Junior Problems

J649. Find all integers n for which

$$(n+3)\left(n^2+3n+3\right)$$

is the product of three primes $p > q > r$ such that $p - r = 6$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J650. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\sqrt{1-bc} + \sqrt{1-ca} + \sqrt{1-ab} \geq \sqrt{6}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J651. Let $ABCD$ be a cyclic quadrilateral such that

$$(AB - BC + CD + DA)(BC + CD + DA - AB) + AC \cdot BD = (AB + AD)(BC + CD).$$

Find $\angle ADC$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J652. Let $ABCD$ be a trapezoid and let X be the intersection of its diagonals. Let r_1, r_2, r_3, r_4 be the inradii of triangles XAD, XDC, XCB, XAB, respectively. Prove that if

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$$

then $ABCD$ is circumscribed about a circle

Proposed by Mihaela Berindeanu, Bucharest, România

J653. Let a, b, c be distinct real numbers. Prove that any two of the equalities

$$3ab + (b-c)(c-a) = \frac{16}{a-b}, \quad 3bc + (c-a)(a-b) = \frac{4}{b-c}, \quad 3ca + (a-b)(b-c) = \frac{-20}{c-a},$$

imply the third. Are there such numbers?

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J654. Is there a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(x)) = x^3 - 2x + 1$ for all $x \in \mathbb{Z}$?

Proposed by Mircea Becheanu, Canada

Senior Problems

S649. Let a, b, c be positive real numbers such that $ab + bc + ca + abc = 4$. Prove that

$$\frac{\sqrt{a}}{a+2} + \frac{\sqrt{b}}{b+2} + \frac{\sqrt{c}}{c+2} \le 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S650. For a triangle ABC let A', B', C' be the tangency points of the ex-circles with sides $[BC], [CA], [AB]$, respectively.

(i) Prove that one can construct a triangle with the vectors $\overline{AA'}, \overline{BB'}, \overline{CC'}$ if and only if triangle ABC is equilateral.

(ii) Does the above property remain true if segments $[AA'], [BB'], [CC']$ are used for the construction?

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

S651. Let a, b, c be real numbers. Prove that

$$3a^2b^2c^2 + 3\sum_{cyc}\left(a^4b^2 + a^2b^4\right) \ge 3abc\sum_{cyc} a^3 + 4\sum_{cyc} a^3b^3.$$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata Roma, Italy

S652. Find the least positive integer m for which there is a positive integer n such that

$$1 + 2 + \cdots + m = (2 + 3 + \cdots + n)^2$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S653. Let $a_1, a_2, \ldots, a_9$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_9 = 2$ and $a_1 \ge a_2 \ge \cdots \ge a_9$. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_9a_1 \le 1.$$

Proposed by Vasile Cârtoaje, Oil-Gas University, Ploieşti, România

S654. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{8(ab + bc + ca)^2}{(a+b)(b+c)(c+a)} \ge 9\sqrt{abc}.$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U649. Evaluate

$$\lim_{n\to\infty}\left(\frac{\left(1^2+n^2\right)\left(2^2+n^2\right)\cdots\left(n^2+n^2\right)}{n!n^n}\right)^{1/n}$$

Proposed by Mircea Becheanu, Canada

U650. Evaluate

$$I=\int_1^e \frac{(\ln x-1)^2-3}{(\ln x+2)^2}dx.$$

Proposed by Mihaela Berindeanu, Bucharest, România

U651. Let x,y,z,t be real numbers such that $x+y+z+t=0$. Prove that

$$\frac{x+1}{x^2+3}+\frac{y+1}{y^2+3}+\frac{z+1}{z^2+3}+\frac{t+1}{t^2+3}\le\frac{4}{3}.$$

Proposed by Marius Stănean, Zalău, România

U652. Let k,n be positive integers and n even. Prove that the polynomial

$$1+x+\cdots+x^{2^k}+x^n$$

has a root on the unit circle if and only if $\nu_2(n)<k$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U653. Find all finite groups which have proper subgroups and these have order 2 or 3 only.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, România

U654. Evaluate

$$\int_0^1 \frac{x\sqrt{x}\,\ln x}{x^2-x+1}\,dx$$

Proposed by Vasile Mircea Popa, Sibiu, România

Olympiad Problems

O649. Find all positive integers n such that

$$n! + 1 = (8a - 3)^2;\ (n + 1)! + 1 = (8b + 3)^2;\ (2n - 1)! + 1 = (8c - 1)^2,$$

for some positive integers a, b, c congruent to 1 $(\bmod 8)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O650. Find the greatest constant λ such that the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \lambda\left(\frac{a^2+b^2+c^2}{(a+b+c)^2} - \frac{1}{3}\right)$$

holds for all nonnegative real numers $a, b, c,$ no two of which are zero.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O651. Find the greatest real number k such that for all nonnegative real numbers a, b, c, no two of wich are zero,

$$\frac{a^3 + ka^2b}{a+b} + \frac{b^3 + kb^2c}{b+c} + \frac{c^3 + kc^2a}{c+a} \geq \frac{(k+1)(ab+bc+ca)}{2}.$$

Proposed by Titu Andreescu, Dallas, USA and Marius Stănean, Zalău, România

O652. For a positive integer n we denote by $S_2(n)$ the sum of digits of n in base 2. Prove that there are integers $l, m \geq 1402$ for which there are infinitely many pairs of odd integers (a, b) with $S_2(a) = l,\ S_2(b) = m$ and $S_2(ab) = 4$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O653. Let $a_1 = a_2 \geq a_3 \geq \cdots \geq a_{n-1} = a_n$ be real numbers where $n \geq 4$. Prove that

$$n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \geq (a_1 + a_2 + \cdots + a_n)^2.$$

Proposed by Vasile Cârtoaje, Oil-Gas University, Ploieşti, România

O654. Let $f(x)$ be a polynomial with integer coefficients whose leading coefficient is positive. Prove that the sequence

$$x_n = \left\{\frac{6^{f(n)}}{n}\right\}$$

is dense in $[0, 1]$. (By $\{a\}$ we mean the fractional part of a.)

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran