Junior Problems

J655. Find all pairs (p, q) of primes such that

$$\left(p^2 q + 1\right)^2 = (n+1)! + n! + 1$$

for some positive integer n.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J656. Prove that for any positive real numbers a, b, c

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{81(a+b)(b+c)(c+a)}{16(a+b+c)^3} \geq 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J657. Let $ABCD$ be a square of side a and center O and M be a point in the plane such that $\max\{MA, MC\} = \frac{1}{\sqrt{2}}(MB + MD)$. Evaluate OM.

Proposed by Mihaela Berindeanu, Bucharest, România

J658. Solve in positive real numbers the system of equations

$$\begin{aligned}
(\log x)(\log y) &= \log(xy) \\
(\log y)(\log z) &= \log(1000yz) \\
(\log z)(\log x) &= \log(zx),
\end{aligned}$$

where log means $\log_{10}$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J659. Find all pairs (m, n) of nonnegative integers such that $2^n + 3^m + 4$ is a perfect square.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J660. Solve in positive integers the equation

$$x^2 - xy + y^2 = \left(\frac{x+y}{2} + 6\right)^2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Senior Problems

S655. Evaluate the product

$$\sin\frac{\pi}{42}\sin\frac{5\pi}{42}\sin\frac{13\pi}{42}\sin\frac{17\pi}{42}\sin\frac{19\pi}{42}\sin\frac{31\pi}{42}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S656. Let ABC be a triangle with semi-perimeter s and area K. Prove that the circle of diameter BC is tangent to the incircle if and only if

$$\frac{1}{s-b}+\frac{1}{s-c}=\frac{s}{K}$$

Proposed by Mihaela Berindeanu, Bucharest, România

S657. Solve in prime numbers the equation

$$p^6+q^6+r^6+2=s^3-t^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S658. Let ABC be a triangle whose side lengths satisfy the condition $a+b+c=2$. Prove that

$$\sqrt{\frac{a}{a^2+bc}}+\sqrt{\frac{b}{b^2+ca}}+\sqrt{\frac{c}{c^2+ab}}\geq 2.$$

Proposed by Vasile Cîrtoaje, Petroleum-Oil University, Ploieşti, România

S659. Let a,b,c be non-negative real numbers such that $a+b+c=1$. Find the greatest constant λ such that the following inequality holds

(a) $a^2+b^2+c^2+\lambda abc\leq 1$.

(b) $a^3+b^3+c^3+\lambda abc\leq 1$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S660. Let a,b,c be positive real numbers such that

$$1-2(a+b+c)+3(ab+bc+ca)-4abc=0.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+56(a+b+c)\geq 54.$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U655. Let n be a positive integer non congruent to $1 \pmod 3$ and let z be a root of unity of order $2^{n+1}-1$. Evaluate

$$\left(\mathrm{Re}(z)-1/2\right)\left(\mathrm{Re}(z^2)-1/2\right)\left(\mathrm{Re}(z^4)-1/2\right)\cdots\left(\mathrm{Re}(z^{2^n})-1/2\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U656. Let $K\subset L$ be fields such that the equation $z^2-z+1=0$ has no roots in K.Assume that $x, y\in L\setminus K$ such that $x^2+2y, y^2+2x, xy\in K$. Prove that $K(x)=K(y)$.

Proposed by Mircea Becheanu, Canada and Mihaela Berindeanu, România

U657. Evaluate

$$\lim_{n\to\infty}\ln^2(n+k)\cdot\frac{\sum\limits_{1\le i<j\le n}3^{-(i+j)}}{\sum\limits_{1\le i<j\le n}(ij)^{-1}}$$

Proposed by Mihaela Berindeanu, Bucharest, România

U658. Find all the quadruples (a,b,c,d) of positive integers that simultaneously verify the equations

$$3a!+b=19^c \quad\text{and}\quad 3a+4b!=19^d.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, România

U659. Evaluate

$$\lim_{n\to\infty} n\sin\left(\left((2\pi n)^p+2^p\pi^p n^{p-1}p\right)^{1/p}\right)-n\pi(1-p).$$

Proposed by Perfetti Paolo, Università degli studi di Tor Vergata Roma, Italy

U660. Let $n\ge 2$ be an integer and $a\in(-2,2)$ a real number.

(1) Prove that the matrix equation $X^2+aX+I_n=O_n$ has a solution in the ring $M_n(\mathbb{C})$.

(2) Prove that the equation has a solution in $M_n(\mathbb{R})$ if and only if n is even.

(3) Prove that in each of the above situations the equation has infinitely many solutions.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Olympiad Problems

O655. Let a, b, and d_n, $n = 1, 2, \dots$ be integers such that a is nonzero and relatively prime to every d_n, $n \geq 1$, and, also, d_n and d_{n+1} are relatively prime for every $n \geq 2$. Prove that there exists a permutation $c_1, c_2, \dots$ of the numbers $an + b$, $n = 1, 2, \dots$ such that d_n divides $c_1 + \cdots + c_n$ for any positive integer n.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O656. Let $n \geq 2$ be an integer. Find the largest integer k such that

$$\left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}$$

for all nonnegative real numbers a_i which satisfy $a_1 \geq a_2 \geq \cdots \geq a_n$.

Proposed by Vasile Cîrtoaje, Petroleum-Oil University, Ploieşti, România

O657. Let $p_1 = 2, p_2 = 3, \dots$ be the increasing sequence of all primes. Prove that a sequence $q_1, q_2, \dots$ of distinct primes can be found such that $q_1 + \cdots + q_n$ is divisible by p_n for every $n \geq 1$.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O658. Prove that there are infinitely many positive integers a such that

$$a! + (a+2)! \,|\, (2a+2)!$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O659. Let ABC be a triangle and let M be any point inside the triangle. Denote by x, y, z the distance from M to sides BC, CA, AB respectively. Prove that

$$\min\{MA, MB, MC\} \leq \frac{\sqrt{abc(ayz + bzx + cxy)}}{2S} \leq \max\{MA, MB, MC\}$$

where S is the area of triangle ABC.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O660. Let a, b, c be positive real numbers such that

$$1 - 2(a+b+c) + 3(ab+bc+ca) - 4abc = 0.$$

Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 32 \geq \frac{33}{a+b+c}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România