Junior problems

J643. Find all positive integers n such that

$$\sqrt{\binom{12n}{n+1}+1}-\sqrt{\binom{12n}{n-1}+1}=40.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Jean Heibig, ISAE-SUPAERO, Toulouse, France

As this is a strictly increasing sequence, $n = 2$ is the only solution.

Let us write $l_n = \sqrt{C_{n+1}^{12n}+1}$ and $r_n = \sqrt{C_{n-1}^{12n}+1}$. Therefore, $l_{n+1}/l_n > r_{n+1}/r_n > 1$ and $l_n > r_n$ (as $n+1 < (12n)/2$).

Also solved by G. C. Greubel, Newport News, VA, USA; Sundaresh Harige, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Michel Faleiros Martins, São Paulo, SP, Brazil; Polyahedra, Polk State College, USA; Andrew Hwang, Langley High School, McLean, VA, USA.

J644. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{\sqrt{a+2a^4}} + \frac{1}{\sqrt{b+2b^4}} + \frac{1}{\sqrt{c+2c^4}} \geq \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

Proposed by Mircea Becheanu, Canada

Solution by Polyahedra, Polk State College, USA

Applying the given condition and Jensen's inequality to the convex function $1/\sqrt{x}$, we have

$$\frac{1}{\sqrt{a+2a^4}} + \frac{1}{\sqrt{b+2b^4}} + \frac{1}{\sqrt{c+2c^4}} = \frac{bc}{\sqrt{bc+2a^2}} + \frac{ca}{\sqrt{ca+2b^2}} + \frac{ab}{\sqrt{ab+2c^2}}$$

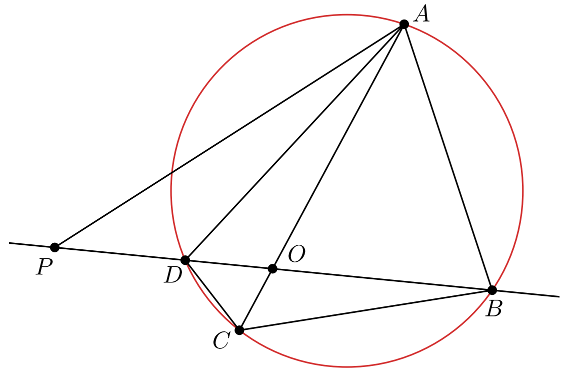
$$\geq \frac{bc+ca+ab}{\sqrt{\dfrac{bc\left(bc+2a^2\right)+ca\left(ca+2b^2\right)+ab\left(ab+2c^2\right)}{bc+ca+ab}}} = \sqrt{bc+ca+ab} = \sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J645. Let $ABCD$ be a cyclic quadrilateral and let O be the intersection of the diagonals AC and BD. Let P be a point on line BD such that $\angle PAD = \angle CAD$ and $BP^2 = 4OC(AO + AP)$. Show that O is the midpoint of the segment BP.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Polyahedra, Polk State College, USA



Since $AP/AO = DP/OD$ and $OC \cdot AO = OD \cdot BO$,

$$0 = BP^2 - 4OC(AO + AP) = BP^2 - 4BO \cdot OD - 4BO \cdot DP = BP^2 - 4BO \cdot OP = (BO - OP)^2,$$

so $BO = OP$.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Faleiros Martins, São Paulo, SP, Brazil; Andrew Hwang, Langley High School, McLean, VA, USA; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Anderson Torres, Brazil.

J646. Let a, b be positive integers such that $\gcd(a, b) = 1$. Prove that $\gcd(a^2 + b^2, a^3 + b^3) \in \{1, 2\}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Michel Faleiros Martins, São Paulo, SP, Brazil

Suppose that $p^k | \gcd(a^2 + b^2, a^3 + b^3)$ for some $k \geq 1$ and a prime p. Then, $p^k | (a+b)(a^2+b^2) - (a^3+b^3)$ yields $p^k | ab(a+b)$. If $p|a$, then $p|(a^2+b^2) - a^2$, so $p|b^2$ and $p|b$, a contradiction, since $\gcd(a, b) = 1$. Similarly if $p|b$. Therefore, $p^k|(a+b)$. Hence, $p^k|(a+b)^2 - (a^2+b^2)$ gives $p^k|2ab$ and finally $p^k|2$. We get $p = 2$ and $k = 1$. Thus, $\gcd(a^2 + b^2, a^3 + b^3) \in \{1, 2\}$.

Also solved by Ivko Dimitric, PSU Fayette, Lemont Furnace, PA, USA; Polyahedra, Polk State College, USA; Andrew Hwang, Langley High School, McLean, VA, USA; Jean Heibig, ISAE-SUPAERO, Toulouse, France; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Sundaresh Harige, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Srijan Sundar, Oxford, UK; Daniel Pascuas, Barcelona, Spain; Arkady Alt, San Jose, CA, USA.

J647. Let a, b, c, p, q, r be positive real numbers such that $q + r \geq 2p$ and $a + b + c = p + q + r$. Prove that

$$\frac{a}{\sqrt{pa+qb+rc}} + \frac{b}{\sqrt{pb+qc+ra}} + \frac{c}{\sqrt{pc+qa+rb}} \geq \sqrt{3}$$

Proposed by Mircea Becheanu, Canada

Solution by the author

We will use the following general inequality: *Lemma:* Let a, b, c, x, y, z be positive real numbers. Then

$$\left(\frac{a}{\sqrt{x}} + \frac{b}{\sqrt{y}} + \frac{c}{\sqrt{z}}\right)^2 \geq \frac{(a+b+c)^3}{ax+by+cz}.$$

Proof of the Lemma. Using Cauchy-Schwartz inequality we have

$$\frac{a}{\sqrt{x}} + \frac{b}{\sqrt{y}} + \frac{c}{\sqrt{z}} = \frac{a^2}{a\sqrt{x}} + \frac{b^2}{b\sqrt{y}} + \frac{c^2}{c\sqrt{z}} \geq \frac{(a+b+c)^2}{a\sqrt{x}+b\sqrt{y}+c\sqrt{z}}.$$

Again by Cauchy-Schwartz inequality we have

$$a\sqrt{x} + b\sqrt{y} + c\sqrt{z} = \sqrt{a}\sqrt{ax} + \sqrt{b}\sqrt{by} + \sqrt{c}\sqrt{cz} \leq \sqrt{a+b+c}\sqrt{ax+by+cz}.$$

This proves the Lemma.

Denoting by S the left hand of the given inequality , we have to show that $S^2 \geq 3$. From the Lemma we have

$$S^2 \geq \frac{(a+b+c)^3}{p(a^2+b^2+c^2)+(q+r)(ab+bc+ca)}.$$

Then we have to show that

$$(a+b+c)^3 \geq 3p(a^2+b^2+c^2) + 3(q+r)(ab+bc+ca).$$

Using $a + b + c = p + q + r$ we have:

$$(a+b+c)^3 = (p+q+r)(a+b+c)^2 =$$

$$(p+q+r)\left(\sum a^2 + 2\sum ab\right) \geq 3p\sum a^2 + 3(q+r)\sum ab.$$

Last inequality is equivalent with

$$(q+r-2p)\sum a^2 \geq (q+r-2p)\sum ab,$$

and this is obvious.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Michel Faleiros Martins, São Paulo, SP, Brazil; Polyahedra, Polk State College, USA; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J648. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$a + b + c + 3abc \geq \frac{4}{a + b + c}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

The inequality is

$$(a + b + c)^3 + 3abc(a + b + c)^2 \geq 4(a + b + c)$$

Schür's inequality of order 3 $a(a - b)(a - b) + b(b - a)(b - c) + c(c - a)(c - b) \geq 0$ is also

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca) = 4(a + b + c)$$

Hence we get

$$4(a + b + c) - 9abc + 3abc(a + b + c)^2 \geq 4(a + b + c) \iff (a + b + c)^2 \geq 3$$

and this follows by $(a + b + c)^2 \geq 3(ab + bc + ca) = 3$. The equality holds for $(a, b, c) = (1, 1, 0)$ and cyclic.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Polyahedra, Polk State College, USA; Andrew Hwang, Langley High School, McLean, VA, USA; Srijan Sundar, Oxford, UK; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Mihaly Bencze, Braşov, Romania; Sarah Seales Phoenix, AZ, USA; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

Senior problems

S643. Let $n \geq 2$ and $P(x) = x^{2n+1} + a_1 x^{2n} + \cdots + a_{2n+1}$ be a polynomial with real coefficients satisfying

$$a_2 + a_4 + \cdots + a_{2n} = \frac{3\left(3^{2n} - 1\right)}{2}.$$

Let $x_1, x_2, \ldots, x_{2n+1}$ be the roots of $P(x)$. Given that

$$(x_1 - 1)(x_2 + 1)(x_3 - 1)(x_4 + 1)\cdots(x_{2n+1} - 1) = 3^n \text{ and}$$
$$(x_1 + 1)(x_2 - 1)(x_3 + 1)(x_4 - 1)\cdots(x_{2n+1} + 1) = 3^{n+1},$$

evaluate $a_1 + a_3 + \cdots + a_{2n+1}$. Give an example of such a polynomial $P(x)$ in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Michel Faleiros Martins, São Paulo, SP, Brazil

We can write $P(x) = (x - x_1)(x - x_2)\cdots(x - x_{2n+1})$. Let $A_n = a_1 + a_3 + \cdots a_{2n+1}$ and $B_n = a_2 + a_4 + \cdots a_{2n}$. Then, using the given properties, we get

$$P(1) = 1 + A_n + B_n,$$
$$P(-1) = -1 + A_n - B_n,$$
$$P(1)P(-1) = (1 - x_1)(1 - x_2)\cdots(1 - x_{2n+1})(-1 - x_1)(-1 - x_2)\cdots(-1 - x_{2n+1}) = 3^{2n+1}.$$

Thus,

$$\begin{aligned} 4A_n^2 &= (P(1) + P(-1))^2 = (P(1) - P(-1))^2 + 4P(1)P(-1) \\ &= (2 + 2B_n)^2 + 4 \cdot 3^{2n+1} \\ &= (3^{2n+1} - 1)^2 + 4 \cdot 3^{2n+1} \\ &= (3^{2n+1} + 1)^2. \end{aligned}$$

We obtain that $A_n \in \{\pm\frac{3^{2n+1}+1}{2}\}$. Now we give two examples, one for each possibility.

Let's set $x_1 = x_2 = \cdots = x_{2n+1} = r$. Then, $(r-1)^{n+1}(r+1)^n = 3^n$ and $(r-1)^n(r+1)^{n+1} = 3^{n+1}$. Hence, $\frac{r+1}{r-1} = 3$ gives $r = 2$. Thus, the polynomial $P(x) = (x-2)^{2n+1}$ satisfies $A_n = \frac{P(1)+P(-1)}{2} = -\frac{3^{2n+1}+1}{2}$ and $B_n = \frac{P(1)-P(-1)-2}{2} = \frac{3(3^{2n}-1)}{2}$.

Next, we set $x_1 = x_2 = \cdots = x_{2n-2} = 2$, $x_{2n-1} = s$, $x_{2n} = t$, $x_{2n+1} = u$, and $A_n = \frac{3^{2n+1}+1}{2}$. Then, $P(-1) = -1 + A_n - B_n = 1$, and we have

$$\begin{cases} (s-1)(t+1)(u-1) = 3 \\ (s+1)(t-1)(u+1) = 9 \\ (s+1)(t+1)(u+1) = -\frac{1}{3^{2n-2}}. \end{cases}$$

We get $\frac{t-1}{t+1} = -3^{2n}$, so $t = \frac{1-3^{2n}}{1+3^{2n}}$. Then,

$$\begin{cases} (s-1)(u-1) = \frac{3+3^{2n+1}}{2} \\ (s+1)(u+1) = -\frac{9+3^{2-2n}}{2}. \end{cases}$$

We obtain $su = \frac{3^{2n+1}-3^{2-2n}-10}{4}$ and $s+u = -\frac{3^{2n+1}+3^{2-2n}+12}{4}$. So s and u are solutions of $(x-t)(x-u) = 0$, a quadratic equation with real coefficients. Therefore, the polynomial $P(x) = (x-2)^{2n-2}(x-s)(x-t)(x-u)$ has all the desired properties.

Also solved by Andrew Hwang, Langley High School, McLean, VA, USA; Srijan Sundar, Oxford, UK; Corneliu Mănescu-Avram, Ploieşti, Romania; Sai Gokul Atmakur, Indian Institute of Technology, Bhubaneswar, India; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Sundaresh Harige, India; Theo Koupelis, Cape Coral, FL, USA.

S644. Let $a \geq b \geq c \geq d \geq e \geq 0$ such that $ab + bc + cd + de + ea = 5$. Prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 + 5(a + b + c + d + e) \geq 30$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

Solution by the author

Denote

$$x = \frac{a+b}{2}, \qquad y = \frac{d+e}{2}, \qquad x \geq c \geq y.$$

Since

$$a^2 + b^2 \geq 2x^2, \qquad d^2 + e^2 \geq 2y^2,$$

it suffices to show that

$$2(x^2 + y^2) + 10(x + y) + c^2 + 5c \geq 30.$$

Moreover, since $c^2 \geq 2c - 1$, it suffices to show that

$$2(x^2 + y^2) + 10(x + y) + 7c \geq 31.$$

We will show first that

$$x^2 + y^2 + xy + c(x + y) \geq 5.$$

Indeed, we have

$$4[x^2 + y^2 + xy + c(x + y) - 5] = (a + b)^2 + (d + e)^2 + (a + b)(d + e)$$
$$+2c(a + b + d + e) - 4(ab + bc + cd + de + ea)$$
$$= (a - b)^2 + (d - e)^2 + a(d + 2c - 3e) + b(d + e - 2c) + 2c(e - d)$$
$$\geq b(d + 2c - 3e) + b(d + e - 2c) + 2c(e - d) = 2b(d - e) + 2c(e - d) = 2(d - e)(b - c) \geq 0.$$

So, it suffices to show that

$$2(x^2 + y^2) + 10(x + y) + \frac{7(5 - x^2 - y^2 - xy)}{x + y} \geq 31.$$

Denoting

$$s = \frac{x+y}{2}, \qquad p = xy \qquad (p \leq s^2),$$

the desired inequality becomes

$$8s^2 - 4p + 20s + \frac{7(5 - 4s^2 + p)}{2s} \geq 31,$$
$$16s^3 + 12s^2 - 62s + 35 \geq p(8s - 7).$$

For $8s - 7 \leq 0$, it suffices to show that $16s^3 + 12s^2 - 62s + 35 \geq 0$. Indeed,

$$16s^3 + 12s^2 - 62s + 35 = s(4s - 3)^2 + (1 - s)(35 - 36s) > 0.$$

Also, for $8s - 7 \geq 0$, we have

$$16s^3 + 12s^2 - 62s + 35 - p(8s - 7) \geq 16s^3 + 12s^2 - 62s + 35 - s^2(8s - 7)$$
$$= 8s^3 + 19s^2 - 62s + 35 = (s - 1)^2(8s + 35) \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = d = e = 1$.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Polyahedra, Polk State College, USA; Jean Heibig, ISAE-SUPAERO, Toulouse, France; Sai Gokul Atmakur, Indian Institute of Technology, Bhubaneswar; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S645. Let ABC be a triangle with sidelengths $BC = a$, $CA = b$, $AB = c$, inradius r, circumradius R, and area K. Let A', B', C' be the tangency points of the incircle of the triangle with sides BC, CA, AB, respectively.

(i) Prove that one can construct a triangle Δ with sidelengths $a \cdot AA', b \cdot BB', c \cdot CC'$.

(ii) Let K' denote the area of Δ. Prove that

$$9r^2 \le \frac{K'}{K} \le \frac{9}{4}R^2.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by the author

(i) It is sufficient to prove the vectorial relation

$$a \cdot \overline{AA'} + b \cdot \overline{BB'} + c \cdot \overline{CC'} = \overline{0}. \quad (1)$$

If $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of triangle ABC, then we have $BA' = s-b, CB' = s-c, AC' = s-a$, hence

$$\overline{AA'} = \overline{AB} + \overline{BA'} = \overline{AB} + \frac{s-b}{a}\overline{BC}.$$

It follows

$$a \cdot \overline{AA'} = a \cdot \overline{AB} + (s-b)\overline{BC}, \quad (2)$$

and similarly $b \cdot \overline{BB'} = b \cdot \overline{BC} + (s-c)\overline{CA}, c \cdot \overline{CC'} = c \cdot \overline{CA} + (s-a)\overline{AB}$. Summing the last three relations we obtain (i).

(ii) The relation (2) is equivalent to $a \cdot \overline{AA'} = (s-c)\overline{AB} + (s-b)\overline{AC}$. Using the properties of the cross product we have

$$2K' = |a \cdot \overline{AA'} \times c \cdot \overline{CC'}| = |((s-c)\overline{AB} + (s-b)\overline{AC}) \times ((s-a)\overline{AB} - c \cdot \overline{AC})| =$$

$$((s-a)(s-b) + c(s-c))|\overline{AC} \times \overline{AB}| = \frac{1}{4}(2ab + 2bc + 2ca - a^2 - b^2 - c^2)2K =$$

$$(ab + bc + ca - s^2)2K = (s^2 + r^2 + 4Rr - s^2)2K = (r^2 + 4Rr)2K,$$

where we have used the well-known relation $ab + bc + ca = s^2 + r^2 + 4Rr$. Therefore

$$\frac{K'}{K} = r^2 + 4Rr,$$

and the inequalities follow by applying the well-known Euler's inequality $R \ge 2r$.

Also solved by Sicheng Du, Shenzhen Middle School, Shenzhen, China; Sai Gokul Atmakur, Indian Institute of Technology, Bhubaneswar, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S646. Let a, b, c be positive real numbers such that $ab + bc + ca = 2(a + b + c)$. Prove that

$$\frac{a}{b^2+4} + \frac{b}{c^2+4} + \frac{c}{a^2+4} \geq \frac{3}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

By the AM-GM inequality we have

$$\begin{aligned}
\frac{a}{b^2+4} + \frac{b}{c^2+4} + \frac{c}{a^2+4} &= \frac{1}{4}\left(a - \frac{ab^2}{b^2+4}\right) + \frac{1}{4}\left(b - \frac{bc^2}{c^2+4}\right) + \frac{1}{4}\left(c - \frac{ca^2}{a^2+4}\right) \\
&= \frac{a+b+c}{4} - \frac{1}{4}\left(\frac{ab^2}{b^2+4} + \frac{bc^2}{c^2+4} + \frac{ca^2}{a^2+4}\right) \\
&\geq \frac{a+b+c}{4} - \frac{1}{4}\left(\frac{ab^2}{4b} + \frac{bc^2}{4c} + \frac{ca^2}{4a}\right) \\
&= \frac{a+b+c}{4} - \frac{ab+bc+ca}{16} \\
&= \frac{a+b+c}{8}.
\end{aligned}$$

On the other hand, from the given condition combining the basic result we obtain

$$2(a+b+c) = ab + bc + ca \leq \frac{1}{3}(a+b+c)^2.$$

Consequently $a + b + c \geq 6$. Combining these relations we get the desired inequality.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S647. Let ABC be an acute triangle with the circumcircle Γ. The tangents to the circle Γ in B and C intersect at the point X and AX cuts for the second time the circle Γ in Y. If M is the midpoint of side BC and D is the point of intersection between YM and the perpendicular from A to BC, show that $\triangle ABC \equiv \triangle DBC$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by the author

Denote: $AX \cap BC = \{Z\}$

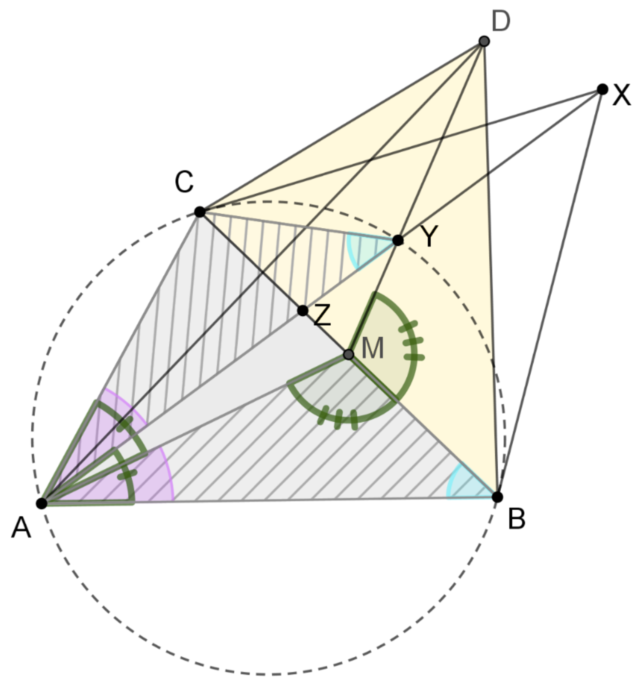


Figura 1:

- Calculating the ratio $\dfrac{BZ}{ZC}$, prove that AX is symmedian

$$\frac{BZ}{ZC} = \frac{\sigma(BXZ)}{\sigma(CXZ)} = \frac{BX \cdot XZ \cdot \sin(BXZ) \cdot \frac{1}{2}}{XZ \cdot CX \cdot \sin(CXZ) \cdot \frac{1}{2}} \text{ where } CX = BX \text{ (as tangents to } \Gamma) \Rightarrow \frac{BZ}{CZ} = \frac{\sin(BXZ)}{\sin(CXZ)}$$

Apply the law of sines in $\triangle ABX$, $\triangle ACX$ and calculate the ratio $\dfrac{\sin(BXZ)}{\sin(CXZ)}$

$$\left.\begin{aligned} \frac{AB}{\sin(BXZ)} &= \frac{AX}{\sin(ABX)} = \frac{AX}{\sin(B+A)} = \frac{AX}{\sin C} \\ \frac{AC}{\sin(CXZ)} &= \frac{AX}{\sin(ACX)} = \frac{AX}{\sin(C+A)} = \frac{AX}{\sin B} \end{aligned}\right\} \Rightarrow$$

$$\frac{\sin(BXZ)}{\sin(CXZ)} = \frac{AB \sin C}{AC \sin B} = \frac{AB^2}{AC^2} \Rightarrow AX = \text{ symmedian}$$

- Show that $\triangle ABC \equiv \triangle ADC$
  AX = symmedian $\Rightarrow$ AX and AM are isogonal lines $\Rightarrow$ $\measuredangle(ZAB) = \measuredangle(MAC)$ and $\measuredangle(YAC) = \measuredangle(MAB)$.

$$\left.\begin{array}{l} YAC \equiv \measuredangle MAB \\ \measuredangle AYC = \measuredangle ABC = \dfrac{\overset{\frown}{AC}}{2} \end{array}\right\} \Rightarrow \triangle BAM \sim \triangle AYC \Rightarrow \frac{BM}{YC} = \frac{AM}{AC}$$

$$\left.\begin{array}{l} \dfrac{BM}{YC} = \dfrac{AM}{AC} \\ BM = MC \end{array}\right\} \Rightarrow \frac{MC}{YC} = \frac{AM}{AC}$$

  From $\measuredangle YCM = \measuredangle YAB = \measuredangle MAC \Rightarrow \triangle AMC \sim \triangle CNY \Rightarrow \measuredangle AMC \equiv \measuredangle CMY \Rightarrow \measuredangle AMB \equiv \measuredangle BMD$
  So. in ADM, MD is both bisector and altitude at the same time $\Rightarrow AMD$ is an isosceles triangle and D is the symmetrical point of A with respect to $BC \Rightarrow \triangle ABC \equiv \triangle ADC$.

Also solved by Farmonov Sukhrobjon, Uzbekistan, Michel Faleiros Martins, São Paulo, SP, Brazil; Andrew Hwang, Langley High School, McLean, VA, USA; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Titu Zvonaru, Comănești, Romania.

S648. Let a,b,c,d be nonnegative real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{a^2+4}+\frac{b}{b^2+4}+\frac{c}{c^2+4}+\frac{d}{d^2+4}\le\frac{1}{5}+\frac{ab+ac+ad+bc+bd+cd}{10}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

Solution by the author

Without loss of generality, we may assume that $a\le b\le c\le d$. The inequality can be rewritten as

$$\sum_{cyc}\frac{a}{a^2+4}+\frac{a^2+b^2+c^2+d^2}{20}\le 1.$$

On other hand we have

$$\frac{x}{x^2+4}+\frac{x^2}{20}-\frac{11x}{50}-\frac{3}{100}=\frac{(x-1)^2(5x^2-12x-12)}{100(x^2+4)}\le 0$$

for $x\in[0,3]$. Therefore we have two cases:

1. If $d\le 3$ then

$$\sum_{cyc}\frac{a}{a^2+4}+\frac{a^2+b^2+c^2+d^2}{20}\le\frac{11(a+b+c+d)}{50}+\frac{12}{100}=1.$$

The equality holds when $a=b=c=d=1$.

2. If $d>3$ then $a+b+c<1$ so $a,b,c\in[0,1)$. We rewrite the inequality as follows

$$\sum_{cyc}\frac{4a}{a^2+4}+\frac{a^2+b^2+c^2+d^2}{5}\le 4,$$

$$\frac{4d}{d^2+4}+\frac{a^2+b^2+c^2+d^2}{5}\le 1+\frac{(2-a)^2}{a^2+4}+\frac{(2-b)^2}{b^2+4}+\frac{(2-c)^2}{c^2+4},$$

$$\frac{4d}{d^2+4}+\frac{d^2}{5}-1\le\frac{(2-a)^2}{a^2+4}+\frac{(2-b)^2}{b^2+4}+\frac{(2-c)^2}{c^2+4}-\frac{a^2+b^2+c^2}{5}.$$

By Cauchy-Scwarz Inequality and $a^2+b^2+c^2\le(a+b+c)^2$, we have

$$\begin{aligned}\frac{(2-a)^2}{a^2+4}+\frac{(2-b)^2}{b^2+4}+\frac{(2-c)^2}{c^2+4}-\frac{a^2+b^2+c^2}{5}&\ge\frac{(6-a-b-c)^2}{a^2+b^2+c^2+12}-\frac{a^2+b^2+c^2}{5}\\&\ge\frac{(d+2)^2}{(a+b+c)^2+12}-\frac{(a+b+c)^2}{5}\\&=\frac{(d+2)^2}{(4-d)^2+12}-\frac{(4-d)^2}{5}.\end{aligned}$$

It remains to show that

$$\frac{4d}{d^2+4}+\frac{d^2}{5}-1\le\frac{(d+2)^2}{(4-d)^2+12}-\frac{(4-d)^2}{5},$$

or after expanding

$$\frac{2(4-d)(d-3)(d^4-5d^3+20d^2-4d+48)}{5(d^2+4)(d^2-8d+28)}\ge 0$$

clearly true for $d\in(3,4]$. The equality holds when $a=b=c=0$, $d=4$.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U643. Let $(x_n)_{n\geq 2}$ be the sequence defined by

$$x_n = \frac{\sqrt[n]{e} - 1}{\sqrt[n^2]{e} - 1} - n.$$

Evaluate $\lim\limits_{n\to\infty} n\left(x_n - \frac{1}{2}\right)$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ştefan Marinescu, Hunedoara, România

Solution by the authors

We shall compute

$$L = \lim_{t\to 0} \frac{2te^t - 2t - 2e^{t^2} + 2 - te^{t^2} + t}{2t^2(e^{t^2} - 1)} = \frac{1}{2}\lim_{t\to 0} \frac{2te^t - 2e^{t^2} + 2 - te^{t^2} - t}{t^4}.$$

Applying the l'Hospital rule and using the above mentioned limit, one obtains

$$2L = \lim_{t\to 0} \frac{2e^t + 2te^t - 4te^{t^2} - e^{t^2} - 2t^2e^{t^2} - 1}{4t^3} = \lim_{t\to 0} \frac{2e^t + 2te^t - 4t - e^{t^2} - 2t^2 - 1}{4t^3} -$$

$$\lim_{t\to 0} \frac{e^{t^2} - 1}{t^2} - \lim_{t\to 0} 2t\frac{e^{t^2} - 1}{4t^2} = \lim_{t\to 0} \frac{2e^t + 2te^t - 4t - e^{t^2} - 2t^2 - 1}{4t^3} - 1 =$$

$$\lim_{t\to 0} \frac{4e^t - 4t - 4 + 2te^t - 2te^{t^2}}{12t^2} - 1 = \lim_{t\to 0} \frac{e^t - t - 1}{3t^2} + \lim_{t\to 0} \frac{1}{6}t\frac{e^t - 1}{t^2} - \lim_{t\to 0} \frac{1}{6}t\frac{e^{t^2} - 1}{t^2} - 1 =$$

$$\frac{1}{6} + \frac{1}{6} - 1 = -\frac{2}{3}.$$

The desired limit is $L = -\frac{1}{3}$.

Remark: Another way to solve the problem is to use the series expansion of the function e^t.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Arkady Alt, San Jose, CA, USA; Srijan Sundar, Oxford, UK; Daniel Pascuas, Barcelona, Spain; Ángel Plaza; Universidad de Las Palmas de Gran Canaria, Spain; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle; Corneliu Mănescu-Avram, Ploieşti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Sundaresh Harige, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U644. Let $f\colon[-1,1]\to\mathbb{R}$ a function three times continuously differentiable such that $f(-1)=f(1)=f''(-1)=0$. Prove that

$$\left(\int_{-1}^{1} f(x)dx\right)^2 \le \frac{104}{315}\int_{-1}^{1}\left(f'''(x)\right)^2 dx$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Solution by the author

$$\begin{aligned}
&\int_{-1}^{1} f(x)dx = xf(x)\Big|_{-1}^{1} - \int_{-1}^{1} xf'(x)dx = -\int_{-1}^{1} xf'(x)dx = \\
&-\left(\frac{x^2}{2}-\frac{1}{2}\right)f'(x)\Big|_{-1}^{1} + \int_{-1}^{1}\left(\frac{x^2}{2}-\frac{1}{2}\right)f''(x)dx = \int_{-1}^{1}\left(\frac{x^2}{2}-\frac{1}{2}\right)f''(x)dx = \\
&= \left(\frac{x^3}{6}-\frac{x}{2}+\frac{1}{3}\right)f''(x)\Big|_{-1}^{1} - \int_{-1}^{1}\left(\frac{x^3}{6}-\frac{x}{2}+\frac{1}{3}\right)f'''(x)dx = \\
&= -\int_{-1}^{1}\left(\frac{x^3}{6}-\frac{x}{2}+\frac{1}{3}\right)f'''(x)dx
\end{aligned}$$

By Cauchy–Schwarz

$$\left(\int_{-1}^{1} f(x)dx\right)^2 \le \int_{-1}^{1}\left(\frac{x^3}{6}-\frac{x}{2}+\frac{1}{3}\right)^2 dx\cdot\int_{-1}^{1}\left(f'''(x)\right)^2 dx = \frac{104}{315}\int_{-1}^{1}\left(f'''(x)\right)^2 dx$$

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Daniel Pascuas, Barcelona, Spain; Thammadol Tansrivorarat, Bristol, UK; G. C. Greubel, Newport News, VA, USA; Joshua Pité, Cambridge Rindge and Latin School, MA, USA; Theo Koupelis, Cape Coral, FL, USA.

U645. Find all non-zero polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P\left((Q(x))^3\right) = x^2P(x)(Q(x))^2,\ \forall x \in \mathbb{R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Matthew Too, Brockport, NY, USA

Let $\deg(P) = m$ and $\deg(Q) = n$ for non-negative integers m and n. Then $\deg(P\left((Q(x))^3\right)) = 3mn$ and $\deg(x^2P(x)(Q(x))^2) = m + 2n + 2$. Setting the degrees equal to each other leads to

$$m + 2n + 2 = 3mn \Longrightarrow n = \frac{m+2}{3m-2}$$

where $m + 2 \geq 3m - 2$, or $m \leq 2$, is required for n to be a non-negative integer. Testing the values $m = 0, 1, 2$ then gives $(m, n) \in \{(1, 3), (2, 1)\}$ as the only possible solutions.

For $(m, n) = (1, 3)$, let $P(x) = a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ with $a_1, b_3 \neq 0$. Substituting these polynomials into the original equation and equating coefficients gives the system

$$\begin{cases} a_1b_3^3 = a_1b_3^2 \\ 3a_1b_2b_3^2 = a_0b_3^2 + 2a_1b_2b_3 \\ 3a_1b_1b_3^2 + 3a_1b_2^2b_3 = a_1b_2^2 + 2a_1b_1b_3 + 2a_0b_2b_3 \\ a_1b_2^3 + 3a_1b_0b_3^2 + 6a_1b_1b_2b_3 = a_0b_2^2 + 2a_1b_1b_2 + 2a_1b_0b_3 + 2a_0b_1b_3 \\ 3a_1b_1b_2^2 + 3a_1b_1^2b_3 + 6a_1b_0b_2b_3 = a_1b_1^2 + 2a_1b_0b_2 + 2a_0b_1b_2 + 2a_0b_0b_3 \\ 3a_1b_0b_2^2 + 3a_1b_1^2b_2 + 6a_1b_0b_1b_3 = a_0b_1^2 + 2a_1b_0b_1 + 2a_0b_0b_2 \\ a_1b_1^3 + 6a_1b_0b_1b_2 + 3a_1b_0^2b_3 = a_1b_0^2 + 2a_0b_0b_1 \\ 3a_1b_0b_1^2 + 3a_1b_0^2b_2 = a_0b_0^2 \\ 3a_1b_0^2b_1 = 0 \\ a_1b_0^3 + a_0 = 0 \end{cases}.$$

The first equation $a_1b_3^2(b_3 - 1) = 0$ implies $b_3 = 1$. This reduces the second equation to $a_0 = a_1b_2$, the third to $b_1 = 0$, the fourth to $b_0 = 0$, and the tenth to $a_0 = 0$, which further implies $b_2 = 0$. Thus, one solution is $P(x) = ax$ and $Q(x) = x^3$ for any $a \in \mathbb{R}_{\neq 0}$.

For $(m, n) = (2, 1)$, let $P(x) = a_2x^2 + a_1x + a_0$ and $Q(x) = b_1x + b_0$ with $a_2, b_1 \neq 0$. Substituting and equating coefficients gives

$$\begin{cases} a_2b_1^6 = a_2b_1^2 \\ 6a_2b_1^5b_0 = a_1b_1^2 + 2a_2b_0b_1 \\ 15a_2b_1^4b_0^2 = a_2b_0^2 + a_0b_1^2 + 2a_1b_0b_1 \\ 20a_2b_1^3b_0^3 + a_1b_1^3 = a_1b_0^2 + 2a_0b_0b_1 \\ 15a_2b_1^2b_0^4 + 3a_1b_1^2b_0 = a_0b_0^2 \\ 6a_2b_1b_0^5 + 3a_1b_1b_0^2 = 0 \\ a_2b_0^6 + a_1b_0^3 + a_0 = 0 \end{cases}.$$

The first equation $a_2b_1^2(b_1 - 1)(b_1 + 1)(b_1^2 + 1) = 0$ implies that $b_1 = 1$ or $b_1 = -1$. We consider the two cases separately.

If $b_1 = 1$, then the second equation reduces to $a_1 = 4a_2b_0$, the third to $a_0 = 6a_2b_0^2$, and the fourth to $4a_2b_0(b_0^2 + 1) = 0$, or equivalently, $b_0 = 0$. Thus, $a_1 = a_0 = 0$ and so $P(x) = ax^2$ and $Q(x) = x$ for any $a \in \mathbb{R}_{\neq 0}$ is a solution.

If $b_1 = -1$, then in a similar manner, the second equation gives $a_1 = -4a_2b_0$, the third $a_0 = 6a_2b_0^2$, and the fourth $4a_2b_0(b_0-1)(b_0+1) = 0$ which implies that $b_0 = 0$ or $b_0 = \pm 1$. However, if $b_0 = \pm 1$, then the fifth equation becomes $a_2 = 0$ which is a contradiction to $a_2 \neq 0$. Thus, $b_0 = 0$ which implies $a_1 = a_0 = 0$ and so $P(x) = ax^2$ and $Q(x) = -x$ for any $a \in \mathbb{R}_{\neq 0}$ is a solution.

Altogether, this means that the solution set consisting of non-zero polynomials is $(P(x), Q(x)) \in \{(ax, x^3), (ax^2, x), (ax$ $a \in \mathbb{R}_{\neq 0}\}$. A quick check shows that these polynomial pairs do satisfy the required equality.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Andrew Hwang, Langley High School, McLean, VA, USA; Srijan Sundar, Oxford, UK; Daniel Pascuas, Barcelona, Spain; Sundaresh Harige, India; Theo Koupelis, Cape Coral, FL, USA.

U646. Let $f:[0,1]\to[0,1)$ be an integrable function. Prove that

$$\lim_{n\to\infty}\int_0^1 f(x)^n\,dx = 0.$$

Proposed by Mihai Piticari and Sorin Rădulescu, România

Solution by Theo Koupelis, Clark College, WA, USA

Let $I=\lim_{n\to\infty}\int_0^1 f(x)^n\,dx$. Clearly $I\ge 0$, because $f(x)\ge 0$. Partitioning the interval $[0,1]$ into m equal intervals and using the Riemann sum $\int_0^1 g(x)\,dx=\lim_{m\to\infty}\sum_{k=1}^m g(x_k^*)\cdot\frac{1}{m}$, where x_k^* is the midpoint of $[x_{k-1},x_k]$, we get

$$I=\lim_{n\to\infty}\lim_{m\to\infty}\sum_{k=1}^m\frac{f(x_k^*)^n}{m}.$$

But $f(x_k^*)\in[0,1)$, and thus $\lim_{n\to\infty}f(x_k^*)^n=0$. Therefore, $I=0$.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Andrew Hwang, Langley High School, McLean, VA, USA; Daniel Pascuas, Barcelona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Soumyadeep Mandal, SVNIT, Surat, India; Sundaresh Harige, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U647. Let $\alpha \in (0,1)$, let $(p_k)_{k\geq 1}$ be the sequence of primes and let $q_n = \prod_{k\leq n} p_k$. Evaluate

$$\lim_{n\to\infty} \frac{\sum_{p|q_n} (\log p)^\alpha}{\omega(q_n)^{1-\alpha}(\log q_n)^\alpha}.$$

($\omega(n)$ denotes the number of distinct primes of a natural number n).

Proposed by Alessandro Ventulo, Milan, Italy

Solution by the author

Let $a_p = 1$ and $b_p = (\log p)^\alpha$. By Hölder's inequality, we have

$$\begin{aligned}\sum_{p|q} (\log p)^\alpha &\leq \left(\sum_{p|q} 1^{\frac{1}{1-\alpha}}\right)^{1-\alpha} \left(\sum_{p|q} ((\log p)^\alpha)^{\frac{1}{\alpha}}\right)^\alpha \\ &= (\omega(q))^{1-\alpha}\left(\log \prod_{p|q} p\right)^\alpha \leq (\omega(q))^{1-\alpha}(\log q)^\alpha.\end{aligned}$$

Since $p_n \geq n$ for all $n \in \mathbb{N}^*$, then $\log p_n \geq \log n$ and

$$\frac{\sum_{k=1}^n (\log k)^\alpha}{n(\log n)^\alpha} \leq \frac{\sum_{p|q_n} (\log p)^\alpha}{n(\log n)^\alpha} \leq \frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha}. \tag{3}$$

Let us prove that $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^n (\log k)^\alpha}{n(\log n)^\alpha} = 1$. Let $a_n = \sum\limits_{k=1}^n (\log k)^\alpha$ and $b_n = n(\log n)^\alpha$. We have

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\log(n+1)^\alpha}{(n+1)(\log(n+1))^\alpha - n(\log n)^\alpha} = 1,$$

so by the Stolz-Cesaro Theorem,

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Now, let us prove that $\lim\limits_{n\to\infty} \frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = 1$. We have

$$\lim_{n\to\infty} \frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = \lim_{n\to\infty} \frac{n^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = \lim_{n\to\infty}\left(\frac{\sum_{p\leq p_n} \log p}{n\log n}\right)^\alpha.$$

By the Prime Number Theorem, $\sum_{p\leq p_n} \log p \sim p_n$ and $n\log n \sim p_n$, so

$$\lim_{n\to\infty} \frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = 1.$$

Using these two limits in (3), by the Squeeze Theorem, we get

$$\lim_{n\to\infty} \frac{\sum_{p|q_n} (\log p)^\alpha}{n(\log n)^\alpha} = 1.$$

Also solved by Daniel Pascuas, Barcelona, Spain.

U648. Let T_n be a sequence defined by $T_1 = a$,

$$T_{n+1} = \frac{T_n^2}{2\sqrt{1+T_n^2}},\ n \ge 1$$

Evaluate $\displaystyle\prod_{n=1}^{\infty} \frac{2+T_n^2}{2T_n^2+2}$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

Solution by the author

$T_n = (\sinh(2^n x)^{-1}$ solves the recurrence

$$\frac{2\sqrt{1+T_n^2}}{T_n^2} = \frac{2\sqrt{1+(\sinh(2^n x))^{-2}}}{(\sinh(2^n x))^{-2}} = \frac{2\cosh(2^n x)}{\sinh(2^n x)(\sinh(2^n x))^{-2}} = \sinh(2^{n+1}x)$$

$$\prod_{n=1}^{N} \frac{2+T_n^2}{2T_n^2+2} = \prod_{n=1}^{N} \frac{2+\frac{1}{(\sinh(2^n x)^2}}{\frac{2}{(\sinh(2^n x)^2}+2} = \prod_{n=1}^{N} \frac{\cosh(2^{n+1}x)}{2(\sinh(2^n x)^2} \frac{(\sinh(2^n x)^2}{(\cosh(2^n x)^2} =$$

$$= \frac{1}{2^N} \frac{\cosh(2^{N+1}a)}{\frac{\sinh(4a)}{2\sinh(2a)} \frac{\sinh(4a)}{2\sinh(2a)} \frac{\sinh(8a)}{2\sinh(4a)} \cdots \frac{\sinh(2^{N+1}a)}{2\sinh(2^N a)}} = \frac{2(\sinh(2a))^2}{\sinh(4a)} \frac{\cosh(2^{N+1}a)}{\sinh(2^{N+1}a)}$$

and the limit for $N \to \infty$ equals

$$\tanh(2a) = \frac{1}{a} \frac{1}{\sqrt{1+\frac{1}{a^2}}} = \frac{1}{\sqrt{1+a^2}}$$

Also solved by Arkady Alt, San Jose, CA, USA; Prodromos Fotiadis, University of Crete, Greece; Michel Faleiros Martins, São Paulo, SP, Brazil; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Cape Coral, FL, USA.

O643. Let a, b, c, λ be positive real numbers such that

$$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda} \le \frac{1}{\lambda}.$$

Prove that

$$a+b+c+\frac{15abc}{ab+bc+ca} \ge 16\lambda.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

Solution by Michel Faleiros Martins, São Paulo, SP, Brazil

Due to homogeneity, we can fix $\lambda = 1$. After clearing denominators, the condition yields

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \le 1 \quad \Leftrightarrow \quad abc \ge a+b+c+2.$$

Let's define $A = ra$, $B = rb$ and $C = rc$ for some $r \in (0,1]$ such that $ABC = A+B+C+2$. This is possible because the polynomial $p(r) = r^3abc - r(a+b+c) - 2$ has a root in $(0,1]$ as a consequence of its continuity and the fact that $p(0) = -2$ and $p(1) = abc - (a+b+c) - 2 \ge 0$. Thus, the inequality can be rewritten as

$$A+B+C+\frac{15ABC}{AB+BC+CA} \ge 16r.$$

So, it suffices to show the last for $r = 1$. Let $x = \frac{1}{A+1}$, $y = \frac{1}{B+1}$ and $z = \frac{1}{C+1}$. Using that $x+y+z = 1$ we have $A = \frac{y+z}{x}$, $B = \frac{x+z}{y}$, $C = \frac{x+y}{z}$. Substituting these we wish to show that

$$\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}+\frac{15}{\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}} \ge 16.$$

Or, equivalently, after substituting $\alpha = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9$, $t = \frac{1}{xyz} = \frac{(x+y+z)^3}{xyz}$, and simplifying it, we get

$$\alpha - 3 + \frac{15(\alpha-1)}{t-2\alpha+3} \ge 16$$

We obtain $4t^2 - (\alpha^2+18\alpha-27)t + 4\alpha^3 \le 0$ after expanding $(x-y)^2(y-z)^2(z-x)^2 \ge 0$ and substituting α and t. Then,

$$t-2\alpha+3 \le \frac{\alpha^2+18\alpha-27+(\alpha-9)\sqrt{(\alpha-1)(\alpha-9)}}{8} - 2\alpha+3 = \frac{(\alpha+3)(\alpha-1)+(\alpha-9)\sqrt{(\alpha-1)(\alpha-9)}}{8}.$$

After clearing denominators, it is enough to show that

$$(\alpha-19)\left[(\alpha+3)(\alpha-1)+(\alpha-9)\sqrt{(\alpha-1)(\alpha-9)}\right]+120(\alpha-1) \ge 0.$$

If $\alpha = 9$ or $\alpha \ge 19$ it is clear. Otherwise, $9 < \alpha < 19$, so that

$$\begin{aligned}(\alpha-1)[120+(\alpha-19)(\alpha+3)] \ge (19-\alpha)(\alpha-9)\sqrt{(\alpha-1)(\alpha-9)} \quad &\Leftrightarrow \\ (\alpha-1)^2(\alpha-9)^2(\alpha-7)^2 \ge (\alpha-1)(\alpha-19)^2(\alpha-9)^3 \quad &\Leftrightarrow \\ (\alpha-1)(\alpha-7)^2 \ge (\alpha-19)^2(\alpha-9) \quad &\Leftrightarrow \\ 32(\alpha-10)^2 \ge 0.&\end{aligned}$$

Therefore, we are done. The equality holds when $(\alpha, t) = (9, 27)$ or $(\alpha, t) = (10, 32)$. In the first case, $x = y = z = \frac{1}{3}$, $A = B = C = 2$, $r = 1$, and finally $a = b = c = 2\lambda$. For the last case, $x+y+z = 1$, $xyz = \frac{1}{32}$, $xy+yz+zx = \frac{\alpha}{t} = \frac{5}{16}$, so, by Vieta's formulas, x, y, z are the roots of $s^3 - s^2 + \frac{5}{16}s - \frac{1}{32} = \left(s-\frac{1}{2}\right)\left(s-\frac{1}{4}\right)^2 = 0$. Then, $A = B = 3$, $C = 1$ and their permutations, $r = 1$, and finally, $a = b = 3\lambda$, $c = \lambda$ and their permutations.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O644. Prove that $k=4$ is the smallest positive constant k such that

$$\left(\frac{ka_1+a_2+\cdots+a_9}{k+8}\right)^2 \geq \frac{a_1a_2+a_2a_3+\cdots+a_9a_1}{9}$$

whenever $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$.

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, România

Solution by the author

For $a_1=\cdots=a_5=1$ and $a_6=a_7=a_8=a_9=0$, the inequality leads to the necessary condition $k\geq 4$. To show that 4 is the smallest value of k, we need to prove that $F(a_1,a_2,\ldots,a_9)\geq 0$, where

$$F(a_1,a_2,\ldots,a_9)=(4a_1+a_2+\cdots+a_9)^2-16(a_1a_2+a_2a_3+\cdots+a_9a_1).$$

We will show that

$$F(a_1,a_2,a_3,\ldots,a_9)\geq F(a_2,a_2,a_3,\ldots,a_9)\geq\cdots\geq F(a_8,a_8,\ldots,a_8,a_9)\geq F(a_9,a_9,\ldots,a_9,a_9)=0,$$

i.e.

$$F(a_i,\ldots,a_i,a_{i+1},\ldots,a_9)\geq F(a_{i+1},\ldots,a_{i+1},a_{i+2},\ldots,a_9),\qquad i\in\{1,2,\ldots,8\}.$$

Write this inequality as follows:

$$[(i+3)a_i+a_{i+1}+\cdots+a_9]^2-16[(i-1)a_i^2+a_ia_{i+1}+\cdots+a_9a_i]\geq$$

$$\geq[(i+4)a_{i+1}+a_{i+2}+\cdots+a_9]^2-16[ia_{i+1}^2+a_{i+1}a_{i+2}+\cdots+a_9a_{i+1}],$$

$$(i+3)(a_i-a_{i+1})[(i+3)a_i+(i+5)a_{i+1}+2a_{i+2}+\cdots+2a_9]\geq$$

$$\geq 16[(i-1)(a_i^2-a_{i+1}^2)+a_{i+1}(a_i-a_{i+1})+a_9(a_i-a_{i+1})]\geq 0,$$

$$(a_i-a_{i+1})E_i\geq 0,$$

where

$$E_i=(i-5)^2a_i+(i^2-8i+15)a_{i+1}+2(i+3)(a_{i+2}+\cdots+a_8)+2(i-5)a_9.$$

Since

$$(i-5)^2a_i+(i^2-8i+15)a_{i+1}\geq(i-5)^2a_{i+1}+(i^2-8i+15)a_{i+1}=2(i-4)(i-5)a_{i+1}\geq 0,$$

it suffices to show that

$$(i+3)(a_{i+2}+\cdots+a_8)+(i-5)a_9\geq 0.$$

This is true for $i\geq 5$, while for $i\leq 4$ we have

$$(i+3)(a_{i+2}+\cdots+a_8)+(i-5)a_9\geq(i+3)(7-i)a_9+(i-5)a_9\geq 3(i+3)a_9+(i-5)a_9=4(i+1)a_9\geq 0.$$

For $k=4$, the equality occurs when $a_1=a_2=\cdots=a_9$, and also when $a_1=\cdots=a_5$ and $a_6=a_7=a_8=a_9=0$.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA.

O645. Find all completely multiplicative functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(a^2 + b^2 + c^2) = f(ab + bc + ca - 2)$$

for any a, b, c positive integers.

Proposed by Titu Andreescu, USA and Vlad Matei, România

Solution by the authors

The only solution is $f \equiv 1$. First let us note that for $f(1) = 1$ since f is completely multiplicative. For $a = b = c = 1$ we obtain that $f(3) = f(1) = 1$ and for $a = 2, b = c = 1$ we obtain $f(6) = f(3)$ thus $f(2) = 1$, using the multiplicativity of the function.

We are now ready to show $f(n) = 1$ by using strong induction on n. The bases cases $n = 1, 2, 3$ are done by the above argument.

Assume that we know it for all $1 \le n < k$ and let us prove it for k. If k is a sum of three positive squares, wlog say $k = a^2 + b^2 + c^2$, then $f(k) = f(a^2 + b^2 + c^2) = f(ab + bc + ca - 2)$. Note that $ab + bc + ca \le a^2 + b^2 + c^2$ and thus $ab + bc + ca - 2 < a^2 + b^2 + c^2$ and thus by the strong induction hypothesis $f(ab + bc + ca - 2) = 1$ and we are done.

If k is a positive square say $k = a^2$ then $f(a^2) = (f(a))^2$ and since $a < a^2$ we have $f(a) = 1$ thus $f(a^2) = 1$.

We remain with k a sum of two positive squares. If k is not prime then once again, since f is multiplicative and we can use the strong induction hypothesis. Thus k is prime and $k \equiv 1 \pmod 4$.

Let us note that for $(a, b) = (1, 2), (4, 5), (6, 13)$ we obtain the following three relations, denoted by $\star$,

$$f(c^2 + 5) = f(3c) = f(3)f(c) = f(c)$$

$$f(c^2 + 41) = f(9(c + 2)) = f(9)f(c + 2) = f(c + 2)$$

$$f(c^2 + 5 \cdot 41) = f(19)f(c + 4) = f(c + 4)$$

To justify $f(19) = 1$ note that $f(19) = f(38) = f(6^2 + 1 + 1) = f(11) = f(3^2 + 1 + 1) = f(5)$. To finish note that $f(5^3) = f(10^2 + 3^2 + 4^2) = f(80)$ thus $(f(5))^3 = f(5) \cdot (f(2))^4$ and we already know $f(2) = 1$. So $f(5) = 1$.

We are now ready to show $f(k) = 1$. Note that at least one of the numbers 5, 41 or $5 \cdot 41$ has to be quadratic residue modulo k. Since $k \equiv 1 \pmod 4$ this means at least one of the numbers $-5, -41, -5 \cdot 41$ is a quadratic residue. Say $\alpha \in \{-5, -41, -5 \cdot 41\}$ is this quadratic residue.

We know that we can find $1 \le c \le \dfrac{k-1}{2}$ such that $k | c^2 + \alpha$. Using $\star$ we conclude there is $\beta \le 4$ such that

$$f(c^2 + \alpha) = f(c + \beta).$$

Since $c + \beta \le \dfrac{k-1}{2} + 4 < k$ if $k \ge 13$ (we have already verified $f(5) = 1$) by the induction hypothesis we have $f(c + \beta) = 1$. It remains to note that

$$f(c^2 + \alpha) = f(k)f(2)f\left(\frac{c^2 + \alpha}{2k}\right) = f(k)$$

since $\dfrac{c^2 + \alpha}{2k} \le \dfrac{(k-1)^2}{8k} + \dfrac{201}{2k} < k$ for $k \ge 13$ and thus $f\left(\frac{c^2+\alpha}{2k}\right) = 1$, using the induction hypothesis.

We are now left with all the values k that cannot be written as a sum of three integer squares. We shall use the following well known fact due to Lagrange: a positive integer x is a sum of three squares if and only if $x \neq 4^y(8z + 7)$.

If k is not prime we can easily finish using the induction hypothesis and the multiplicativity of the function. We remain with k prime and $k \equiv 7 \pmod 8$ from the above.

Note that $3k \equiv 5 \pmod 8$ thus using the above theorem we know that we can find three nonnegative integers r, s, t such that $3k = r^2 + s^2 + t^2$. Note that none of them can be zero. Otherwise wlog say $r = 0$ then $3|s^2 + t^2$ thus $3|s, t$. This implies $3|k$ so $k = 3$ which is false. By the functional equation

$$f(3p) = f(3)f(p) = f(p) = f(r^2 + s^2 + t^2) = f(rs + st + rt - 2).$$

Since $r^2 + s^2 + t^2 \equiv 5 \pmod 8$ and $x^2 \equiv 0, 1, 4 \pmod 8$ for an integer x we conclude wlog that $4|r$, $s \equiv 2 \pmod 4$ and t is odd. Thus $4|rs + st + rt - 2$. Using the multiplicativity of the function

$$f(rs + st + rt - 2) = f(4) \cdot f((rs + st + rt - 2)/4) = 1$$

since $f(4) = 1$ and $\frac{rs + st + rt - 2}{<}\frac{r^2 + s^2 + t^2}{4} = \frac{3k}{4} < k$.

This ends the proof of the induction step and we conclude that the only function is $f \equiv 1$.

Remark: The case when k is a prime which is 1 mod 4 would have been easier to deal with if we knew that in this case k could also be written as a sum of three nonzero squares. Conditional on the Extended Riemann Hypothesis, the set of exceptions, i.e those that cannot be written, is the finite set $\{5, 13, 37\}$. See for example the MathOverflow discussion here https://mathoverflow.net/questions/90914/sums-of-three-non-zero-squares.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Sicheng Du, Shenzhen Middle School, Shenzhen, China.

O646. Let x, y, z be non–zero real numbers such that $x+y+z=xyz$. Prove that

$$\left|x+y+z-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right| \geq 2\sqrt{3}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Let $x, y, z>0$ and define $a=1/x$, $b=1/y$, $c=1/z$. The condition $x+y+z=xyz$ becomes $ab+bc+ca=1$ and the inequality becomes

$$\left|a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right| \geq 2\sqrt{3} \tag{1}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq a+b+c \iff ab+bc+ca \geq (a+b+c)abc \iff (a+b+c)abc \leq 1$$

We know that $ab+bc+ca \geq \sqrt{3abc(a+b+c)}$ hence $\sqrt{3abc(a+b+c)} \leq 1$ and then $abc(a+b+c) \leq 1$. It follows that (1) is actually

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-a-b-c \geq 2\sqrt{3} \iff \frac{ab+bc+ca}{abc}-(a+b+c) \geq 2\sqrt{3}$$
$$\iff \frac{1}{abc}-(a+b+c) \geq 2\sqrt{3}$$

$\sqrt{3abc(a+b+c)} \leq 1$ yields $\dfrac{1}{abc} \geq 3(a+b+c)$ thus we are led to prove

$$3(a+b+c)-(a+b+c) \geq 2\sqrt{2} \iff a+b+c \geq \sqrt{3}$$

but this follows by $(a+b+c)^2 \geq 3(ab+bc+ca)=3$. This proof clearly covers also the $x, y, z<0$ case. The equality holds true when $ab+bc+ca \geq \sqrt{3abc(a+b+c)}$ and $(a+b+c)^2 \geq 3(ab+bc+ca)$ hence $a=b=c$ and $ab+bc+ca=1$ yields $a=b=c=1/\sqrt{3}$. Of course there is also the other equality case $(a, b, c)=-1/\sqrt{3}$. In terms of the variables (x, y, z) we get $(x, y, z)=\sqrt{3}(1,1,1)$ or $(x, y, z)=-\sqrt{3}(1,1,1)$

Let $x, y>0$ and $z<0$. This case covers also the $x>0$ and $y, z<0$ case. As before let's set $a=1/x$, $b=1/y$, $c=1/z$. Form $ab+bc+ca=1$ we get $c=(1-ab)/(a+b)$ and the inequality

$$\left|a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right| \geq 2\sqrt{3} \iff \left|a+b+\frac{1-ab}{a+b}-\frac{1}{a}-\frac{1}{b}-\frac{a+b}{1-ab}\right| \geq 2\sqrt{3} \tag{2}$$

By defining the new variables $a+b=2u$, $ab=v^2$ (2) reads as

$$\left|2u-\frac{1-v^2}{2u}-\frac{2u}{v^2}-\frac{2u}{1-v^2}\right| \geq 2\sqrt{3}$$

which is

$$\left|\frac{4u^2v^2-4u^2v^4+v^2-2v^4+v^6-4u^2}{2uv^2(v^2-1)}\right| \geq 2\sqrt{3}$$

that is

$$\left|\frac{v^2(1-v^2)^2+4u^2(v^2-v^4-1)}{2uv^2(v^2-1)}\right| \geq 2\sqrt{3} \tag{3}$$

$u \geq v$ by the AGM and $v>1$ by $c<0$.

$$4u^2(v^4-v^2+1) \geq 4v^2(v^4-v^2+1) \geq v^2(1-2v^2+v^4) \iff 3v^4-2v^2+3 \geq 0$$

and this holds true by $v>1$ hence (3) is

$$f(u) \doteq 4u^2(v^4+1-v^2)-v^2(1-v^2)^2-4\sqrt{3}uv^2(v^2-1) \geq 0$$

The minimum of the parabola $f(u)$ occurs for $u = \dfrac{4\sqrt{3}v^2(v^2-1)}{4(v^4+1-v^2)}$ but it cannot be reached by u because

$$\frac{4\sqrt{3}v^2(v^2-1)}{8(v^4+1-v^2)} \leq v \iff \frac{v(\sqrt{3}v^3-\sqrt{3}v-2v^4+2v^2-2)}{2(v^4-v^2+1)} \leq 0$$

This may be seen by observing that $v \geq 1$ and

$$\sqrt{3}v^3-\sqrt{3}v-2v^4+2v^2-2 = -2-v(v^2-1)(2v-\sqrt{3}) < 0$$

It follows that

$$f(u) \geq f(v) = \frac{1}{3}(3v+\sqrt{3})^2(v-\sqrt{3})^2v^2 \geq 0$$

and the equality cases occur when $u = v = \sqrt{3}$ hence $a = b = \sqrt{3}$, $c = \dfrac{1-ab}{a+b} = -1/\sqrt{3}$ yielding $(x,y,z) = (1/\sqrt{3}, 1/\sqrt{3}, -\sqrt{3})$ and clearly the opposite $(x,y,z) = (-1/\sqrt{3}, -1/\sqrt{3}, \sqrt{3})$

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Sicheng Du, Shenzhen Middle School, Shenzhen, China; Theo Koupelis, Cape Coral, FL, USA.

O647. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{a^2+3} + \frac{b}{b^2+3} + \frac{c}{c^2+3} \leq \frac{1}{4} + \frac{ab+bc+ca}{6}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

Solution by the author

The inequality can be rewritten as

$$\sum_{cyc} \frac{3a}{3a^2+(a+b+c)^2} \leq \frac{3}{4(a+b+c)} + \frac{9(ab+bc+ca)}{2(a+b+c)^3},$$

or

$$\sum_{cyc} \frac{a(a+b+c)}{3a^2+(a+b+c)^2} \leq \frac{1}{4} + \frac{3(ab+bc+ca)}{2(a+b+c)^2},$$

or

$$\sum_{cyc} \left[\frac{1}{4} - \frac{a(a+b+c)}{3a^2+(a+b+c)^2}\right] \geq \frac{1}{2} - \frac{3(ab+bc+ca)}{2(a+b+c)^2},$$

or

$$\sum_{cyc} \left[\frac{b^2+c^2-2ab-2ac+2bc}{4(3a^2+(a+b+c)^2)}\right] \geq \frac{1}{2} - \frac{3(ab+bc+ca)}{2(a+b+c)^2},$$

or

$$\sum_{cyc} \left[\frac{(b-c)^2+2b(c-a)+2c(b-a)}{3a^2+(a+b+c)^2}\right] \geq \sum_{cyc} \frac{(b-c)^2}{(a+b+c)^2},$$

or

$$\sum_{cyc} \left(\frac{1}{3a^2+(a+b+c)^2} + \frac{6a(b+c)}{(3b^2+(a+b+c)^2)(3c^2+(a+b+c)^2)}\right)(b-c)^2 \geq \sum_{cyc} \frac{(b-c)^2}{(a+b+c)^2},$$

Denote

$$S_a = \frac{1}{3a^2+(a+b+c)^2} + \frac{6a(b+c)}{(3b^2+(a+b+c)^2)(3c^2+(a+b+c)^2)} - \frac{1}{(a+b+c)^2}$$

and similarly we define S_b and S_c. Without loss of generality, we may assume that $a \geq b \geq c$. Clearing denominators and expanding, we have

$$S_b = \frac{\beta}{(3a^2+(a+b+c)^2)(3b^2+(a+b+c)^2)(3c^2+(a+b+c)^2)(a+b+c)^2}$$

$$S_c = \frac{\gamma}{(3a^2+(a+b+c)^2)(3b^2+(a+b+c)^2)(3c^2+(a+b+c)^2)(a+b+c)^2}$$

$$a^2S_b + b^2S_a = \frac{\delta}{(3a^2+(a+b+c)^2)(3b^2+(a+b+c)^2)(3c^2+(a+b+c)^2)(a+b+c)^2}$$

$$\begin{aligned}\frac{\beta}{3b} =& 2a^5 + 2a^4(2b+5c) + 2a^3(4b^2+11bc+10c^2) + a^2(11b^3+36b^2c+27bc^2+20c^3) + \\ & 2a(b+c)(2b^3+12b^2c+6bc^2+5c^3) - b^5 + 4b^4c + 11b^3c^2 + 8b^2c^3 + 4bc^4 + 2c^5 \geq 0.\end{aligned}$$

$$\begin{aligned}\frac{\gamma}{3c} =& 2a^5 + 2a^4(5b+2c) + 2a^3(10b^2+11bc+4c^2) + a^2(20b^3+27b^2c+36bc^2+11c^3) + \\ & 2a(b+c)(5b^3+6b^2c+12bc^2+2c^3) + 2b^5 + 4b^4c + 8b^3c^2 + 11b^2c^3 + 4bc^4 - c^5 \geq 0.\end{aligned}$$

$$\begin{aligned}\frac{\delta}{3ab} =&2a^6 + a^5(3b + 10c) + 2a^4(6b^2 + 13bc + 10c^2) + 2a^3(11b^3 + 32b^2c + 19bc^2 + 10c^3)+\\ &2a^2(6b^4 + 32b^3c + 36b^2c^2 + 15bc^3 + 5c^4) + a(3b^5 + 26b^4c + 38b^3c^2 + 30b^2c^3 + 8bc^4 + 2c^5)+\\ &2b^6 + 10b^5c + 20b^4c^2 + 20b^3c^3 + 10b^2c^4 + 2bc^5 \geq 0.\end{aligned}$$

Therefore $S_b \geq 0$, $S_c \geq 0$ and $a^2S_b + b^2S_a \geq 0$. According to SOS method the inequality is proven. The equality holds when $a = b = c = 1$ or $a = b = 0$, $c = 3$ and its cyclic permutations.

Also solved by Michel Faleiros Martins, São Paulo, SP, Brazil; Theo Koupelis, Cape Coral, FL, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O648. Let $A(x), B(x)$ be polynomials with integer coefficients for which there are polynomials $P_1(x), Q_1(x)$ with integer coefficients such that

$$P_1(x)A(x) + Q_1(x)B(x) = 1. \tag{1}$$

Prove that for each positive integer n there are integer polynomials $P_n(x), Q_n(x)$ such that

$$P_n(x)A(x)^n + Q_n(x)B(x)^n = 1. \tag{2}$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Daniel Pascuas, Barcelona, Spain

First, we prove the case $n = 2$, from which the remaining cases will easily follow. By squaring identity (1) we get that

$$1 = P_1(x)^2A(x)^2 + Q_1(x)^2B(x)^2 + 2P_1(x)A(x)Q_1(x)B(x).$$

Now we multiply (1) by $2P_1(x)A(x)Q_1(x)B(x)$. We obtain that

$$2P_1(x)A(x)Q_1(x)B(x) = P_0(x)A(x)^2 + Q_0(x)B(x)^2,$$

where $P_0(x) = 2P_1(x)Q_1(x)B(x)$ and $Q_0(x) = 2P_1(x)A(x)Q_1(x)$ are polynomials with integer coefficients. Therefore the polynomials with integer coefficients $P_2(x) = P_1(x)^2 + P_0(x)$ and $Q_2(x) = Q_1(x)^2 + Q_0(x)$ satisfy (2) for $n = 2$.

By iteration, the case $n = 2$ gives the case $n = 2^k$, for any positive integer k. Finally, if n is a positive integer, then $n < 2^n$, so the polynomials with integer coefficients $P_n(x) = P_{2^n}(x)A(x)^{2^n-n}$ and $Q_n(x) = Q_{2^n}(x)B(x)^{2^n-n}$ clearly satisfy (2).

Also solved by Srijan Sundar, Oxford, UK; Jean Heibig, ISAE-SUPAERO, Toulouse, France.