\psi(x)$ are both non-decreasing. Observe:

$$\psi(x) - 2\psi(\sqrt{x}) = \vartheta(x) - \vartheta(x^{\frac{1}{2}}) + \vartheta(x^{\frac{1}{3}}) - \vartheta(x^{\frac{1}{4}}) + \vartheta(x^{\frac{1}{5}}) - \cdots \tag{49}$$

Similarly

$$\log\lfloor x \rfloor ! - 2\log\lfloor x/2 \rfloor ! = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) + \psi\left(\frac{x}{5}\right) - \cdots \tag{50}$$

Since $\vartheta(x)$ is non-decreasing, the alternating sum $\vartheta(x^{\frac{1}{2}}) - \vartheta(x^{\frac{1}{3}}) + \vartheta(x^{\frac{1}{4}}) - \vartheta(x^{\frac{1}{5}}) + \cdots \geq 0$ as each $\vartheta(x^{\frac{1}{2k}}) - \vartheta(x^{\frac{1}{2k+1}}) \geq 0$. It follows that $\vartheta(x) \geq \psi(x) - 2\psi(\sqrt{x})$ by (49). Combined with Definition 2, it follows that

$$\psi(x) \geq \vartheta(x) \geq \psi(x) - 2\psi(\sqrt{x}) \tag{51}$$

By similar logic, we find that

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \geq \log\lfloor x \rfloor! - 2\log\lfloor x/2 \rfloor! \geq \psi(x) - \psi\left(\frac{x}{2}\right) \tag{52}$$

Observe that the value of $\lfloor x/2 \rfloor$ is constant over $[2k, 2k+2)$ for each $k \in \mathbb{N}$. Similarly, $\lfloor x \rfloor$ is constant over $[k, k+1)$. Thus, we consider the intervals $[2k, 2k+1)$ and $[2k+1, 2k+2)$ for arbitrary $k \in \mathbb{N}$. For all $x \in [2k, 2k+1)$, we find that $\lfloor x \rfloor = 2k$ and $\lfloor x/2 \rfloor = k$. Hence, If we look at the limit as x approaches $2k+1$ for $x \in [2k, 2k+1)$, we find

$$\begin{aligned} \lim_{x \to 2k+1^-} \log\lfloor x \rfloor! - 2\log\lfloor x/2 \rfloor! &= \log(\Pi(2k)) - 2\log\left(\Pi\left(k\right)\right) \\ &= \log(\Pi((2k+1)-1)) - 2\log\left(\Pi\left(\frac{(2k+1)-1}{2}\right)\right) \end{aligned} \tag{53}$$

Observe $\log(\Pi(x-1)) - 2\log\left(\Pi\left(\frac{x-1}{2}\right)\right)$ is increasing (the proof of this fact requires Real Analysis, which is beyond the scope of this paper; however, note that this fact is empirically evident in Figure 5). It follows that

$$\log(\Pi(x-1)) - 2\log\left(\Pi\left(\frac{x-1}{2}\right)\right) \leq \log\lfloor x \rfloor! - 2\log\lfloor x/2 \rfloor! \tag{54}$$

since the right-hand side is equivalent to (53) when $x = 2k+1$. If we look at the same limit, but for $x \in [2k+1, 2k+2)$, we find:

$$\log\lfloor x \rfloor! - 2\log\lfloor x/2 \rfloor! \leq \log(\Pi(x)) - 2\log\left(\Pi\left(\frac{x-1}{2}\right)\right) \tag{55}$$

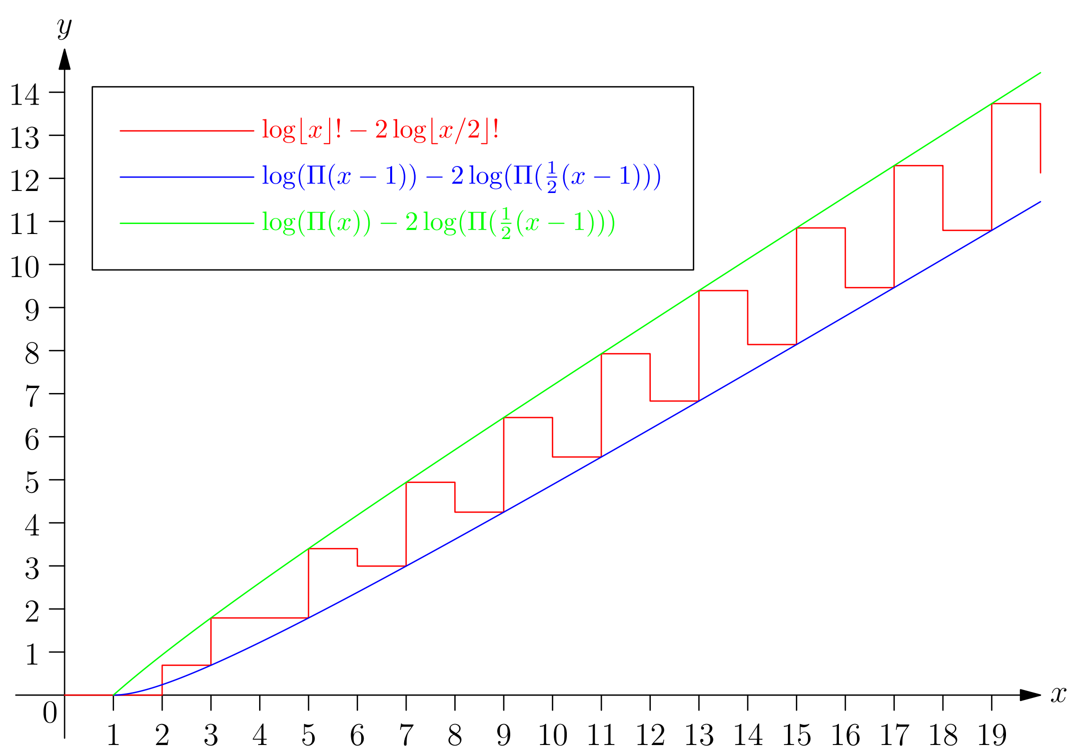

Figure 5: Upper and Lower bounds of $\log\lfloor x \rfloor! - 2\log\lfloor x/2 \rfloor!$

Since $\log(\Pi(x-1)) - 2\log(\Pi(\frac{1}{2}(x-1)))$ and $\log(\Pi(x)) - 2\log(\Pi(\frac{1}{2}(x-1)))$ are continuously increasing, (54) and (55) are lower and upper bounds respectively for $\log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor!$ for every $x \in [2k, 2k+2) \forall k \in \mathbb{N}$.

We will now refine the upper and lower bounds via Stirling's Approximation, due to its rapid convergence to $\Pi(x)$ for sufficiently large x. Beginning with (54):

$$\log(\Pi(x-1)) - 2\log(\Pi(\frac{1}{2}(x-1))) \approx (x-1)\log(x-1) - (x-1) + \frac{1}{2}\log(2\pi(x-1)) \\ - (x-1)\log((x-1)/2) + (x-1) - \log(\pi(x-1)) \tag{56}$$

$$= (x-1)\log(2) + \frac{1}{2}\log\left(\frac{2}{\pi(x-1)}\right) \tag{57}$$

$$> 0.693x - \frac{1}{2}\log(x-1) - \frac{1}{2}\log(2\pi) \tag{58}$$

Observe that for sufficiently large x, $\log(x-1) + \frac{1}{2}\log(2\pi)$ becomes proportionally insignificant compared to x. Hence, there exists $x_{\min}$ for which $\frac{2}{3}x < x\log(2) - \frac{1}{2}\log(x-1) - \frac{1}{2}\log(2\pi)$ for all $x \geq x_{\min}$:

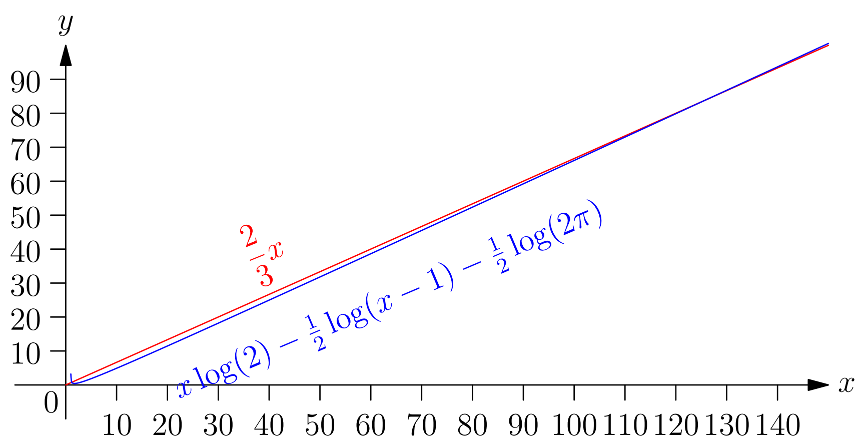

Figure 6: Comparison of $\frac{2}{3}x$ and $x\log(2) - \frac{1}{2}\log(x-1) - \frac{1}{2}\log(2\pi)$

Graphically, $x_{\min} \approx 130$. Hence:

$$\frac{2}{3}x < \log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor!, \quad \text{for all } x > 130 \tag{59}$$

We can similarly refine (55):

$$\log(\Pi(x)) - 2\log(\Pi(\frac{1}{2}(x-1))) \approx x\log(x) - x + \frac{1}{2}\log(2\pi x) \\ - (x-1)\log((x-1)/2) + (x-1) - \log(\pi(x-1)) \tag{60}$$

$$= x\log(x) - (x-1)\log((x-1)/2) - 1 + \frac{1}{2}\log\left(\frac{2x}{\pi(x-1)^2}\right) \tag{61}$$

$$= x\log\left(\frac{x}{x-1}\right) + x\log(2) + \frac{1}{2}\log\left(\frac{x}{2\pi}\right) - 1 \tag{62}$$

Because $y^2 \le e^y$ for all $y > 0$, $\log(y) \le \frac{y}{2}$. Hence, we find:

$$\le x\log\left(\frac{x}{x-1}\right) + x\left(\log(2) + \frac{1}{8\pi}\right) - 1 \tag{63}$$

$$< 0.733x + x\log\left(\frac{x}{x-1}\right) - 1 \tag{64}$$

$$< \frac{3}{4}x \tag{65}$$

Note that $\lim_{x\to\infty} x\log\left(\frac{x}{x-1}\right) = \lim_{x\to\infty} \frac{\log(x)-\log(x-1)}{x^{-1}} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x-1}}{-x^{-2}} = \lim_{x\to\infty} \frac{x^2}{x^2-x} = 1$. Hence, for sufficiently large x, $x\log\left(\frac{x}{x-1}\right) - 1$ becomes negligible so (65) follows.

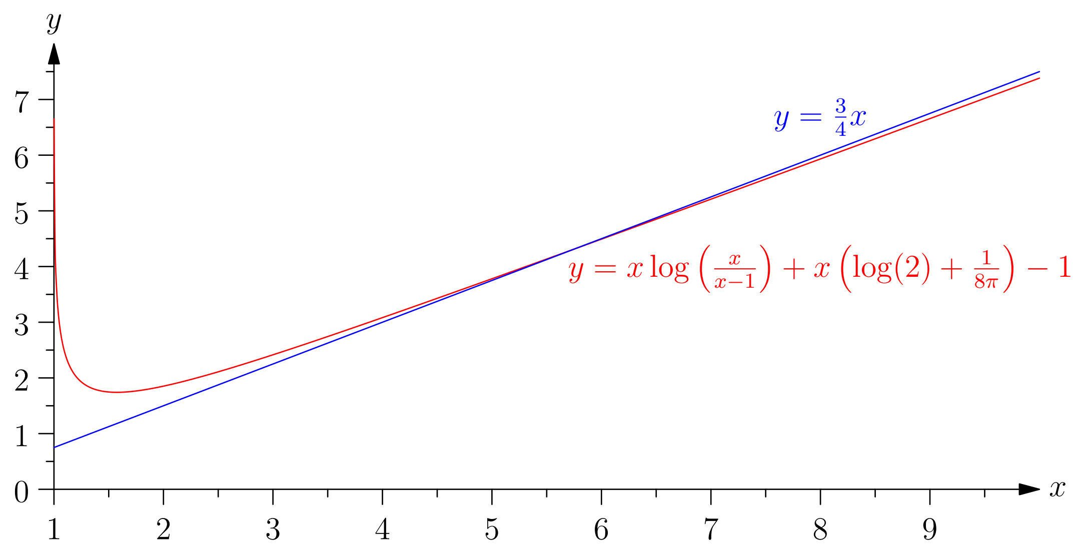

Figure 7: Comparison of $\frac{3}{4}x$ and $x\log\left(\frac{x}{x-1}\right) + x\left(\log(2) + \frac{1}{8\pi}\right) - 1$

Graphically, this occurs for $x > 6$, but we can verify the following to be true for all $6 \ge x > 0$:

$$\log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor! < \frac{3}{4}x, \quad \text{for all } x > 0 \tag{66}$$

Due to (52):

$$\psi(x) - \psi(x/2) < \frac{3}{4}x, \quad \text{for all } x > 0 \tag{67}$$

and from (59)

$$\frac{2}{3}x < \psi(x) - \psi(x/2) + \psi(x/3), \quad \text{for all } x > 130 \tag{68}$$

Manipulating (67):

$$\psi(x) = \left(\psi(x) - \psi\left(\frac{1}{2}\right)\right) + \left(\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{4}\right)\right) + \left(\psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{8}\right)\right) + \cdots \tag{69}$$

$$< \frac{3}{4}x\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \tag{70}$$

$$= \frac{3}{4}x\cdot\frac{1}{\frac{1}{2}} \tag{71}$$

$$= \frac{3}{2}x, \quad \text{for all } x>0 \tag{72}$$

We can now determine a lower bound for $\vartheta(x)$. Starting with (68):

$$\frac{2}{3}x < \psi(x)-\psi(x/2)+\psi(x/3) \tag{73}$$

$$\leq \vartheta(x)+2\psi(\sqrt{x})-\vartheta(x/2)+\psi(x/3), \quad \text{for all } x>130 \tag{74}$$

where (74) follows from manipulating (51) into the form $\psi(x)\leq\vartheta(x)+2\psi(\sqrt{x})$ and $\psi(x/2)\geq\vartheta(x/2)\implies-\psi(x/2)\leq-\vartheta(x/2)$. We will now make use of (72) with (74):

$$\vartheta(x)-\vartheta(x/2)+3\sqrt{x}+\frac{1}{2}x > \frac{2}{3}x \tag{75}$$

$$\vartheta(x)-\vartheta(x/2) > \frac{1}{6}x-3\sqrt{x}, \quad \text{for all } x>130 \tag{76}$$

For sufficiently large x, observe $\frac{1}{6}x-3\sqrt{x}>0$. Solving $\frac{1}{6}x=3\sqrt{x}\implies x=18\sqrt{x}$ implies $x=324$. Thus,

$$\vartheta(2x)-\vartheta(x)>0, \quad \text{for all } x>162 \tag{77}$$

Hence, due to the definition of $\vartheta(x)\equiv\sum_{p\leq x}\log(p)$, $\vartheta(2x)-\vartheta(x)>0$ implies the existence of a prime in $[x,2x]$. Hence, we have proved Bertrand's Postulate, as it is easily verified via inspection to be true for $x\leq 162$ [4].

5 Conclusion

We have seen a highly non-trivial connection between two fields that—on the surface—seem to be opposites of each other: number theory, which concerns the *discrete* integers, and calculus, which largely concerns *continuous* curves and spaces. In particular, one may observe the following noteworthy connection: problems regarding the distribution of primes can be converted into problems of analysis via the joint use of Chebyshev functions, Theorem 3 and Stirling's approximation. Such a method helped in proving Bertrand's Postulate, and the inclined reader may refer to [2] to see this same method used in proving the Prime Number Theorem. Lastly, the inclined reader may wish to see a simple, elegant proof of Bertrand's postulate due to Paul Erdős in [1].

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