Junior problems

J649. Find all integers n for which

$$(n+3)(n^2+3n+3)$$

is the product of three primes $p > q > r$ such that $p - r = 6$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution 1 by Corneliu Mănescu-Avram, Ploieşti, Romania

The answer is 8. From $p - r = 6$ it follows that $r > 2$, so $q = r + 2$ or $q = r + 4$. Let

$$N = (n+3)(n^2+3n+3) = pqr$$

Then, $N = (n+2)^3 + 1$.

Case 1: $q = r + 2$. We have $N = r(r+2)(r+6) = (r+2)^3 + 2r^2 - 8 < (r+3)^3$ and the equality is impossible, since 1 is odd, whereas $2r^2 - 8$ is even.

Case 2: $q = r + 4$. We have $N = r(r+4)(r+6) = (r+3)^3 + r^2 - 3r - 27 < (r+4)^3$, whence $r^2 - 3r - 27 - 1 = (r+4)(r-7) = 0$, therefore $r = 7$. The solution is

$$(p, q, r) = (13, 11, 7), \quad n = 8.$$

Solution 2 by Polyahedra, Polk State College, USA

If $n + 3 \leq 0$, then $n^2 + 3n + 3 = n(n+3) + 3 > 0$, so $pqr \leq 0$, an impossibility. Also, it is clear that $n \notin \{-2, -1, 0, 1\}$ and $r \notin \{2, 3\}$. Thus $n \geq 2$ and $r \geq 5$. Since

$$qr - p \geq (p-4)(p-6) - p = (p-3)(p-8) > 0,$$

$n + 3$ cannot be the product of two or three primes. If $n + 3 = p$, then $qr \leq (n+1)(n-3) < n^2 + 3n + 3$. If $n + 3 \leq p - 4$, then $pq \geq (n+7)(n+3) > n^2 + 3n + 3$. Therefore, $n + 3 = q = p - 2$, and we get $n^2 + 3n + 3 = pr = (n+5)(n-1)$, so $n = 8$. Finally, when $n = 8$, the expression equals $11 \cdot 7 \cdot 13$.

Also solved by Farmonov Sukhrobjon, Uzbekistan; Sundaresh. H. R, Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA.

J650. Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\sqrt{1-ab}+\sqrt{1-bc}+\sqrt{1-ca} \geq \sqrt{6}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by Polyahedra, Polk State College, USA

Squaring both sides we see that the inequality is equivalent to

$$2\left(\sqrt{(1-bc)(1-ca)}+\sqrt{(1-ca)(1-ab)}+\sqrt{(1-ab)(1-bc)}\right) \geq 3+ab+bc+ca.$$

By the AM-GM and the Cauchy-Schwarz inequalities,

$$2\sqrt{(1-bc)(1-ca)} \geq \sqrt{(2-b^2-c^2)(2-c^2-a^2)} = \sqrt{(1+a^2)(1+b^2)} \geq 1+ab.$$

Summing with the other two analogous inequalities completes the proof.

Equality holds if and only if $a=b=c=\pm\dfrac{\sqrt{3}}{3}$.

Solution 2 by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

The worst case clearly is with $a, b, c \geq 0$. Hölder yields $(ab \leq (a^2+b^2)/2 \leq 1)$

$$\sum_{\text{cyc}} \sqrt{1-ab} \sum_{\text{cyc}} \sqrt{1-ab} \sum_{\text{cyc}} \frac{1}{1-ab} \geq 27$$

hence it suffices to prove

$$\frac{27}{\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}} \geq 6 \iff \sum_{\text{cyc}} \frac{1}{1-ab} \leq \frac{9}{2}$$

Upon expanding we get

$$2\sum_{\text{cyc}}(1-ab)(1-bc) \leq 9(1-ab)(1-bc)(1-ca) \tag{1}$$

that is

$$9(abc)^2+5(ab+bc+ca) \leq 3+7abc(a+b+c)$$

Let's define the new variables $a+b+c=3u$, $ab+bc+ca=3v^2$, $abc=w^3$; We get

$$9(w^3)^2-21uw^3+15v^2-3 \leq 0$$

This is a convex parabola in w^3 hence we must check at the extreme values of w^3 which occurs when $w^3=0$ or when $c=b$.

$w^3=0$ hence $c=0$ (or cyclic).

$$15v^2 \leq 3 \iff 5ab \leq 3$$

which holds true by $ab \leq (a^2+b^2)/2 \leq 1/2$.

If $c=b$ we get $a=\sqrt{1-2b^2}$, $0 \leq b \leq 1/\sqrt{2}$ and (1) becomes

$$10\sqrt{1-2b^2}b-14\sqrt{1-2b^2}b^3 \leq 3+2b^2+18b^6-23b^4 \tag{2}$$

$$10b-14b^3 \geq 0 \iff 2b(5-7b^2) \geq 2b(5-\frac{7}{2})>0$$

$$3+2b^2+18b^6-23b^4 \geq 3+12b^4-23b^4 = 3-11b^4 \geq 3-\frac{11}{4}>0$$

hence we can square both sides of (2) getting

$$-(18b^4-25b^2+9)(2b^4-b^2+1)(-1+3b^2)^2 \leq 0$$

which is true with equality cases $a=b=c=\pm 1/\sqrt{3}$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Clark College, Washington, USA.

J651. Let $ABCD$ be a cyclic quadrilateral such that

$$(AB - BC + CD + DA)(BC + CD + DA - AB) + AC \cdot BD = (AB + AD)(BC + CD).$$

Find $\angle ADC$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

By Ptolemy's theorem, $AC \cdot BD = AB \cdot CD + BC \cdot DA$, so the given condition becomes $AB \cdot BC + CD \cdot DA = AB^2 + BC^2 - CD^2 - DA^2$. By the law of cosines,

$$AB^2 + BC^2 + 2AB \cdot BC \cos \angle ADC = AC^2 = CD^2 + DA^2 - 2CD \cdot AD \cos \angle ADC.$$

Therefore,

$$\cos \angle ADC = \frac{CD^2 + DA^2 - AB^2 - BC^2}{2(AB \cdot BC + CD \cdot AD)} = -\frac{1}{2},$$

so $\angle ADC = 120^\circ$.

Also solved by Sundaresh. H. R, Shivamogga, Karnataka, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Farmonov Sukhrobjon, Uzbekistan; Anderson Torres, Brazil; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, Washington, USA; Daniel Văcaru, Economic College Maria Teiuleanu, Pitești, Romania.

J652. Let $ABCD$ be a trapezoid and let X be the intersection of its diagonals. Let r_1, r_2, r_3, r_4 be the inradii of triangles XAD, XDC, XCB, XAB, respectively. Prove that if

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$$

then $ABCD$ is circumscribed about a circle.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution 1 by the author

Denote: S_1, S_2, S_3, S_4 respectively p_1, p_2, p_3, p_4 the area, respectively the semi-perimeter of triangles AXD, DXC, BXC, AXB.

Observation: $ABCD$ is a trapezoid and $AC \cap BD = \{X\} \implies S_1 = S_3 = S$.

Using the formula $S = rp$ we have $r = \frac{p}{S}$, so $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$ becomes $\frac{p_1}{S_1} + \frac{p_3}{S_3} = \frac{p_2}{S_2} + \frac{p_4}{S_4}$ or

$$\frac{p_1 + p_3}{S} = \frac{p_2}{S_2} + \frac{p_4}{S_4}. \tag{1}$$

Since $\triangle XDC \sim \triangle XAB$, we have

$$\frac{XC}{XA} = \frac{p_2}{p_4} = \frac{S}{S_4} \implies \frac{p_4}{S_4} = \frac{p_2}{S}$$

and

$$\frac{XC}{XA} = \frac{S}{S_4} = \frac{S_2}{S} = \frac{p_2}{p_4} \implies \frac{p_2}{S_2} = \frac{p_4}{S}.$$

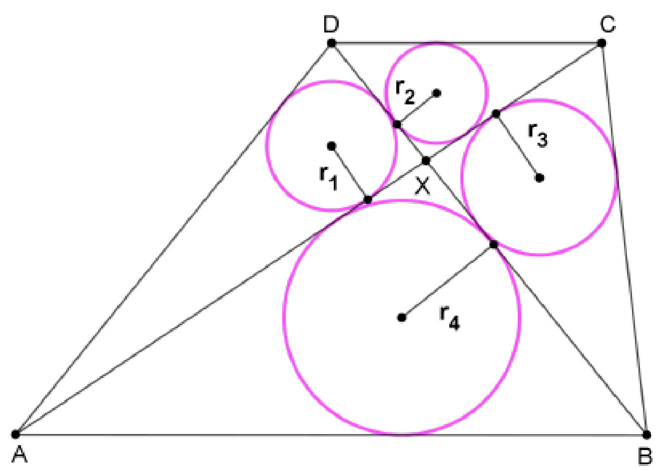
So, (1) becomes

$$\frac{p_1 + p_3}{S} = \frac{p_2 + p_4}{S} \implies p_1 + p_3 = p_2 + p_4,$$

i.e.

$$XD + XA + AD + XC + XB + BC = XA + XB + AB + XC + XD + DC,$$

so $AD + BC = DC + AB$. It follows that the trapezoid is circumscribed about a circle.



Solution 2 by Polyahedra, Polk State College, USA

Suppose that $AB \parallel CD$ and $CD : AB = k$. Then $k = r_2 : r_4 = XD : XB = XC : XA = [XAD] : [XAB] = [XBC] : [XAB]$. Therefore,

$$\frac{1}{r_1} = \frac{AD + XA + kXB}{kr_4(AB + XA + XB)}, \quad \frac{1}{r_3} = \frac{BC + XB + kXA}{kr_4(AB + XA + XB)},$$

so

$$\frac{1+k}{kr_4} = \frac{1}{r_2} + \frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_3} = \frac{AD + BC + (1+k)(XA + XB)}{kr_4(AB + XA + XB)},$$

which implies that $AD + BC = (1+k)AB = AB + CD$, completing the proof.

Solution 3 by Anderson Torres, Brazil

It is a well-known fact that $S = pr$, or $\frac{1}{r} = \frac{p}{S}$. Further, we have $\sin\angle AXD = \sin\angle BXC = \sin\angle AXB = \sin\angle CXD$

$$\frac{AX + XD + DA}{2 \cdot [AXD]} + \frac{BX + XC + CB}{2 \cdot [BXC]} = \frac{AX + XB + BA}{2 \cdot [AXB]} + \frac{CX + XD + DC}{2 \cdot [CXD]}$$

$$\frac{AX + XD + DA}{AX \cdot XD \cdot \sin AXD} + \frac{BX + XC + CB}{BX \cdot XC \cdot \sin BXC} = \frac{AX + XB + BA}{AX \cdot XB \cdot \sin AXB} + \frac{CX + XD + DC}{CX \cdot XD \cdot \sin CXD}$$

$$\frac{AX + XD + DA}{AX \cdot XD} + \frac{BX + XC + CB}{BX \cdot XC} = \frac{AX + XB + BA}{AX \cdot XB} + \frac{CX + XD + DC}{CX \cdot XD}$$

$$\frac{1}{AX} + \frac{1}{DX} + \frac{AD}{AX \cdot DX} + \frac{1}{BX} + \frac{1}{CX} + \frac{BC}{BX \cdot CX} = \frac{1}{AX} + \frac{1}{BX} + \frac{AB}{AX \cdot BX} + \frac{1}{CX} + \frac{1}{DX} + \frac{CD}{CX \cdot DX}$$

$$\frac{1}{AX} + \frac{1}{DX} + \frac{AD}{AX \cdot DX} + \frac{1}{BX} + \frac{1}{CX} + \frac{BC}{BX \cdot CX} = \frac{1}{AX} + \frac{1}{BX} + \frac{AB}{AX \cdot BX} + \frac{1}{CX} + \frac{1}{DX} + \frac{CD}{CX \cdot DX}$$

$$\frac{AD}{AX \cdot DX} + \frac{BC}{BX \cdot CX} = \frac{AB}{AX \cdot BX} + \frac{CD}{CX \cdot DX}$$

$$AB \cdot CX \cdot DX + CD \cdot AX \cdot BX = AD \cdot BX \cdot CX + BC \cdot AX \cdot DX$$

Now let's suppose with negligible loss of generality AB is parallel to CD. Then the triangles AXB and CXD are similar, allowing us to write $CX = k \cdot AX$, $DX = k \cdot BX$ and $CD = k \cdot AB$:

$$AB \cdot CX \cdot DX + CD \cdot AX \cdot BX = AD \cdot BX \cdot CX + BC \cdot AX \cdot DX$$

$$AB \cdot k \cdot AX \cdot k \cdot BX + CD \cdot AX \cdot BX = AD \cdot BX \cdot k \cdot AX + BC \cdot AX \cdot k \cdot BX$$

$$k^2 \cdot AB + CD = k \cdot (AD + BC)$$

$$k^2 \cdot AB + k \cdot AB = k \cdot (AD + BC)$$

$$k \cdot AB + AB = AD + BC$$

$$CD + AB = AD + BC$$

Therefore $ABCD$ is circumscribed about a circle.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Farmonov Sukhrobjon, Uzbekistan; Theo Koupelis, Clark College, Washington, USA.

J653. Let a, b, c be distinct real numbers. Prove that any two of the equalities

$$3ab + (b-c)(c-a) = \frac{16}{a-b}, 3bc + (c-a)(a-b) = \frac{4}{b-c}, 3ca + (a-b)(b-c) = \frac{-20}{c-a},$$

imply the third. Are there such numbers?

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution 1 by the author

Assume that

$$3ab(a-b) + (b-c)(c-a)(a-b) = 16$$

$$3bc(b-c) + (c-a)(a-b)(b-c) = 4$$

$$3ca(c-a) + (a-b)(b-c)(c-a) \neq -20.$$

Summing up we get, after cancellations, $0 \neq 0$, a contradiction. Hence the conclusion. There are such numbers a, b, c, for example $a = 3$, $b = 2$, $c = 1$.

Solution 2 by Polyahedra, Polk State College, USA

This follows immediately from the identity

$$(a-b)[3ab + (b-c)(c-a)] + (b-c)[3bc + (c-a)(a-b)] + (c-a)[3ca + (a-b)(b-c)] = 0.$$

Yes, for example, $(a, b, c) = (3, 2, 1)$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Sundaresh. H. R, Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA.

J654. Is there a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(x)) = x^3 - 2x + 1$ for all $x \in \mathbb{Z}$?

Proposed by Mircea Becheanu, Canada

Solution 1 by the author

The answer is no. Assume by contradiction that such a function exists. We remark first that f has no fixed points. Indeed, if $f(x) = x$ it follows $f(f(x)) = f(x) = x = x^3 - 2x + 1$. This is impossible because the equation $x^3 - 3x + 1 = 0$ has no integer solution.

From the condition of the problem we have $f(f(0)) = 1$ and $f(f(1)) = 0$. From these we obtain:

$$f(1) = f(f(f(0))) = f(0)^3 - 2f(0) + 1 \tag{1}$$

$$f(0) = f(f(f(1))) = f(1)^3 - 2f(1) + 1. \tag{2}$$

After subtracting (1)-(2) one obtains:

$$f(1) - f(0) = (f(0) - f(1))(f(0)^2 + f(0)f(1) + f(1)^2 - 2). \tag{3}$$

If $f(1) = f(0)$ one obtains $f(f(1)) = f(f(0))$ and this is a contradiction. Then, from (3) we have

$$f(0)^2 + f(0)f(1) + f(1)^2 = 1.$$

We write this in the equivalent form:

$$f(0)^2 + f(1)^2 + (f(0) + f(1))^2 = 2. \tag{4}$$

Because f has no fixed points the possible solutions of (4) are:
Case 1. $f(0) = 1$ and $f(1) = 0$. Then $f(1) = f(f(0)) = 1$ and this is a contradiction.
Case 2. $f(0) = 1$ and $f(1) = -1$. Then $f(1) = f(f(0)) = 1$, a contradiction.
Case 3. $f(0) = -1$ and $f(1) = 0$. In this case we have $f(0) = f(f(1)) = 0$, contradiction. The conclusion is, such a function does not exist.

Solution 2 by Polyahedra, Polk State College, USA

No. Suppose that f is such a function. First, $f(f(0)) = 0^3 - 2(0) + 1 = 1$ and $f(f(1)) = 1^3 - 2(1) + 1 = 0$. If $f(1) = 1$, then $1 = f(f(1)) = 0$, an absurdity. If $f(1) = 0$, then $f(0) = f(f(1)) = 0$, so $0 = f(f(0)) = 1$, an absurdity again. Therefore, $f(1) = a \in \mathbb{Z} \setminus \{0, 1\}$. Then $f(0) = f(f(f(1))) = f(f(a)) = a^3 - 2a + 1$ and

$$a = f(1) = f(f(f(0))) = f(0)^3 - 2f(0) + 1 = \left(a^3 - 2a + 1\right)^3 - 2\left(a^3 - 2a + 1\right) + 1,$$

which is equivalent to

$$a(a-1)\left(a^3 - 3a + 1\right)\left(a^4 + a^3 - 2a^2 + 3\right) = 0.$$

By the rational root theorem, $a^3 - 3a + 1$ and $a^4 + a^3 - 2a^2 + 3$ have no rational roots, so a cannot be an integer.

S649. Let a, b, c be positive real numbers such that $ab + bc + ca + abc = 4$. Prove that

$$\frac{\sqrt{a}}{a+2} + \frac{\sqrt{b}}{b+2} + \frac{\sqrt{c}}{c+2} \leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author
The given condition can be rewritten as

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} = 1$$

or

$$\frac{a}{a+2} + \frac{b}{b+2} + \frac{c}{c+2} = 1.$$

Now we use Cauchy-Schwarz inequality to obtain

$$\begin{aligned} 1 &= \left(\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2}\right)\left(\frac{a}{a+2} + \frac{b}{b+2} + \frac{c}{c+2}\right) \\ &\geq \left(\frac{\sqrt{a}}{a+2} + \frac{\sqrt{b}}{b+2} + \frac{\sqrt{c}}{c+2}\right)^2. \end{aligned}$$

The conclusion follows.

Solution 2 by Theo Koupelis, Clark College, Washington, USA
Using AM-GM Inequality we get

$$\frac{\sqrt{a}}{a+2} \leq \frac{a+1}{2(a+2)} = \frac{1}{2}\left(1 - \frac{1}{a+2}\right),$$

with similar expressions for the other variables. Thus, it is sufficient to show that

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} \geq 1,$$

or, after clearing denominators and simplifying, $ab + bc + ca + abc \leq 4$, which is satisfied by the given condition. Equality occurs when $a = b = c = 1$. From the above we get that the desired inequality holds true for $ab + bc + ca + abc = k$, where $0 < k \leq 4$.

Note: In general, for positive real numbers that satisfy $ab + bc + ca + abc = k$, where $0 < k \leq 4$, we have

$$\frac{\sqrt{a}}{a+2} + \frac{\sqrt{b}}{b+2} + \frac{\sqrt{c}}{c+2} \leq \frac{3\sqrt{x_0}}{x_0+2},$$

where $x_0 \in (0, 1]$ is the solution of the cubic $3x_0^2 + x_0^3 = k$. Equality occurs when $a = b = c = x_0$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Anderson Torres, Brazil; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Jiang Lianjun,Quanzhou Middle School,Guilin,China; Arkady Alt, San Jose, CA, USA; Farmonov Sukhrobjon, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S650. For a triangle ABC let A', B', C' be the tangency points of the ex-circles with the sides $[BC], [CA]$, and $[AB]$, respectively.

1) Prove that one can construct a triangle with the vectors $\overline{AA'}, \overline{BB'}, \overline{CC'}$ if and only if the triangle ABC is equilateral.

2) Remain the above property true if the construction is with the segments $[AA'], [BB'], [CC']$?

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Solution by the author

1) One can construct a triangle with the vectors $\overline{AA'}, \overline{BB'}, \overline{CC'}$ if and only if one of the following vectorial relations holds

$$\overline{AA'} + \overline{BB'} + \overline{CC'} = \overline{0}. \tag{1}$$

If $a = BC, b = CA, c = AB$ and $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter of triangle ABC, then we have $BA' = s-c, CB' = s-c, AC' = s-a$, hence

$$\overline{AA'} = \overline{AB} + \overline{BA'} = \overline{AB} + \frac{s-c}{a}\overline{BC} = \overline{AB} + \frac{s-c}{a}(\overline{AC} - \overline{AB}) =$$

$$(1 - \frac{s-c}{a})\overline{AB} + \frac{s-c}{a}\overline{AC}.$$

Similarly, we have

$$\overline{BB'} = -\overline{AB} + \frac{s-c}{b}\overline{AC}, \overline{CC'} = \frac{s-b}{c}\overline{AB} - \overline{AC}.$$

The relation $\overline{AA'} + \overline{BB'} + \overline{CC'} = \overline{0}$ is equivalent to $(\frac{s-b}{c} - \frac{s-c}{a})\overline{AB} + (\frac{s-c}{b} + \frac{s-c}{a} - 1)\overline{AC} = \overline{0}$, hence

$$\begin{cases} \dfrac{s-b}{c} - \dfrac{s-c}{a} = 0 \\ \dfrac{s-c}{b} + \dfrac{s-c}{a} - 1 = 0. \end{cases} \tag{2}$$

The system (2) is equivalent to

$$\begin{cases} a^2 + c^2 = ab + bc \\ a^2 + b^2 = ac + bc, \end{cases}$$

hence $c^2 - b^2 = a(b-c)$, and we obtain $b = c = a$.

2) The following example shows that the above property is not true if the construction is with the segments $[AA'], [BB'], [CC']$. Consider the isosceles right-angled triangle BAC. We have $AA' = a/2$, and the triangle inequalities are obviously satisfied by the segments $[AA'], [BB'], [CC']$. Therefore, the construction is possible but the triangle ABC is not equilateral.

Also solved by Theo Koupelis, Clark College, Washington, USA.

S651. Let a, b, c be real numbers. Prove that

$$3a^2b^2c^2 + 3\sum_{cyc}(a^4b^2 + a^2b^4) \geq 3abc\sum_{cyc} a^3 + 4\sum_{cyc} a^3b^3.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution by Corneliu Mănescu-Avram, Ploieşti, Romania

Denote $p = a + b + c, q = ab + bc + ca, r = abc$. The inequality becomes

$$3r^2 + 3(p^2 - 2q)(q^2 - 2pr) - 9r^2 - 3p^3r + 9pqr - 9r^2 - 4(q^3 - 3pqr + 3r^2) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 27r^2 + 3r(3p^3 - 11pq) + 10q^3 - 3p^2q^2 \leq 0.$$

By the Schur's inequality, $4pq - p^3 \leq 9r \leq pq$. If $p = 3$, then we can write the inequality as

$$3q^2 + (81 - 33q)q + 10q^3 - 3p^2q^2 \leq 0 \Leftrightarrow 10q^3 - 57q^2 + 81q \leq 0$$

$$\Leftrightarrow q(q - 3)(10q - 27) \leq 0$$

which is true.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Jiang Lianjun, Quanzhou Middle School, Guilin, China; Theo Koupelis, Clark College, Washington, USA.

S652. Find the least positive integer m for which there is a positive integer n such that

$$1 + 2 + \cdots + m = (2 + 3 + \cdots + n)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Henry Ricardo, Westchester Area Math Circle

We consider the equation $m(m+1)/2 = s_n^2$, where $s_n = 2 + 3 + \cdots + n$. Rearranging this equation, we get $(2m+1)^2 = 8s_n^2 + 1$; and, letting $x = 2m+1$, $y = s_n$, we get the Pell equation

$$x^2 - 8y^2 = 1.$$

Note that $(x, y) = (1, 0)$ is a solution.

It is well-known that if we know at least one nontrivial solution of Pell's equation, all other solutions can be obtained as a result of consecutive iterations $(x, y) \to (3x + 8y, x + 3y)$. Therefore, the first few ordered pairs of solutions are

$$(1, 0) \to (3, 1) \to (17, 6) \to (99, 35) \to (577, 204).$$

It is easily seen that the smallest value of $m = (x-1)/2$ for which an appropriate value of n exists is $m = 49$ with corresponding $n = 8$:

$$(2 \cdot 49 + 1)^2 \;-\; 8(35)^2 \;=\; 1.$$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sundaresh. H. R, Shivamogga, Karnataka, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Theo Koupelis, Clark College, Washington, USA.

S653. Let $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ such that $a_1 + a_2 + \cdots + a_9 = 2$. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_9a_1 \leq 1.$$

Proposed by Vasile Cârtoaje, Oil-Gas University, Ploieşti, România

Solution by the author

Write the inequality as $F(a_1, a_2, \ldots, a_9) \geq 0$, where

$$F(a_1, a_2, \ldots, a_9) = (a_1 + a_2 + \cdots + a_9)^2 - 4(a_1a_2 + a_2a_3 + \cdots + a_9a_1).$$

We will show that

$$F(a_1, a_2, a_3, \ldots, a_9) \geq F(a_2, a_2, a_3 \ldots, a_9) \geq 0.$$

The left inequality is equivalent to

$$(a_1 + a_2 + a_3 + \cdots + a_9)^2 - (2a_2 + a_3 + \cdots + a_9)^2 \geq 4(a_1a_2 + a_2a_3 + \cdots + a_9a_1) - 4(a_2^2 + a_2a_3 + \cdots + a_9a_2),$$

$$(a_1 - a_2)(a_1 + 3a_2 + 2a_3 + \cdots + 2a_9) \geq 4(a_1 - a_2)(a_2 + a_9),$$

$$(a_1 - a_2)(a_1 - a_2 + 2a_3 + \cdots + 2a_8 - 2a_9) \geq 0,$$

while the right inequality is equivalent to $G(a_2, a_3, \ldots, a_8, a_9) \geq 0$, where

$$G(a_2, a_3, \ldots, a_8, a_9) = (2a_2 + a_3 + \cdots + a_8 + a_9)^2 - 4(a_2^2 + a_2a_3 + \cdots + a_8a_9 + a_9a_2).$$

We will show that

$$G(a_2, a_3, \ldots, a_8, a_9) \geq G(a_2, a_3, \ldots, a_8, 0) \geq \cdots \geq G(a_2, 0, \ldots, 0, 0) = 0.$$

We have

$$G(a_2, a_3, \ldots, a_8, a_9) - G(a_2, a_3, \ldots, a_8, 0) =$$

$$= (2a_2 + a_3 + \cdots + a_8 + a_9)^2 - (2a_2 + a_3 + \cdots + a_8)^2 - 4(a_2^2 + a_2a_3 + \cdots + a_8a_9 + a_9a_2) + 4(a_2^2 + a_2a_3 + \cdots + a_7a_8)$$

$$= a_9(4a_2 + 2a_3 + \cdots + 2a_8 + a_9) - 4a_9(a_8 + a_2) = a_9(2a_3 + \cdots + 2a_7 - 2a_8 + a_9) \geq 0,$$

$$G(a_2, a_3, \ldots, a_7, a_8, 0) - G(a_2, a_3, \ldots, a_7, 0, 0) =$$

$$= (2a_2 + a_3 + \cdots + a_8)^2 - (2a_2 + a_3 + \cdots + a_7)^2 - 4(a_2^2 + a_2a_3 + \cdots + a_7a_8) + 4(a_2^2 + a_2a_3 + \cdots + a_6a_7)$$

$$= a_8(4a_2 + 2a_3 + \cdots + 2a_7 + a_8) - 4a_7a_8 = a_8(4a_2 + 2a_3 \cdots + 2a_6 - 2a_7 + a_8) \geq 0$$

and, similarly,

$$G(a_2, a_3, \ldots, a_7, 0, 0) - G(a_2, a_3, \ldots, a_6, 0, 0, 0) = a_7(4a_2 + 2a_3 + 2a_4 + 2a_5 - 2a_6 + a_7) \geq 0,$$

$$G(a_2, a_3, a_4, a_5, a_6, 0, 0, 0) - G(a_2, a_3, a_4, a_5, 0, 0, 0, 0) = a_6(4a_2 + 2a_3 + 2a_4 - 2a_5 + a_6) \geq 0,$$

$$G(a_2, a_3, a_4, a_5, 0, 0, 0, 0) - G(a_2, a_3, a_4, 0, 0, 0, 0, 0) = a_5(4a_2 + 2a_3 - 2a_4 + a_5) \geq 0,$$

$$G(a_2, a_3, a_4, 0, 0, 0, 0, 0) - G(a_2, a_3, 0, 0, 0, 0, 0, 0) = a_4(4a_2 - 2a_3 + a_4) \geq 0,$$

$$G(a_2, a_3, 0, 0, 0, 0, 0, 0) - G(a_2, 0, 0, 0, 0, 0, 0, 0) = a_3^2 \geq 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = 1$ and $a_3 = \cdots = a_9 = 0$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, Washington, USA.

S654. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{8(ab+bc+ca)^2}{(a+b)(b+c)(c+a)} \geq 9\sqrt{abc}$$

Proposed by Marius Stănean, Zalău, România

Solution 1 by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let's define the new variables $a+b+c=3u$, $ab+bc+ca=3v^2$, $abc=w^3$. The inequality reads as

$$\frac{72v^4}{9uv^2-w^3} \geq 9(w^3)^{1/2} \iff 72v^4-81v^2w^{3/2}+9w^{9/2} \geq 0 \qquad (1)$$

$3=a+b+c \geq 3(abc)^{1/3}$ hence $w \leq 1$. Moreover the AGM also yields $w \leq v \leq 1$. This implies that (1) is a decreasing function of w

$$\left(72v^4-81v^2w^{3/2}+9w^{9/2}\right)_w = \frac{-243}{2}v^2w^{1/2}+\frac{81}{2}w^{7/2} = \frac{81w^{\frac{1}{2}}}{2}(w^3-v^2) \leq$$
$$\leq \frac{81w^{\frac{1}{2}}}{2}(v^3-v^2) = \frac{81v^2w^{\frac{1}{2}}}{2}(v-1) \leq 0$$

It follows that we have to check the inequality for the minimum value of w and this occurs when $c=0$ (or cyclic) or $b=c$ (or cyclic). If $c=0$ the inequality trivially holds true. If $c=b$, $a=3-2b$ the inequality becomes

$$\frac{4(2b(3-2b)+b^2)^2}{(3-b)^2b} \geq 9\sqrt{(3-2b)b^2}$$

Upon squaring both sides we get

$$\frac{81b^2(2b^3-7b^2+13)(b-1)^2}{(b-3)^4} \geq 0$$

and $2b^3-7b^2+13 \geq (2b^3-7b^2+13)_{b=3/2} = -4$ (as a function of $b \leq 3/2$ it decreases).

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Homogenizing the desired inequality we get

$$8(ab+bc+ca)^2\sqrt{a+b+c} \geq 9\sqrt{3abc}(a+b)(b+c)(c+a). \qquad (*)$$

Setting $x=b/a$, and $y=c/a$, squaring $(*)$, and simplifying we get

$$64(x+y+xy)^4(1+x+y) \geq 243xy(1+x)^2(1+y)^2(x+y)^2.$$

Expanding and rearranging, we get

$$\begin{aligned} y^4(x^2-1)^2(25x+38y+25)&+x^4(y^2-1)^2(25y+38x+25)\\ &+(x^2-y^2)^2(25x+25y+38)\\ &+13xy\left[x^4(y-1)^2+y^4(x-1)^2+(x-y)^2\right]\\ &+(x^5y^4+x^4y^5+x^4+x^5+y^4+y^5-6x^3y^3)\\ &+40xy(x^3+y^3+x^3y^3-3x^2y^2) \geq 0, \end{aligned}$$

which is obvious because every term is nonenegative by AM-GM. Equality occurs when $a=b=c$.

Also solved by Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Jiang Lianjun,Quanzhou Middle School,Guilin,China.

U649. Evaluate

$$\lim_{n\to\infty}\left(\frac{(1^2+n^2)(2^2+n^2)\cdots(n^2+n^2)}{n!n^n}\right)^{1/n}.$$

Proposed by Mircea Becheanu, Canada

Solution by Matthew Too, Brockport, NY, USA

Let Q_n be the given sequence. Then

$$\begin{aligned}\ln Q_n &= \frac{1}{n}\ln\left[\frac{1}{n!}\prod_{i=1}^{n}\left(\frac{i^2+n^2}{n}\right)\right] = \frac{1}{n}\left[\sum_{i=1}^{n}\ln\left(\frac{i^2+n^2}{n}\right)-\ln n!\right]\\ &= \frac{1}{n}\left[\sum_{i=1}^{n}\ln\left(1+(i/n)^2\right)+n\ln n-\ln n!\right]\\ &= \frac{1}{n}\left[\sum_{i=1}^{n}\ln\left(1+(i/n)^2\right)+n-O(\ln n)\right]\\ &= \frac{1}{n}\sum_{i=1}^{n}\ln\left(1+(i/n)^2\right)+1-O\left(\frac{\ln n}{n}\right)\end{aligned}$$

where we used Stirling's formula to simplify $\ln n!$. Notice that the first term of the previous result corresponds to a Riemann sum, so

$$\begin{aligned}\lim_{n\to\infty}\ln Q_n &= 1+\int_0^1\ln(1+x^2)\,dx = 1+\left[x\ln(1+x^2)\right]_0^1-\int_0^1\frac{2x^2}{1+x^2}\,dx\\ &= 1+\ln 2-2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx = 1+\ln 2-2\left[x-\arctan x\right]_0^1\\ &= \ln 2+\frac{\pi}{2}-1.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\left(\frac{(1^2+n^2)(2^2+n^2)\cdots(n^2+n^2)}{n!n^n}\right)^{1/n} = \exp\left(\lim_{n\to\infty}\ln Q_n\right) = 2e^{\frac{\pi}{2}-1}.$$

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Farmonov Sukhrobjon, Uzbekistan; Sundaresh. H. R, Shivamogga, Karnataka, India; Aryan Desai; Corneliu Mănescu-Avram, Ploieşti, Romania; Henry Ricardo, Westchester Area Math Circle; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal; Monil Patel, University of Calgary, Canada; Theo Koupelis, Clark College, Washington, USA.

U650. Evaluate

$$\int_1^e \frac{(\ln x - 1)^2 - 3}{(\ln x + 2)^2} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Sukhrobjon Farmonov, Uzbekistan

For each $x > 0$ not equal to e^{-2}, we have

$$\frac{(\ln x - 1)^2 - 3}{(\ln x + 2)^2} = \frac{(\ln x + 2 - 3)^2 - 3}{(\ln x + 2)^2} = 1 - \frac{6}{\ln x + 2} + \frac{6}{(\ln x + 2)^2}.$$

Thus

$$I = e - 1 - 6\int_1^e \frac{1}{\ln x + 2} dx + 6\int_1^e \frac{1}{(\ln x + 2)^2} dx.$$

Integrating by part we get

$$\int_1^e \frac{1}{\ln x + 2} dx = \left.\frac{x}{\ln x + 2}\right|_1^e + \int_1^e \frac{1}{(\ln x + 2)^2} dx.$$

We proved that

$$I = 2 - e.$$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Sundaresh. H. R, Shivamogga, Karnataka, India; Ankush Kumar Parcha , New-Delhi, India; Aryan Desai; Corneliu Mănescu-Avram, Ploieşti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal; Matthew Too, Brockport, NY, USA; Yunyong Zhang, Chinaunicom, Yunnan, China; Theo Koupelis, Clark College, Washington, USA.

U651. Let x, y, z, t be real numbers such that $x+y+z+t=0$. Prove that

$$\frac{x+1}{x^2+3}+\frac{y+1}{y^2+3}+\frac{z+1}{z^2+3}+\frac{t+1}{t^2+3}\le\frac{4}{3}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

Let $a=x+1$, $b=y+1$, $c=z+1$, $d=t+1$. We have $a+b+c+d=4$ and the inequality becomes

$$\frac{a}{a^2-2a+4}+\frac{b}{b^2-2b+4}+\frac{c}{c^2-2c+4}+\frac{d}{d^2-2d+4}\le\frac{4}{3},$$

or

$$\sum_{cyc}\left(\frac{1}{2}-\frac{a}{a^2-2a+4}\right)\ge\frac{2}{3},$$

or

$$\sum_{cyc}\frac{(a-2)^2}{a^2-2a+4}\ge\frac{4}{3}.$$

By Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\sum_{cyc}\frac{(a-2)^2}{a^2-2a+4}=&\sum_{cyc}\frac{(a-2)^4}{(a-2)^2(a^2-2a+4)}\\ \ge&\frac{\left[(a-2)^2+(b-2)^2+(c-2)^2+(d-2)^2\right]^2}{\sum_{cyc}(a-2)^2(a^2-2a+4)}\\ =&\frac{(a^2+b^2+c^2+d^2)^2}{\sum_{cyc}(a^4-6a^3+16a^2-24a+16)}.\end{aligned}$$

Hence, it remains to prove that

$$3(a^2+b^2+c^2+d^2)^2\ge 4\sum_{cyc}(a^4-6a^3+16a^2-24a+16),$$

$$6\sum_{sym}a^2b^2-\sum_{cyc}a^4+24\sum_{cyc}a^3-64\sum_{cyc}a^2+128\ge 0.$$

Let $P(X)=X^4-4X^3+6pX^2-4qX+r$ be the polynomial of degree 4 whose roots are a, b, c, d. We have the following identities

$$\begin{aligned}\sum_{sym}a^2b^2=&36p^2-32q+2r,\\ \sum_{cyc}a^2=&16-2p,\\ \sum_{cyc}a^3=&64-72p+12q,\\ \sum_{cyc}a^4=&4\sum_{cyc}a^3-6p\sum_{cyc}a^2+4q\sum_{cyc}a-4r\\ =&72p^2-384p+64q-4r+256.\end{aligned}$$

Thus, our inequality becomes

$$9p^2-36p+2q+r+24\ge 0.$$

On the other hand

$$(a-b)^2(c-d)^2+(a-c)^2(b-d)^2+(a-d)^2(b-c)^2 \geq 0 \Longleftrightarrow r \geq 4q-3p^2.$$

Therefore it suffices to prove that

$$9p^2-36p+2q+4q-3p^2+24 \geq 0,$$

that is

$$p^2-6p+q+4 \geq 0.$$

By Roll's Theorem, $P'(X)=4\left(X^3-3X^2+3pX-q\right)$ has three real roots u,v,w. We have $u+v+w=3$ and setting $uv+vw+wu=3(1-t^2)$, $t \geq 0$, we have the following result

$$q \geq (1-2t)(t-1)^2.$$

We still have to prove that

$$(1-t^2)^2-6(1-t^2)+(1-2t)(t+1)^2+4 \geq 0,$$

that is

$$t^2(t-1)^2 \geq 0,$$

clearly true. The equality holds when $a=b=c=d=1$ which means $x=y=z=t=0$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

We have

$$\frac{x+1}{x^2+3}-\frac{1}{2}=-\frac{(x-1)^2}{2(x^2+3)},$$

with similar expressions for the other variables. Therefore, it is sufficient to show that

$$\sum_{cyc} \frac{(x-1)^2}{2(x^2+3)} \geq \frac{2}{3}.$$

From the given condition we get $x=-(y+z+t)$ and thus $x^2=(y+z+t)^2$. Using AM-GM we get $4x^2 \leq 3(x^2+y^2+z^2+t^2)$, and thus $4(x^2+3) \leq 3(4+\sum_{cyc} x^2)$, with similar expressions for the other variables. Therefore, it is sufficient to show that

$$\frac{\sum_{cyc}(x-1)^2}{4+\sum_{cyc} x^2} \geq 1,$$

which is obvious because $\sum_{cyc} x=0$. Equality occurs when $x=y=z=t=0$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Daniel Văcaru, Economic College Maria Teiuleanu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U652. Let k, n be a positive integer, such that n is even. Prove that the following polynomial has a root on unit circle, if and only if $\nu_2(n) < k$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We shall prove a more general argument that if a, b, c are integers satisfying $0 \le a < b \le c$, then the polynomial $1 + \ldots + x^a + x^b + \ldots + x^c$ has a root on the unit circle if and only if $c = a + b$ or at least one of the inequalities $\gcd(a+1, c-b+1) > 1$, $\gcd(c+1, c-b+a+2) > 1$, $\nu_2(c-a+b) > \nu_2(b)$ holds. First, selecting $a = d-2 \ge 1$ and $b = c = n > a$, we see that $c \ne a+b$ and $\gcd(a+1, c-b+1) = \gcd(d-1, 1) = 1$. Inserting $c = n$, $c-b+a+2 = d$, $|c-a-b| = d-2$, $b = n$ we find that $\gcd(c+1, c-b+a+2) = \gcd(n+1, d)$. Let $n > d-2 \ge 1$ be positive integers. Then, the polynomial $1 + x + \ldots + x^{d-2} + x^n$ has a root on the unit circle if and only if at least one of the inequalities $\gcd(n+1, d) > 1$ or $\nu_2(d-2) > \nu_2(n)$ holds. We first prove this nice lemma.

Lemma. Let z_1, z_2, z_3, z_4 be complex numbers of modululs 1 satisfying $z_1 + z_2 + z_3 + z_4 = 0$. Then, $z_1 + z_j = 0$ for some $j \in \{2, 3, 4\}$.

Proof. Assume z_1, z_2, z_3, z_4 are in clockwise order on the unit circle $|z| = 1$. We then prove $z_1 + z_3 = z_2 + z_4 = 0$. Let ℓ be a line passing through the origin and the midpoint of the line segment connecting z_1 and z_2. By projecting the sum $z_1 + z_2 + z_3 + z_4 = 0$ onto ℓ, we deduce that ℓ passes through the midpoint of the line segment connecting z_3 and z_4. Furthermore, the distance between the origin and these midpoints must be equal. Therefore, the points z_1, z_2, z_3, z_4 are the consecutive vertices of a rectangle. Of course, we have the degenerate situations, namely, $z_1 = z_2$, $z_3 = z_4$ and $z_1 = z_4$, $z_2 = z_3$ are also possible. However, in all these cases, we deduce that z_1 and z_3 are on a diameter of the unit circle, so $z_1 + z_3 = 0$. □

For $c = a + b$, then $1 + \ldots + x^a + x^b + \ldots + x^c = (1 + \ldots + x^a)(1 + x^b)$ is a product of cyclotomic polynomials, so all of its roots are roots of unity. Assume now $c \ne a + b$. If r, $|r| = 1$, is a root of $f(x)$, then $(1-r)f(r) = 1 - r^{a+1} + r^b - r^c + 1 = 0$. It is clear that $r \ne 1$. Now, we have three different possibilities: (i) $1 - r^{a+1} = r^b - r^{c+1} = 0$, (ii) $1 - r^{c+1} = -r^{a+1} + r^b = 0$ and (iii) $1 + r^b = r^{a+1} + r^{c+1} = 0$. In the first case, $r^{a+1} = r^{c-b+1} = 1$ and hence if $d = \gcd(a+1, c-b+1) = 1$ it follows that $r = 1$, otherwise, $r^d = 1$ and we are done. In the second case, $r^{c+1} = r^{b-a+1} = 1$ so, if $g = \gcd(c+1, b-a+1) = 1$, then $r = 1$, otherwise, $r^g = 1$. Finally, if $r^b + 1 = r^{c-a} + 1 = 0$ and $\nu_2(b) = \nu_2(c-a)$, then $r^{\gcd(b,c-a)} = -1$ and we can be sure that $\gcd(b, c-a) > 1$, otherwise, it would be absurd.

U653 . Find all finite groups which have proper subgroups and these have order 2 or 3 only.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

Solution by the author

We have three cases.

(i) We determine finite groups G that admit only proper subgroups of order 2. For any $x \in G \setminus \{e\}$ we have $\text{ord}(x) = 2$, so G is abelian and has 2^n elements. In this case, the group G is isomorphic to the group $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}$, which admits the subgroup $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n-1 \text{ times}} \times \{0\}$ of order 2^{n-1}. So, $2^{n-1} = 2$, from which we get $n = 2$ and $\text{ord}(G) = 4$. In this case we have $G \simeq \mathbb{Z}_4$ or $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) We determine the finite groups G that admit proper subgroups of order 2 and of order 3. For any $x \in G \setminus \{e\}$ we have $\text{ord}(x) = 2$ or $\text{ord}(x) = 3$, so G has $2^n \cdot 3^m$ elements. If $n > 1$, as $(2,3) = 1$, it follows from Sylow's theorem that G admits a subgroup of order 2^n, contradiction! Analogous for $m > 1$. Hence $n = m = 1$, so $\text{ord}(G) = 6$ and then $G \simeq \mathbb{Z}_6$ or $G \simeq S_3$. Both $\mathbb{Z}_6$ and S_3 satisfy the conditions of the problem.

Another way: For any $x \in G \setminus \{e\}$ we have $\text{ord}(x) = 2$ or $\text{ord}(x) = 3$. As $2 \mid \text{ord}(G)$ and $3 \mid \text{ord}(G)$, it follows that $\text{ord}(G) = 6p$, $p \in \mathbb{N}^*$.

If G is commutative, considering $a, b \in G$ with $\text{ord}(a) = 2$ and $\text{ord}(b) = 3$, we have $\text{ord}(ab) = 6$, i.e. the subgroup generated by ab has order 6. Therefore $\text{ord}(G) = 6$ and then $G \simeq \mathbb{Z}_6$.

If G is non-commutative, then $Z(G) = \{e\}$ (otherwise we find two elements a, b of order 2 and 3 respectively that commute, so $\text{ord}(ab) = 6$ and then $G \simeq \mathbb{Z}_6$, i.e. G is commutative, contradiction). Writing the class equation for group G we have:

$$|G| = |Z(G)| + \sum_{x \in G \setminus Z(G)} \frac{|G|}{|C_x|},$$

where C_x is the centralizer of x in G. Then

$$|G| = |Z(G)| + \sum_{\text{ord}(x)=2} \frac{|G|}{|C_x|} + \sum_{\text{ord}(x)=3} \frac{|G|}{|C_x|} \implies 6p = 1 + \frac{6p}{2} + \frac{6p}{3},$$

so $p = 1$ and then $\text{ord}(G) = 6$. Since G is non-commutative, it follows that $G \simeq S_3$.

(iii) We determine the finite groups G that admit only proper subgroups of order 3. For any $x \in G \setminus \{e\}$ we have $\text{ord}(x) = 3$, so G has 3^n elements. If G is commutative, then it is isomorphic to the group $\underbrace{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}$, which admits the subgroup $\underbrace{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3}_{n-1 \text{ times}} \times \{0\}$ of order 3^{n-1}. So, $3^{n-1} = 3$, from which we get $n = 2$ and $\text{ord}(G) = 9$. In this case we have $G \simeq \mathbb{Z}_9$ or $G \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

If G is non-commutative with 3^n elements, from $Z(G) \leq G$ we deduce $|Z(G)| = 1$ or $|Z(G)| = 3$.

For $|Z(G)| = 1$, from the class equation for the group G we get

$$|G| = |Z(G)| + \sum_{x \in G \setminus Z(G)} \frac{|G|}{|C_x|} \implies 3^n = 1 + 3^{n-1} \cdot k,$$

so $n = 1$. We have $|G| = 3$, so G is commutative, contradiction.

For $|Z(G)| = 3$, from the class equation for the group G we get

$$|G| = |Z(G)| + \sum_{x \in G \setminus Z(G)} \frac{|G|}{|C_x|} \implies 3^n = 3 + 3^{n-1} \cdot k,$$

and then $n \in \{1, 2\}$. We have $|G| = 3$ or $|G| = 9$. As 3 is prime we deduce that G is commutative, contradiction.

U654. Evaluate

$$\int_0^1 \frac{x\,\sqrt{x}\,\ln(x)}{x^2-x+1}\,dx$$

Proposed by Vasile Mircea Popa, Sibiu, România

Solution by the author

Let us denote:

$$I=\int_0^1 \frac{x\,\sqrt{x}\,\ln(x)}{x^2-x+1}\,dx=\int_0^1 \frac{(1+x)\,x\,\sqrt{x}\,\ln(x)}{1+x^3}\,dx;$$

$$A=\int_0^1 \frac{x^{\frac{3}{2}}\,\ln(x)}{1+x^3}\,dx;\quad B=\int_0^1 \frac{x^{\frac{5}{2}}\,\ln(x)}{1+x^3}\,dx;$$

We consider integral A. We have, successively:

$$A=\int_0^1 \left(\sum_{n=0}^{\infty} x^{6n+\frac{3}{2}}\,\ln(x)-\sum_{n=0}^{\infty} x^{6n+\frac{9}{2}}\,\ln(x)\right)dx$$

$$A=\sum_{n=0}^{\infty}\left(\int_0^1 x^{6n+\frac{3}{2}}\,\ln(x)\,dx-\int_0^1 x^{6n+\frac{9}{2}}\,\ln(x)\,dx\right)$$

We will to use the following relation:

$$\int_0^1 x^a\,\ln(x)\,dx=-\frac{1}{(a+1)^2}\,,\,where\;a\in\mathbf{R},\;a\geq 0$$

We obtain:

$$A=\sum_{n=0}^{\infty}\left[\frac{1}{\left(6n+\frac{11}{2}\right)^2}-\frac{1}{\left(6n+\frac{5}{2}\right)^2}\right]=\sum_{n=0}^{\infty}\left[\frac{\frac{1}{36}}{\left(n+\frac{11}{12}\right)^2}-\frac{\frac{1}{36}}{\left(n+\frac{5}{12}\right)^2}\right]$$

We now use the following relation:

$$\psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\psi_1(x)$ is the trigamma function.

We obtained the value of the integral A:

$$A=\frac{1}{36}\left[\psi_1\left(\frac{11}{12}\right)-\psi_1\left(\frac{5}{12}\right)\right]$$

We consider the integral B. By proceeding similarly to the integral A, we obtain:

$$B=\frac{1}{36}\left[\psi_1\left(\frac{13}{12}\right)-\psi_1\left(\frac{7}{12}\right)\right]$$

Combining the results:

$$I=A+B=\frac{1}{36}\left[-\psi_1\left(\frac{5}{12}\right)-\psi_1\left(\frac{7}{12}\right)+\psi_1\left(\frac{11}{12}\right)+\psi_1\left(\frac{13}{12}\right)\right]$$

We use the reflection formula:

$$\psi_1\left(x\right)+\psi_1\left(1-x\right)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\psi_1\left(\frac{5}{12}\right)+\psi_1\left(\frac{7}{12}\right)=\frac{\pi^2}{\sin^2(5\pi/12)}=\frac{4\pi^2}{2+\sqrt{3}}$$

We use the recurrence formula:

$$\psi_1\left(x+1\right)=\psi_1\left(x\right)-\frac{1}{x^2}$$

We obtain:

$$\psi_1\left(\frac{11}{12}\right)+\psi_1\left(\frac{13}{12}\right)=\psi_1\left(\frac{11}{12}\right)+\psi_1\left(\frac{1}{12}+1\right)=$$

$$=\psi_1\left(\frac{11}{12}\right)+\psi_1\left(\frac{1}{12}\right)-144=\frac{\pi^2}{\sin^2(\pi/12)}-144=\frac{4\pi^2}{2-\sqrt{3}}-144$$

Result:

$$I=\frac{1}{36}\left(-\frac{4\pi^2}{2+\sqrt{3}}+\frac{4\pi^2}{2-\sqrt{3}}-144\right)$$

$$I=\frac{2\pi^2\sqrt{3}}{9}-4$$

Thus, the problem is solved.

Also solved by Ankush Kumar Parcha, New-Delhi, India; Aryan Desai; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal; Yunyong Zhang, Chinaunicom, Yunnan, China; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, Washington, USA.

O649. Find all positive integers n such that

$$n!+1=(8a-3)^2;(n+1)!+1=(8b+3)^2;(2n-1)!+1=(8c-1)^2,$$

for some positive integers a, b, c congruent to $1 \mod 8$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Anderson Torres, Brazil

Let's write $c = 8d+1$. Then $(2n-1)! = (8(8d+1)-1)^2 - 1 = (64d+6)(64d+8) = 16(32d+3)(8d+1)$, implying $2n-1 < 8$ (since $8! = 40320 = 128 \cdot 315$ can't be written as $16 \cdot i$ with i odd).

It implies $n \geq 4$. Testing manually, we find only 4 as solution (with $a = 1, b = 1, c = 1$).

Solution by the author

Let $a = 8u+1$, $b = 8v+1$, $c = 8w+1$, for some nonnegative integers u, v, w. We have

$$n!+1+(n+1)!+1+(2n-1)!+1 = (64u+5)^2+(64v+11)^2+(64w+7)^2,$$

implying

$$n!+(n+1)!+(2n-1)! = 64^2(u^2+v^2+w^2)+128(u+v+w)+192.$$

For $n \geq 8$, the left-hand side is divisible by 2^7, while the right-hand side is only divisible by 2^6, a contradiction. Hence $n \leq 7$. Because $n!+1$ is not a perfect square for $n = 1, 2, 3$ and $(2n-1)!+1$ is not a perfect square for $n = 5, 6, 7$, we are left with $n = 4$. And

$$4!+1 = (8 \cdot 1 - 3)^2, \qquad 5!+1 = (8 \cdot 1 + 3)^2, \qquad (2 \cdot 4 - 1) = (8 \cdot 9 - 1)^2,$$

with 1, 1, 8 congruent to 1 modulo 9. Thus $n = 4$.

Also solved by Sundaresh. H. R, Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA.

O650. Find the greatest constant λ such that the inequality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}+\lambda\left(\frac{a^2+b^2+c^2}{(a+b+c)^2}-\frac{1}{3}\right)$$

holds for all non-negative real numers a, b, c, no two of which are zero.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author
Using the known identity

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}=\sum_{\text{cyc}} \frac{(b-c)^2}{2(a+b)(a+c)}$$

we can rewrite the inequality as

$$\sum_{\text{cyc}} \frac{(b-c)^2}{2(a+b)(a+c)} \geq \frac{\lambda}{3(a+b+c)^2} \cdot \sum_{\text{cyc}}(b-c)^2$$

or

$$\sum_{\text{cyc}}\left(\frac{3}{(a+b)(a+c)}-\frac{2\lambda}{(a+b+c)^2}\right)(b-c)^2 \geq 0.$$

The standard form of this inequality is

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0$$

where

$$S_a=\frac{3}{(a+b)(a+c)}-\frac{2\lambda}{(a+b+c)^2},$$
$$S_b=\frac{3}{(b+c)(b+a)}-\frac{2\lambda}{(a+b+c)^2},$$
$$S_c=\frac{3}{(c+a)(c+b)}-\frac{2\lambda}{(a+b+c)^2}.$$

Now we take $c=a$, the above inequality gives us

$$2S_a(a-b)^2 \geq 0,$$

or equivalently

$$S_a=\frac{3}{2a(a+b)}-\frac{2\lambda}{(2a+b)^2} \geq 0,$$
$$\frac{3(2a+b)^2}{4a(a+b)} \geq \lambda,$$
$$\frac{3(2+x)^2}{4(1+x)} \geq \lambda \quad (\text{where } x=\frac{b}{a}),$$
$$3=\min_{x \geq 0} \frac{3(2+x)^2}{4(1+x)} \geq \lambda.$$

Next, we will show that the inequality holds for $\lambda=3$, i.e.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{1}{2}+\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}.$$

This inequality is written in S.O.S fomr with

$$S_a = \frac{1}{(a+b)(a+c)} - \frac{2}{(a+b+c)^2},$$

$$S_b = \frac{1}{(b+c)(b+a)} - \frac{2}{(a+b+c)^2},$$

$$S_c = \frac{1}{(c+a)(c+b)} - \frac{2}{(a+b+c)^2}.$$

Due to the symmetry of the inequality we can assume that $a \le b \le c$. Then $S_a \ge S_b \ge S_c$. Now we have

$$\begin{aligned} S_b + S_c &= \frac{1}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) - \frac{4}{(a+b+c)^2} \\ &\ge \frac{4}{(b+c)(2a+b+c)} - \frac{4}{(a+b+c)^2} \\ &= \frac{4a^2}{(b+c)(2a+b+c)(a+b+c)^2} \\ &\ge 0. \end{aligned}$$

This tells us that $S_a \ge S_b \ge 0$. From here and note that $(c-a)^2 \ge (a-b)^2$ we obtain

$$\begin{aligned} S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 &\ge S_b(c-a)^2 + S_c(a-b)^2 \\ &\ge (S_b + S_c)(a-b)^2 \\ &\ge 0. \end{aligned}$$

The conclusion follows.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Arkady Alt, San Jose, CA, USA; Jiang Lianjun,Quanzhou Middle School,Guilin, China; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, Washington, USA.

O651. Find the greatest real number k such that for all nonnegative real numbers a, b, c, no two of wich are zero,

$$\frac{a^3+ka^2b}{a+b}+\frac{b^3+kb^2c}{b+c}+\frac{c^3+kc^2a}{c+a}\geq\frac{(k+1)(ab+bc+ca)}{2}.$$

Proposed by Titu Andreescu, Dallas, USA and Marius Stănean, Zalău, România

Solution by the authors

We have

$$\sum_{cyc}\frac{a^3}{a+b}+\sum_{cyc}\frac{b^3}{a+b}=2(a^2+b^2+c^2-(ab+bc+ca),$$

$$\sum_{cyc}\frac{a^3}{a+b}-\sum_{cyc}\frac{b^3}{a+b}=\frac{(ab+bc+ca)(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)},$$

so if we setting $X=(a-b)(b-c)(c-a)$

$$\sum_{cyc}\frac{a^3}{a+b}=a^2+b^2+c^2-\frac{ab+bc+ca}{2}+\frac{(ab+bc+ca)X}{2(a+b)(b+c)(c+a)}.$$

Similarly, we have

$$\sum_{cyc}\frac{a^2b}{a+b}+\sum_{cyc}\frac{b^2a}{a+b}=ab+bc+ca,$$

$$\sum_{cyc}\frac{a^2b}{a+b}-\sum_{cyc}\frac{b^2a}{a+b}=-\frac{(ab+bc+ca)(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)},$$

so

$$\sum_{cyc}\frac{a^2b}{a+b}=\frac{ab+bc+ca}{2}-\frac{(ab+bc+ca)X}{2(a+b)(b+c)(c+a)}.$$

Hence, the inequality can be rewritten as

$$\sum_{cyc}a^2-\sum_{cyc}ab+\frac{(ab+bc+ca)X}{2(a+b)(b+c)(c+a)}\geq\frac{k(ab+bc+ca)X}{2(a+b)(b+c)(c+a)},$$

or

$$\frac{2(a+b)(b+c)(c+a)(a^2+b^2+c^2-ab-bc-ca)}{ab+bc+ca}\geq(k-1)X.$$

We have three cases:

1. If two of the variables are equal, then the inequality is true for any real k.

2. $a>b>c$ which implies $X<0$, so the inequality is true for any real k.

3. $c>b>a$ so $X>0$. If $k\leq 1$ the inequality is obviously true, so we will look for $k>1$. Denote

$$f(a,b,c)=\frac{(a+b)(b+c)(c+a)Y}{(ab+bc+ca)X},$$

where we setting $Y=(a-b)^2+(b-c)^2+(c-a)^2$. We have

$$\begin{aligned}f(a,b,c)-f(a-a,b-a,c-a)=&\frac{(a+b)(b+c)(c+a)Y}{(ab+bc+ca)X}-\frac{(b-a)(b+c-2a)(c-a)Y}{(b-a)(c-a)X}\\=&\frac{a(3ab+3ac+2bc)Y}{(ab+bc+ca)X}\geq 0.\end{aligned}$$

Therefore, it suffices to find minimum value of

$$\begin{aligned}f(0,u,v) =&\frac{2uv(u+v)(u^2-uv+v^2)}{u^2v^2(v-u)}\\ =&\frac{2(u+v)(u^2-2uv+v^2)}{uv(v-u)}\\ =&\frac{2(t+1)(t^2-t+1)}{t(t-1)} = \frac{2(t^3+1)}{t^2-t}\\ =&2t+2+\frac{4}{t-1}-\frac{2}{t},\end{aligned}$$

where $t = \dfrac{v}{u} > 1$.

Denote $g(t) = 2t+2+\frac{4}{t-1}-\frac{2}{t}$, for $t > 1$. We have

$$g'(t) = 2 - \frac{4}{(t-1)^2} + \frac{2}{t^2} = \frac{2(t^4-2t^3-2t+1)}{t^2(t-1)^2},$$

$$g''(t) = \frac{8}{(t-1)^3} - \frac{4}{t^3} > 0, \quad \text{for } t > 1$$

We were looking for the critical points of g

$$g'(t) = 0 \Longleftrightarrow t^4 - 2t^3 - 2t + 1 = 0$$

which is a reciprocal equation. Let $y = t + \frac{1}{t}$, then

$$y^2 - 2y - 2 = 0 \Longrightarrow y = 1 + \sqrt{3}$$

$$\Longrightarrow t + \frac{1}{t} = 1 + \sqrt{3} \Longrightarrow t = \frac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2} > 1.$$

This is the minimum point of the function g and its minimum value is

$$g\left(\frac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}\right) = 2\sqrt{9+6\sqrt{3}}.$$

In conclusion

$$k = 1 + 2\sqrt{9+6\sqrt{3}}.$$

The equality holds, in this case, when

$$a = 0, \quad \frac{c}{b} = \frac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}.$$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Clark College, Washington, USA.

O652. For a positive integer n we denote by $S_2(n)$ the sum of digits of n in base 2. Prove that there are integers $l, m \geq 1402$ for which there are infinitely many pairs of odd integers (a, b) with $S_2(a) = l$, $S_2(b) = m$ and $S_2(ab) = 4$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let $f(x) = x^9 + 1$. Observe that

$$f(x) = (x+1)(x^2 - x + 1)(x^6 - x^3 + 1) = (x^2 - x + 1)(x^7 + x^6 - x^4 - x^3 + x + 1).$$

Let $a_0 = a_0(n) = 2^{2n} - 2^n + 1$, $b_0 = b_0(n) = 2^{7n} + 2^{6n} - 2^{4n} - 2^{3n} + 2^n + 1$. Then $a_0 b_0 = 2^{9n} + 1$. Moreover, $S(a_0) = n + 1$, $S(b_0) = 3n + 2$. Let $l = S(a_0)$ and $m_1 = S(b_0)$. Choosing n large enough, it follows that $a^{(N)} = a_0$, $b^{(N)} = b_0(2^N + 1) = 2^N b_0 + b_0$. Choose N large enough to find $S(a^{(N)}) = l$ and $S(b^{(N)}) = 2S(b_0) = 2m_1 = m$. Hence,

$$S\left(a^{(N)} b^{(N)}\right) = S\left((2^N + 1)a_0 b_0\right) = S\left(2^{N+9n} + 2^N + 2^{9n} + 1\right) = 4.$$

O653. Let $a_1 = a_2 \geq a_3 \geq \cdots \geq a_{n-1} = a_n$ be real numbers where $n \geq 4$. Prove that

$$n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \geq (a_1 + a_2 + \cdots + a_n)^2.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution by the author

For $n = 4$, the inequality reduces to an identity. Consider next $n > 4$, denote

$$S = \frac{a_2 + a_{n-1}}{2}, \qquad s = \frac{a_3 + \cdots + a_{n-2}}{n-4},$$

and write the inequality as follows:

$$n[a_2^2 + a_{n-1}^2 + a_2a_{n-1} + (a_2a_3 + \cdots + a_{n-2}a_{n-1})] \geq [2(a_2 + a_{n-1}) + (a_3 + \cdots + a_{n-2})]^2,$$

$$n[4S^2 - a_2a_{n-1} + (a_2a_3 + \cdots + a_{n-2}a_{n-1})] \geq [4S + (n-4)s]^2.$$

Since the sequences $(a_2, \ldots, a_{n-2})$ and $(a_3, \ldots, a_{n-1})$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \geq (a_2 + \cdots + a_{n-2})(a_3 + \cdots + a_{n-1}),$$

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \geq [a_2 + (n-4)s][a_{n-1} + (n-4)s],$$

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \geq a_2a_{n-1} + 2(n-4)sS + (n-4)^2s^2.$$

So, it suffices to show that

$$n\left[4S^2 - a_2a_{n-1} + \frac{a_2a_{n-1} + 2(n-4)sS + (n-4)^2s^2}{n-3}\right] \geq [4S + (n-4)s]^2,$$

which is equivalent to

$$4(n-3)S^2 - 6(n-4)sS + 3(n-4)s^2 \geq na_2a_{n-1}.$$

Since $a_2 \geq s \geq a_{n-1}$, we have

$$(s - a_2)(s - a_{n-1}) \leq 0, \qquad a_2a_{n-1} \leq 2sS - s^2.$$

Therefore, it suffices to show that

$$4(n-3)S^2 - 6(n-4)sS + 3(n-4)s^2 \geq n(2sS - s^2),$$

that is equivalent to

$$(n-3)(S-s)^2 \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Also solved by Theo Koupelis, Clark College, Washington, USA.

O654. Let $f(x)$ be a polynomial with integer coefficients whose leading coefficient is positive. Prove that the sequence

$$x_n = \left\{ \frac{6^{f(n)}}{n} \right\}$$

is dense in $[0,1]$. (By $\{a\}$ we mean the fractional part of a.)

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We prove that for all n large enough, the sequence $x_n = \left\{ \frac{6^{f(n)}}{n} \right\}$ can either take zero or numbers of the form $\frac{r}{m}$, where $m \geq 2$ is an integer coprime to 6 and $r \in \{2^i \cdot 3^j : i,j \geq 0, i,j \in \mathbb{Z}\}$ modulo m. For this reason, we need some important lemmas.

Lemma 1. Let $t \geq 1$, $u \geq 0$, $v \geq 1$, $a \geq 2$ and for every $f(x)$ with integer coefficients and positive leading coefficient there are infinitely many n such that $f(va^n) - n \equiv u \pmod{t}$.

Proof. By induction on t. □

Lemma 2. Let $f(x)$ be a polynomial with positive leading coefficient. Then for any integers $t \geq 1$, $k \geq 1$, $m \geq 1$, $u1, \ldots, u_k \geq 0$, $v_1, \ldots, v_k \geq 1$, $k \geq 1$ and any k positive integers $P_1, \ldots, P_k > 1$ there is a vector of positive integers $(n_1, \ldots, n_k)$ such that $v_i f(mP_1^{n_1} \cdot \ldots \cdot P_k^{n_k}) - n_i \equiv u_i \pmod{t}$ for $i = 1, \ldots, k$ and $\min(n_1, \ldots, n_k) \geq K$.

Note that $\left\{ \frac{6^{f(n)}}{n} \right\} = 0$ for each $n = 6^s$ where s is large enough. So, $0 \in (x_n)_{n \geq 1}$, and the value 0 is attained for infinitely many n. Now assume that $w \neq 0$, $w \in (x_n)_{n \geq 1}$. Evidently, w must be a rational number in $(0,1)$. Suppose $w = \left\{ \frac{6^{f(s)}}{s} \right\}$ for some positive integer $s \geq n_f$. We claim that the equation $w = \left\{ \frac{6^{f(n)}}{n} \right\}$ has infinitely many solutions in positive integers. Indeed, set $n = s \cdot 6^{\ell}$. Notice that

$$\frac{6^{f(n)}}{n} - \frac{6^{f(s)}}{s} = \frac{6^{f(s \cdot 6^{\ell})}}{s \cdot 6^{\ell}} - \frac{6^{f(s)}}{s} = \frac{6^{f(s)}}{s}(6^{f(s \cdot 6^{\ell}) - f(s) - \ell} - 1).$$

We prove that by suitable choice of ℓ the above expression is an integer. Let s_0 be the largest odd divisor of s and set $s_1 = \frac{s}{s_0}$. We will prove that $s_1 \mid 6^{f(s)}$ and $s_0 \mid 6^{f(s \cdot 6^{\ell}) - f(s) - \ell}$ for infinitely many ℓ. Now, s_1 must be a product non-negative powers of 2 and we can assume $\nu_2(s_1) = l \geq 1$. Then, $2^{2l} \mid 2^s$. Indeed, $s \geq s_1 \geq 2^l \geq 2l$. Applying this argument to powers of 3 in s_1 we deduce that $s_1^2 \mid 6^s$.

Moreover, since $\gcd(s_0, 6) = 1$ it suffices to show that $\varphi(s_0) \mid f(s \cdot 6^{\ell}) - f(s) - \ell$ for infinitely many ℓ. Indeed, in the lemma, put $(t, u, v) = (\varphi(s_0), f(s), s)$. Thus, the equation $w = \left\{ \frac{6^{f(n)}}{n} \right\}$ has infinitely many solutions in positive integers. Next, assume that $w = \frac{r}{m} \in (x_n)_{n \geq 1}$, where $m \geq 2$, $1 \leq r \leq m$, $\gcd(r, m) = 1$. Then, for some $n \geq n_f$ we must have $\frac{r}{m} = \left\{ \frac{6^{f(s)}}{s} \right\}$. Write $s = s_0 s_1$, where s_0 is the largest divisor of s, $\gcd(s, 6) = 1$. We claim that $s_0 = m$. Indeed, we have $s_1 \mid 6^{f(s)}$. Letting $L = \left\lfloor \frac{6^{f(s)}}{s} \right\rfloor$, we find that

$$\left\{ \frac{6^{f(s)}}{s} \right\} = \frac{r}{m} = \frac{6^{f(s)}}{s} - L = \frac{6^{f(s)}}{s_0 s_1} - L = \frac{\frac{6^{f(s)}}{s_1} - L s_0}{s_0}.$$

Here the numerator $\frac{6^{f(s)}}{s_1} - Ls_0$ is coprime to s_0, hence $r = \frac{6^{f(s)}}{s_1} - Ls_0$, $m = s_0$.
In order to complete the proof, it remains to show that only $r \in \{2^i \cdot 3^j : i, j \geq 0, i, j \in \mathbb{Z}\}$ modulo m occur as numerators of the rational numbers $w = \dfrac{r}{m} \in (x_n)_{n\geq 1}$ and all $\{2^i \cdot 3^j : i, j \geq 0, i, j \in \mathbb{Z}\}$ modulo m occur as numerators. The first assertion is clear because, $1 \leq r < m$ and $\dfrac{6^{f(s)}}{s_1} \in \{2^i \cdot 3^j : i, j \geq 0, i, j \in \mathbb{Z}\}$ modulo m, so that $r = \dfrac{6^{f(s)}}{s_1} - Ls_0 = \dfrac{6^{f(s)}}{s_1} - Lm \in \{2^i \cdot 3^j : i, j \geq 0, i, j \in \mathbb{Z}\}$ modulo m.
To prove the second assertion, assume that $r \in \{2^i \cdot 3^j : i, j \geq 0, i, j \in \mathbb{Z}\}$ modulo m. Then, for some integers $u_1, u_2, T \geq 0$ we have $r = 2^{u_1} \cdot 3^{u_2} - T_m$. Let $t = \varphi(m)$, $P_1 = 2$, $P_2 = 3$ in the lemma. Then there is a vector of positive integers (n_1, n_2) such that $v_i f(m2^{n_1} \cdot 3^{n_2}) - n_i \equiv u_i \pmod{\varphi(m)}$ for $i = 1, 2$. Therefore, if $n = m2^{n_1} \cdot 3^{n_2}$ we have

$$\frac{6^{f(n)}}{n} = \frac{2^{f(n)} \cdot 3^{f(n)}}{m2^{n_1} \cdot 3^{n_2}} = \frac{2^{f(n)-n_1} \cdot 3^{f(n)-n_2}}{m}.$$

Since $\gcd(6, m) = 1$, the numerator of the last fraction equals $2^{u_1} \cdot 3^{u_2}$ modulo m, which is r modulo m. Thus $\dfrac{6^{f(n)}}{n} = \dfrac{r}{m} + B$ for some integer B. Thus, for each n we obtain $\left\{\dfrac{6^{f(n)}}{n}\right\} = \dfrac{r}{m}$.