On the Calculation of a Series Involving the Product of the Tail $\zeta(3)$ and Harmonic Numbers $H_n H_n^{(2)}$

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Abstract

The infinite series

$$\sum_{n=1}^{\infty} H_n \left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) \left(\zeta(3) - 1 - \frac{1}{2^3} - \ldots - \frac{1}{n^3}\right)$$

where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the nth harmonic number and $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is Apéry's constant is evaluated in closed form.

1 Introduction

The the generalized harmonic number $H_n^{(r)}$ is defined to be

$$H_n^{(r)} := \sum_{j=1}^{n} \frac{1}{j^r} = 1 + \frac{1}{2^r} + \ldots + \frac{1}{n^r}.$$

Of course, $H_n^{(1)} \equiv H_n$. We evaluate the infinite series from [1, Problem 7.91, p. 214]

$$\sum_{n=1}^{\infty} H_n \left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) \left(\zeta(3) - 1 - \frac{1}{2^3} - \ldots - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} H_n H_n^{(2)} \left(\zeta(3) - H_n^{(3)}\right)$$

in terms of the Riemann zeta function (see [2, 25.2]), defined as

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} = 1 + \frac{1}{2^s} + \ldots + \frac{1}{n^s} + \ldots, \quad \forall \Re(s) > 1.$$

Special values of the Riemann zeta function can be obtained from [2, 25.6]. The infinite series can be calculated quickly using the results from [1, Problem 7.104(a), p.220] and [1, Problem 2.56, p. 2.56]. In this paper, we will make use of some special functions. One of these functions is the classical gamma function, which is traditionally defined to be

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt, \quad \forall \Re(z) > 0$$

(see [2, 5.2.2]) and satisfies the recursive formula

$$\Gamma(z+1) = z\Gamma(z) \tag{1}$$

(see [3, 6.1.15, p. 256]). Special values of the gamma function can be obtained from [2, 5.4] and other interesting properties of the gamma function can be found in [3, pp. 255-257].

Our calculations will involve the polygamma functions $\psi^{(m)}(z)$. The polygamma function is defined to be

$$\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z)$$

(see [3, 6.4.1, p. 260]) and satisfies the recurrence formula

$$\psi^{(m+1)}(z+1) = \psi^{(m)}(z) + \frac{(-1)^m m!}{z^{m+1}}, \quad \forall m \in \{0, 1, \ldots\} \tag{2}$$

(see [3, 6.4.6, p. 260]). The polygamma function also has integer values

$$\begin{aligned} \psi^{(m)}(n+1) &= (-1)^m m! \left(H_n^{(m+1)} - \zeta(m+1) \right) \\ &= (-1)^m m! \left(-\zeta(m+1) + 1 + \frac{1}{2^{m+1}} + \ldots + \frac{1}{n^{m+1}} \right) \end{aligned} \tag{3}$$

(see [3, 6.4.3]). The digamma function $\psi^{(0)}(z) \equiv \psi(z)$ is simply a special case of the polygamma function and satisfies the recursive formula

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{4}$$

(see [3, 6.3.5, p.258]). Other properties of the polygamma function and digamma function can be examined in [2, 5.2(i), 5.15] and [3, pp. 258-260]. We will also make use of the beta function $\mathrm{B}(x, y)$, defined to be

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \forall \Re(x), \Re(y) > 0 \tag{5}$$

(see [2, 5.12.1]) and the polylogarithm, whose power series is expressed as

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad \forall \Re(s) > 1, |z| \le 1$$

(see [2, 25.12(ii)]). Notice that $\mathrm{Li}_s(1) = \zeta(s)$ and $\mathrm{Li}_1(z) \equiv -\ln(1-z)$. Polylogarithms can also be expressed in terms of integrals of other polylogarithms:

$$\mathrm{Li}_{s+1}(z) = \int_0^z \frac{\mathrm{Li}_s(t)}{t}\, dt \tag{6}$$

(see [4, 7.2, p. 189]). The dilogarithm $\mathrm{Li}_2(z)$ is a specific case of the polylogarithm. The interested reader may examine more properties of the polylogarithm in [4]. Naturally, the calculation of our integrals may occasionally require the evaluation of some Euler sums. We will utilize the following formulae from [5, pp. 103-105] and [6] for Euler sums in our calculation:

$$2\sum_{n=1}^{\infty} \frac{H_n}{n^m} = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1), \quad \forall m \in \{2, 3, \ldots\}, \tag{7}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} =& \zeta(m) \left(\frac{1}{2} - \frac{(-1)^p}{2} \binom{m-1}{p} - \frac{(-1)^p}{2} \binom{m-1}{q} \right) \\ &+ \frac{1-(-1)^p}{2} \zeta(p)\zeta(q) + (-1)^p \sum_{k=1}^{\lfloor p/2 \rfloor} \binom{m-2k-1}{q-1} \zeta(2k)\zeta(m-2k) \\ &+ (-1)^p \sum_{k=1}^{\lfloor q/2 \rfloor} \binom{m-2k-1}{p-1} \zeta(2k)\zeta(m-2k), \quad m = p+q \text{ is odd} \end{aligned} \tag{8}$$

where $\zeta(1) := 0$ whenever it occurs.

2 Some lemmas and the calculation of our series

To make our calculation more readable, we will define our series to be

$$\mathcal{S} := \sum_{n=1}^{\infty} H_n H_n^{(2)} \left(\zeta(3) - H_n^{(3)}\right).$$

In order to evaluate the series, we must establish some integral equalities. Our lemmas will involve several integrals involving logarithms and polylogarithms.

Lemma 1. *The following equalities hold:*

$$\begin{aligned}
&(a) \qquad \int_0^1 \frac{(\mathrm{Li}_2(\mathrm{t}))^2}{t}\,dt = \frac{\pi^2}{3}\zeta(3) - 3\zeta(5)\\
&(b) \qquad \int_0^1 \frac{\mathrm{Li}_2(t)\mathrm{Li}_2(1-t)}{t}\,dt = \frac{9}{2}\zeta(5) - \frac{\pi^2}{3}\zeta(3)\\
&(c) \qquad \int_0^1 \frac{\ln^3(t)\ln^2(1-t)}{(1-t)^2}\,dt = \frac{\pi^4}{6} - 12\zeta(3) + 4\pi^2\zeta(3) - 48\zeta(5)\\
&(d) \qquad \int_0^1 \frac{\ln^2(t)\ln(1-t)}{(1-t)^2}\,dt = \frac{\pi^2}{3} - 4\zeta(3)\\
&(e) \qquad \int_0^1 \frac{\ln^2(t)\mathrm{Li}_2(t)}{1-t}\,dt = \pi^2\zeta(3) - 11\zeta(5)\\
&(f) \qquad \int_0^1 \frac{\ln^3(1-t)}{t^2}\,dt = -6\zeta(3)\\
&(g) \qquad \int_0^1 \frac{\ln^3(1-t)}{t}\,dt = -\frac{\pi^4}{15}\\
&(h) \qquad \int_0^1 \frac{\ln(1-t)\mathrm{Li}_2(t)}{t}\,dt = -\frac{\pi^4}{72}\\
&(i) \qquad \int_0^1 \frac{\ln^2(1-t)\mathrm{Li}_2(t)}{t}\,dt = \frac{\pi^2}{3}\zeta(3) - \zeta(5).
\end{aligned}$$

Proof. To evaluate the integral in (a), we will convert the dilogarithm into its power series, interchange the order of summation and integration, and perform integration by parts:

$$\begin{aligned}
\int_0^1 \frac{(\mathrm{Li}_2(\mathrm{t}))^2}{t}\,dt &= \int_0^1 \frac{t^{n-1}}{n^2}\mathrm{Li}_2(t)\,dt\\
&= \sum_{n=1}^{\infty} \frac{1}{n^2}\int_0^1 t^{n-1}\mathrm{Li}_2(t)\,dt\\
&= \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\left[\frac{t^n}{n}\mathrm{Li}_2(t)\right]_{t=0}^1 + \frac{1}{n}\int_0^1 t^{n-1}\ln(1-t)\,dt\right)\\
&= \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{\zeta(2)}{n} - \frac{H_n}{n^2}\right).
\end{aligned}$$

We made use of $\frac{d}{dt}\mathrm{Li}_2(t) = -\frac{\ln(1-t)}{t}$ and $\int_0^1 t^{n-1}\ln(1-t)\,dt = -\frac{H_n}{n}$ (see [7, eq. 1.4, p. 2]). We may further simplify by making use of the Euler sum formula eq. (7) and the special values of the Riemann zeta function found in [2, 25.6] to obtain the desired result.

To evaluate the integral in (b), we utilize the reflection formula

$$\mathrm{Li}_2(t) + \mathrm{Li}_2(1-t) = \frac{\pi^2}{6} - \ln(t)\ln(1-t)$$

(see [4, 1.11, p.5]) to split the integral apart into

$$\int_0^1 \frac{\mathrm{Li}_2(t)\mathrm{Li}_2(1-t)}{t}\,dt = \frac{\pi^2}{6}\int_0^1 \frac{\mathrm{Li}_2(t)}{t}\,dt - \int_0^1 \frac{\ln(t)\ln(1-t)\mathrm{Li}_2(t)}{t}\,dt - \int_0^1 \frac{(\mathrm{Li}_2(t))^2}{t}\,dt.$$

The integral is simplified using eq. (6) coupled with $\mathrm{Li}_s(1) = \zeta(s)$, the parameterized integral function from [8, eq. 1.204, p. 57]

$$\begin{aligned}\int_0^1 \frac{\ln(1-x)\ln^m(x)\mathrm{Li}_2(x)}{x}\,dx =& \frac{(-1)^{m-1}}{2}m!((m+5)\zeta(m+4) - \zeta(2)\zeta(m+2)) \\ &- \frac{3}{2}(-1)^{m-1}m!\sum_{k=1}^{m}\zeta(k+1)\zeta(m-k+3) \\ &+ (-1)^{m-1}m!\sum_{k=1}^{m}\sum_{n=1}^{\infty}\frac{H_n^{(k+1)}}{n^{m-k+3}}, \quad \forall m \in \{1,2,\ldots\}.\end{aligned}$$

alongside eq. (8), and the integral in part (a). The integral in (c) is a special case of the beta function:

$$\begin{aligned}\int_0^1 \frac{\ln^3(t)\ln^2(1-t)}{(1-t)^2}\,dt &= \int_0^1 \lim_{\substack{x\to 1\\ y\to -1}} (t^{x-1}(1-t)^{y-1})\ln^3(t)\ln^2(1-t)\,dt \\ &= \int_0^1 \lim_{\substack{x\to 1\\ y\to -1}} \frac{\partial^5}{\partial x^3 \partial y^2}(t^{x-1}(1-t)^{y-1})\,dt \\ &= \lim_{\substack{x\to 1\\ y\to -1}} \frac{\partial^5}{\partial x^3 \partial y^2}\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \\ &= \lim_{\substack{x\to 1\\ y\to -1}} \frac{\partial^5}{\partial x^3 \partial y^2}\mathrm{B}(x,y).\end{aligned}$$

One may evaluate the limit using some CAS like Wolfram Mathematica or manually. If done manually, we would make use of the gamma function representation of the beta function in eq. (5) in order to take partial derivatives and apply the limits. Using the recursive formulae from eq. (1), (2), and (4) and each functions' integer values found in [3, 6.16, 6.3.2, 6.4.2] with the zeta function's special values from [2, 25.6].

Our calculation of the integral in (d) is also very similar:

$$\begin{aligned}\int_0^1 \frac{\ln^2(t)\ln(1-t)}{(1-t)^2}\,dt &= \int_0^1 \lim_{\substack{x\to 1\\ y\to -1}} (t^{x-1}(1-t)^{y-1})\ln^2(t)\ln(1-t)\,dt \\ &= \lim_{\substack{x\to 1\\ y\to -1}} \frac{\partial^3}{\partial x^2 \partial y}\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \\ &= \lim_{\substack{x\to 1\\ y\to -1}} \frac{\partial^3}{\partial x^2 \partial y}\mathrm{B}(x,y).\end{aligned}$$

The remaining limits and partial derivatives can be calculated using some CAS like Wolfram Mathematica or computed manually in the same manner as the calculation of (c).

To evaluate the integral from (e), we will express the dilogarithm as its power series representation, interchange the order of summation and integration, then integrate by substituting $t \mapsto e^{-t}$:

$$\begin{aligned}\int_0^1 \frac{\ln^2(t)\mathrm{Li}_2(t)}{1-t}\,dt &= \int_0^1 \frac{\ln^2(t)}{1-t}\sum_{n=1}^{\infty}\frac{t^n}{n^2}\,dt \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 \frac{\ln^2(t)t^n}{1-t}\,dt \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty} \frac{t^2 e^{-(n+1)t}}{1-e^{-t}}\,dt.\end{aligned}$$

We make use of

$$\psi^{(m)}(z) = (-1)^{m+1} \int_0^\infty \frac{t^m e^{-zt}}{1-e^{-t}}\, dt$$

(see [3, 6.4.1, p. 260]) and eq. (3) to obtain

$$2\zeta(3) \sum_{n=1}^\infty \frac{1}{n^2} - 2 \sum_{n=1}^\infty \frac{H_n^{(3)}}{n^2}.$$

The remaining series can be reduced using the definition of the zeta function and eq. (8).

To evaluate (f), we integrate by substituting $t \mapsto 1-t$:

$$\begin{aligned}
\int_0^1 \frac{\ln^3(1-t)}{t^2}\, dt &= \int_0^1 \frac{ln^3(t)}{(1-t)^2}\, dt \\
&= \int_0^1 \ln^3(t) \sum_{n=1}^\infty n t^{n-1}\, dt \\
&= \sum_{n=1}^\infty n \int_0^1 t^{n-1} \ln^3(t)\, dt \\
&= -6 \sum_{n=1}^\infty \frac{1}{n^3} \\
&= -6\zeta(3).
\end{aligned}$$

We made use of the power series $\frac{1}{(1-t)^2} = \sum_{n=1}^\infty n t^{n-1}$ and the integral $\int_0^1 t^{n-1} \ln^3(t)\, dt = -\frac{6}{n^4}$, which can be proven via repeated integration by parts.

We evaluate (g) in a similar way:

$$\begin{aligned}
\int_0^1 \frac{\ln^3(1-t)}{t}\, dt &= \int_0^1 \frac{\ln^3(t)}{1-t}\, dt \\
&= \sum_{n=1}^\infty \int_0^1 t^{n-1} \ln^3(t)\, dt \\
&= -6 \sum_{n=1}^\infty \frac{1}{n^4} \\
&= -6\zeta(4) \\
&= -\frac{\pi^4}{15}.
\end{aligned}$$

We once again made use of the integral $\int_0^1 t^{n-1} \ln^3(t)\, dt = -\frac{6}{n^4}$ alongside the geometric series formula and special value of the zeta function found in [2, 25.6]. One may alternatively calculate the integrals from (f) and (g) by applying limits and partial derivatives to the beta function, similarly to our calculation of (c).

In our calculation of (h), we will once again use the dilogarithm's power series representation and the parameterized integral $\int_0^1 t^{n-1} \ln(1-t)\, dt = -\frac{H_n}{n}$ from [8, eq. 1.4, p. 2] to get

$$\begin{aligned}
\int_0^1 \frac{\ln(1-t)\mathrm{Li}_2(t)}{t}\, dt &= \int_0^1 \ln(1-t) \sum_{n=1}^\infty \frac{t^{n-1}}{n^2}\, dt \\
&= \sum_{n=1}^\infty \frac{1}{n^2} \int_0^1 t^{n-1} \ln(1-t)\, dt \\
&= -\sum_{n=1}^\infty \frac{H_n}{n^3} \\
&= -\frac{\pi^4}{72}.
\end{aligned}$$

The Euler sum was simplified using eq. (7) and special values of the zeta function found in [2, 25.6]. The integral from (i) can be found in [8, 1.209, p. 59] and reduced further using the special values of the zeta function in [2, 25.6]. Otherwise, the interested reader verify the integral using the power series of the dilogarithm, utilizing the parameterized integral in [7, 1.5, p.2]. The remaining series can be simplified using eq. (8) and the series from [7, p. 293]. Another method of evaluating the integral is computing the antiderivative directly. □

We will also need to compute some useful antiderivatives:

Lemma 2. *The following equalities hold:*

$$(a) \qquad \int \left(\frac{\ln(1-t)}{t}\right)^2 dt = \ln^2(1-t) - \frac{\ln^2(1-t)}{t} + 2\mathrm{Li}_2(t) + \mathrm{C}$$

$$(b) \qquad \int \left(\frac{\ln(t)}{1-t}\right)^2 dt = \frac{t\ln^2(t)}{1-t} - 2\mathrm{Li}_2(1-t) + \mathrm{C}$$

where C *is some constant of integration.*

Proof. To calculate (a), we start by integrating by parts:

$$\begin{aligned}\int \left(\frac{\ln(1-t)}{t}\right)^2 dt &= \frac{\ln^2(1-t)}{t} - \int t\left(-\frac{2\ln(1-t)}{t^2(1-t)} - \frac{2\ln^2(1-t)}{t^3}\right) dt \\ &= \frac{\ln^2(1-t)}{t} + 2\int \frac{\ln(1-t)}{1-t}\,dt + 2\int \frac{\ln(1-t)}{t}\,dt + 2\int \left(\frac{\ln(1-t)}{t}\right)^2 dt \\ &= \frac{\ln^2(1-t)}{t} - \ln^2(1-t) - 2\mathrm{Li}_2(t) + 2\int \left(\frac{\ln(1-t)}{t}\right)^2 dt.\end{aligned}$$

We obtain the antidervative when we solve for $\int \left(\frac{\ln(1-t)}{t}\right)^2 dt$. The integral in (b) is obtained by substituting $u = 1-t$ and following the same calculations performed for (a). □

We are now ready to calculate $\mathcal{S}$.

Proposition 1. *The following equality holds:*

$$\mathcal{S} = \sum_{n=1}^{\infty} H_n H_n^{(2)} \left(\zeta(3) - H_n^{(3)}\right) = \frac{3}{2}\zeta(5) - \frac{\pi^4}{180}.$$

Proof. We know from eq. (3) that

$$\zeta(3) - H_n^{(3)} = -\frac{\psi^{(2)}(n+1)}{2}.$$

We substitute the polygamma function into our series and express it as its integral representation from [3, 6.4.1, p.260]:

$$\begin{aligned}\mathcal{S} &= \frac{1}{2}\sum_{n=1}^{\infty} H_n H_n^{(2)} \int_0^{\infty} \frac{t^2 e^{-(n+1)t}}{1-e^{-t}}\,dt \\ &= \frac{1}{2}\sum_{n=1}^{\infty} H_n H_n^{(2)} \int_0^1 \frac{\ln^2(t)t^n}{1-t}\,dt \\ &= \frac{1}{2}\int_0^1 \frac{\ln^2(t)}{1-t}\sum_{n=1}^{\infty} H_n H_n^{(2)} t^n.\end{aligned}$$

We integrated by substituting $t \mapsto -\ln(t)$ and interchanging the order of summation and integration. Using the generating function from [7, 4.8, p. 284]

$$\sum_{n=1}^{\infty} H_n H_n^{(2)} x^n = \frac{1}{1-x}\left(\frac{1}{2}\ln(x)\ln^2(1-x) + \mathrm{Li}_3(x) + \mathrm{Li}_3(1-x) - \zeta(2)\ln(1-x) - \zeta(3)\right), \quad \forall |x| < 1,$$

we are left with

$$\begin{aligned}\mathcal{S} =& \frac{1}{4}\int_0^1 \frac{\ln^3(t)\ln^2(1-t)}{(1-t)^2}\,dt + \frac{1}{2}\int_0^1 \frac{\ln^2(t)\mathrm{Li}_3(t)}{(1-t)^2}\,dt + \frac{1}{2}\int_0^1 \frac{\ln^2(t)\mathrm{Li}_3(1-t)}{(1-t)^2}\,dt \\ &- \frac{\zeta(2)}{2}\int_0^1 \frac{\ln^2(t)\ln(1-t)}{(1-t)^2}\,dt - \frac{\zeta(3)}{2}\int_0^1 \frac{\ln^2(t)}{(1-t)^2}\,dt.\end{aligned}$$

The resulting expression can be further simplified using (c) and (d) from lemma 1 and applying bounds to (b) in lemma 2. This results in

$$\begin{aligned}\mathcal{S} &= \frac{\pi^4}{72} + \frac{7\pi^2}{6}\zeta(3) - 3\zeta(3) - 12\zeta(5) + \frac{1}{2}\int_0^1 \frac{\ln^2(t)\mathrm{Li}_3(t)}{(1-t)^2}\,dt + \frac{1}{2}\underbrace{\int_0^1 \frac{\ln^2(t)\mathrm{Li}_3(1-t)}{(1-t)^2}\,dt}_{t\mapsto 1-t} \\ &= \frac{\pi^4}{72} + \frac{7\pi^2}{6}\zeta(3) - 3\zeta(3) - 12\zeta(5) + \frac{1}{2}\int_0^1 \frac{\ln^2(t)\mathrm{Li}_3(t)}{(1-t)^2}\,dt + \frac{1}{2}\int_0^1 \frac{\ln^2(1-t)\mathrm{Li}_3(t)}{t^2}\,dt.\end{aligned}$$

We may now begin apply integration by parts on the remaining integrals using (a) and (b) from lemma 2. This results in

$$\begin{aligned}\mathcal{S} =& \frac{\pi^4}{72} + \frac{4\pi^2}{3} - 3\zeta(3) - 12\zeta(5) - \frac{1}{2}\int_0^1 \frac{\ln^2(t)\mathrm{Li}_2(t)}{1-t}\,dt + \int_0^1 \frac{\mathrm{Li}_2(1-t)\mathrm{Li}_2(t)}{t}\,dt \\ &- \int_0^1 \frac{(\mathrm{Li}_2(t))^2}{t}\,dt - \frac{1}{2}\int_0^1 \frac{\ln^2(1-t)\mathrm{Li}_2(t)}{t}\,dt + \frac{1}{2}\int_0^1 \frac{\ln^2(1-t)\mathrm{Li}_2(t)}{t^2}\,dt.\end{aligned}$$

The remaining expression can be further simplified using (a), (b), (e), and (i) from lemma 1, leaving behind

$$\mathcal{S} = \frac{\pi^4}{72} - 3\zeta(3) + \frac{3}{2}\zeta(5) + \frac{1}{2}\int_0^1 \frac{\ln^2(1-t)\mathrm{Li}_2(t)}{t^2}\,dt.$$

We apply integration by parts once again using the antiderivative (b) from lemma 2 to get

$$\mathcal{S} = \frac{\pi^4}{24} - 3\zeta(3) + \frac{3}{2}\zeta(5) - \frac{1}{2}\int_0^1 \frac{\ln^3(1-t)}{t^2}\,dt + \frac{1}{2}\int_0^1 \frac{\ln^3(1-t)}{t}\,dt + \int_0^1 \frac{\ln(1-t)\mathrm{Li}_2(t)}{t}\,dt.$$

Plugging in (f), (g), and (h), we obtain the desired closed form. □

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