Junior problems

J655. Find all pairs (p, q) of primes such that

$$(p^2q+1)^2 = (n+1)! + n! + 1$$

for some positive integer n.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Theo Koupelis, Clark College, Washington, USA

The given equation is equivalent to

$$p^2q \cdot (p^2q+2) = n!(n+2).$$

But p and q are primes, and thus $p^2q \geq 8$; therefore, $n \geq 4$ and $3 \mid n!$. When $p = q = 2$ we get $80 = n!(n!+2)$, which is not possible because $3 \nmid 80$. Thus, at least one of the primes p, q is odd, and therefore p^2q is at best a multiple of 4, and $p^2q(p^2q+2)$ is at best a multiple of 8. But $n!(n!+2)$ is a multiple of 16 for $n \geq 6$, and therefore we only need examine the cases $n \in \{4, 5\}$. For $n = 4$ there is no solution because the quantity $(n+1)! + n! + 1 = 145$ is not a square. For $n = 5$ we get $(p^2q+1)^2 = 841 = 29^2$, and thus $p^2q = 28$; therefore, $(n, p, q) = (5, 2, 7)$ is the only solution.

Also solved by G. C. Greubel, Newport News, VA; Mohamed Amine, Algiers, Algeria; Sundaresh. H. R., Shivamogga, Karnataka, India; Ganghun Kim, The Fessenden School, Newton, MA, USA; Srijan Sundar, Oxford, UK.

J656. Prove that for any positive real numbers a, b, c

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{81(a+b)(b+c)(c+a)}{16(a+b+c)^3}\geq 3$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

By the AM-GM Inequality we have

$$\frac{a+b+c}{b+c}+\frac{b+c+a}{c+a}+\frac{c+a+b}{a+b}+\frac{81(a+b)(b+c)(c+a)}{16(a+b+c)^3}\geq 4\sqrt[4]{\frac{81}{16}}.$$

It follows that

$$\frac{a}{b+c}+1+\frac{b}{c+a}+1+\frac{c}{a+b}+1+\frac{81(a+b)(b+c)(c+a)}{16(a+b+c)^3}\geq 6,$$

or

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{81(a+b)(b+c)(c+a)}{16(a+b+c)^3}\geq 3$$

as desired.

Equality occurs when $a=b=c$.

Solution 2 by William Han, Horace Mann School, Bronx, NY, USA

Introduce the following variables

$$x=\frac{b+c}{a+b+c},\quad y=\frac{c+a}{a+b+c},\quad z=\frac{a+b}{a+b+c}.$$

Note that x, y, z are positive and $x+y+z=2$. It is easy to see that

$$\frac{1}{x}-1=\frac{a}{b+c},\quad \frac{1}{y}-1=\frac{b}{c+a},\quad \frac{1}{z}-1=\frac{c}{a+b}.$$

The inequality to prove can be written in equivalent form

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{81}{16}xyz\geq 6. \tag{1}$$

Denote $\sqrt[3]{xyz}=p$. From the inequality between the harmonic and the geometric mean, we have that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{3}{p}.$$

In order to prove inequality (1), it would suffice to show that

$$\frac{3}{p}+\frac{81}{16}p^3\geq 6.$$

After clearing the denominators and rearranging the terms this inequality becomes

$$81p^4-96p+48\geq 0\iff 3(3p-2)^2(3p^2+4p+4)\geq 0,$$

which is obviously true.

Equality happens when $p=2/3$, that is, when $\sqrt[3]{xyz}=(x+y+z)/3=2/3$ and this is true when $x=y=z=2/3$. In terms of the original variables, equality takes place only when $a=b=c$.

Solution 3 by Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania
Let $p = \sum_{cyc} a$, $q = \sum_{cyc} ab$, $r = abc$.
By the Cauchy-Schwarz Inequality, we have

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{\left(\sum_{cyc} a\right)^2}{2\sum_{cyc} ab} = \frac{p^2}{2q}.$$

Therefore, it suffices to show that

$$\frac{p^2}{2q} + \frac{81(pq-r)}{16p^3} \geq 3$$

The inequality is equivalent to

$$8p^5 + 81pq^2 - 81qr - 48p^3q \geq 0$$

By Schur's Inequality $pq \geq 9r$, we get

$$\begin{aligned} 8p^5 + 81pq^2 - 81qr - 48p^3q \geq 0 \quad &\iff \quad 8p^5 + 81pq^2 - 9pq^2 - 48p^3q \geq 0 \\ &\iff \quad 8p(p^4 - 6pq^2 + 9q^2) \geq 0 \\ &\iff \quad 8p(p^2 - 3q)^2 \geq 0. \end{aligned}$$

and the last inequality is obviously true.

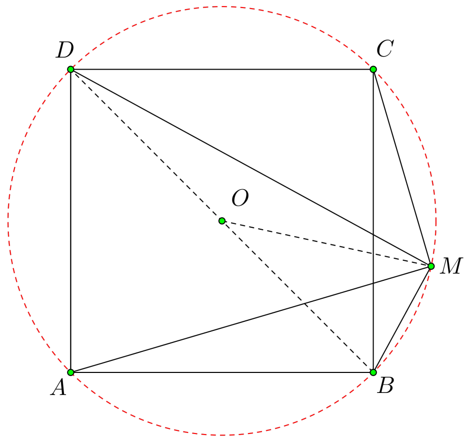
Also solved by Arkady Alt, San Jose, California, USA; Mohamed Amine, Algiers, Algeria; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Daniel Pascuas, Barcelona, Spain; Srijan Sundar, Oxford, UK.

J657. Let $ABCD$ be a square of side a and center O and M be a point in the plane such that $\max\{MA, MC\} = \frac{1}{\sqrt{2}}(MB + MD)$. Evaluate OM.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Kousik Sett, India

Without loss of any generality, we may assume that $\max\{MA, MC\} = MA$.



Using Ptolemy's Inequality on quadrilateral $ABMD$, we obtain

$$AD \cdot MB + AB \cdot MD \geq BD \cdot MA.$$

Since $ABCD$ is a square with side a, we have $AB = AD = a$ and $BD = \sqrt{2}a$. So the above inequality becomes

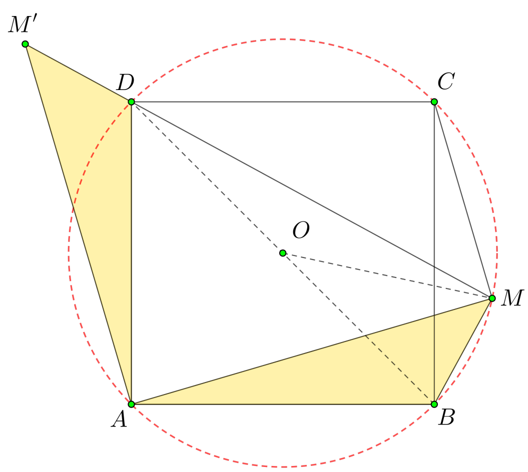
$$MB + MD \geq \sqrt{2}MA,$$

but from the given condition, we see that equality holds. Therefore, $ABMD$ is a cyclic quadrilateral. Hence

$$OM = OB = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}.$$

Solution 2 by Kousik Sett, India

Without loss of any generality, we may assume that $\max\{MA, MC\} = MA$.



Rotate triangle ABM about A by an angle of 90° counterclockwise and let triangle ADM' be the image of triangle ABM. Since $AD = AB$ and $\angle BAD = 90°$, then after the rotation AB will map to AD, BM

to DM'. So, $\angle MAM' = 90°$. Since $\triangle ADM' \cong \triangle ABM$, we have $M'A = MA$. Therefore, by Pythagorean Theorem, we have $MM' = \sqrt{2}MA$.

Consider the three points M, D, and M'. Using the triangle inequality, we have

$$\begin{gathered} MD + DM' \geq MM' \\ \Rightarrow MD + MB \geq \sqrt{2}MA, \end{gathered}$$

but from the given condition, we see that equality holds. Therefore, M, D, and M' are collinear. Since MAM' is a right-angled isosceles triangle and $ABCD$ is a square, we have

$$\angle AMD = \angle AMM' = 45° = \angle ABD.$$

Therefore, A, B, M, and D are concyclic. Hence

$$OM = OB = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}.$$

Solution 3 by Daniel Pascuas, Barcelona, Spain

OM is equal to half of the length of the diagonal of the square $ABCD$, that is, $OM = a\frac{\sqrt{2}}{2}$. Indeed, we are going to show that the set S of points M in the plane satisfying that

$$\max\{MA, MC\} = \tfrac{1}{\sqrt{2}}(MB + MD)$$

is just the circumference centered at O with radius $a\frac{\sqrt{2}}{2}$.

By translating, rotating, and rescaling, we may assume without loss of generality that $a = \sqrt{2}$, O is the origin of coordinates in the Euclidean plane $\mathbb{R}^2$, *i.e.* $O = (0,0)$, $A = (-1,0)$, $B = (0,-1)$, $C = (1,0)$, and $D = (0,1)$. We have to show that $S = \big\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\big\}$.

Let $M = (x,y) \in \mathbb{R}^2$. Then $MB = \sqrt{x^2 + (y+1)^2}$, $MD = \sqrt{x^2 + (y-1)^2}$, and

$$\max\{MA, MC\} = \begin{cases} MA = \sqrt{(x+1)^2 + y^2}, & \text{if } x \geq 0, \\ MC = \sqrt{(x-1)^2 + y^2}, & \text{if } x \leq 0. \end{cases}$$

It follows that $S = S_- \cup S_+$, where S_+ is the set of points $(x,y) \in \mathbb{R}^2$ such that $x \geq 0$ and

$$\sqrt{(x+1)^2 + y^2} = \tfrac{1}{\sqrt{2}}\left(\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-1)^2}\,\right), \tag{1}$$

and S_- is the set of points $(x,y) \in \mathbb{R}^2$ such that $x \leq 0$ and

$$\sqrt{(x-1)^2 + y^2} = \tfrac{1}{\sqrt{2}}\left(\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-1)^2}\,\right).$$

By squaring and simplifying we obtain that (1) can be written as

$$2x = \sqrt{x^2 + (y+1)^2}\sqrt{x^2 + (y-1)^2}. \tag{2}$$

Thus S_+ is the set of points $(x,y) \in \mathbb{R}^2$ which satisfy equation (2). We square again to get that (2) is equivalent to $4(x^2 + y^2) = (x^2 + y^2 + 1)^2$ and $x \geq 0$. Since

$$(x^2 + y^2 + 1)^2 - 4(x^2 + y^2) = (x^2 + y^2)^2 - 2(x^2 + y^2) + 1 = (x^2 + y^2 - 1)^2,$$

we conclude that S_+ is the semicircumference $x^2 + y^2 = 1$, $x \geq 0$. Similarly, we can prove that S_- is the semicircumference $x^2 + y^2 = 1$, $x \leq 0$. Hence we have just shown that S is the unit circumference $x^2 + y^2 = 1$.

Also solved by Theo Koupelis, Clark College, Washington, USA; Ganghun Kim, The Fessenden School, Newton, MA, USA; Srijan Sundar, Oxford, UK.

J658. Solve in positive real numbers the system of equations

$$\begin{aligned}(\log x)(\log y) &= \log(xy)\\(\log y)(\log z) &= \log(1000yz)\\(\log z)(\log x) &= \log(zx),\end{aligned}$$

where log means $\log_{10}$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution 1 by the author

Let $\log x = u$, $\log y = v$, $\log z = w$. Then the system becomes

$$uv = u + v, \quad vw = v + w + 3, \quad wu = w + u,$$

which can be rewritten as

$$(u-1)(v-1) = 1, \quad (v-1)(w-1) = 4, \quad (w-1)(u-1) = 1.$$

By multiplication we get

$$(u-1)^2(v-1)^2(w-1)^2 = 4,$$

implying $(u-1)(v-1)(w-1) = 2$ or $(u-1)(v-1)(w-1) = -2$. In the first case, $u - 1 = \frac{1}{2}$ and $v - 1 = w - 1 = 2$, while in the second, $u - 1 = -\frac{1}{2}$ and $v - 1 = w - 1 = -2$, yielding $(u, v, w) = \left(\frac{3}{2}, 3, 3\right)$ or $(u, v, w) = \left(\frac{1}{2}, -1, -1\right)$.

It follows that $(x, y, z) = (10\sqrt{10}, 1000, 1000)$ or $(x, y, z) = \left(\sqrt{10}, \frac{1}{10}, \frac{1}{10}\right)$. Both triples satisfy the given system.

Solution 2 by the author

If $\log x = 0$, then $x = 1$ and $\log y = \log z = 0$, implying $y = z = 1$ and $\log 1000 = 0$, a contradiction. If $\log z = 0$, then $z = 1$ and $\log x = 0$, leading to the same contradiction. So, from the first and third equations,

$$\frac{\log y}{\log z} = \frac{\log x + \log y}{\log x + \log z},$$

implying

$$(\log y)(\log z) + (\log x)(\log y) = (\log x)(\log z) + (\log y)(\log z).$$

Hence $\log y = \log z$, so $y = z$ and

$$(\log y)^2 = \log(1000y^2),$$

yielding $(\log y)^2 = 3 + 2\log y$. It follows that $\log y = 3$ and $3\log x = 3 + \log x$ or $\log y = -1$ and $-\log x = -1 + \log x$. We get $x = 10\sqrt{10}$ and $y = z = 1000$ or $x = \sqrt{10}$ and $y = z = \frac{1}{10}$.

Editor's Comment.

G. C. Greubel generalized and solved the system

$$\begin{aligned}(\log x)(\log y) &= \log(xy)\\(\log y)(\log z) &= \log(ayz)\\(\log z)(\log x) &= \log(zx),\end{aligned}$$

where $a > 0$. This system can be solved as in the previous two solutions.

Also solved by Andrew Hwang, Langley High School, McLean, Virginia USA; Corneliu Mănescu-Avram, Ploieşti, Romania; G. C. Greubel, Newport News, VA; Kennedy Samarakody, SUNY Brockport; Matthew Too, University of Illinois Urbana-Champaign, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Nihad Hashimov; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Srijan Sundar, Oxford, UK.

J659. Find all pairs (m, n) of nonnegative integers such that $2^n + 3^m + 4$ is a perfect square.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author
We consider the following cases:

- For $n = 0$ then $A = 3^m + 5$. If $m = 0$ then $A = 6$. If $m \geq 1$ then $A \equiv 2 \pmod 3$. Hence A is not a perfect square.
- For $n = 1$ then $A = 3^m + 6$. If $m = 0$ then $A = 7$. If $m = 1$ then $A = 9$ (true). If $m \geq 2$ then $3 \mid A$ but $9 \nmid A$. Hence A is not a perfect square.
- For $n = 2$ then $A = 3^m + 8$. If $m = 0$ then $A = 9$ (true). If $m \geq 1$ then $A \equiv 2 \pmod 3$. Hence A is not a perfect square.
- For $n \geq 3$ then $A \equiv 3^m + 4 \pmod 8$. If m is even, i.e. $m = 2k$ then $A \equiv 3^{2k} + 4 \pmod 8 \equiv 5 \pmod 8$. If m is odd, i.e. $m = 2k+1$ then $A \equiv 3^{2k+1} + 4 \pmod 8 \equiv 7 \pmod 8$. Hence A is not a perfect square.

So we get $(m, n) \in \{(0, 2), (1, 1)\}$.

Also solved by G. C. Greubel, Newport News, VA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; William Han, Horace Mann School, Bronx, NY, USA; Srijan Sundar, Oxford, UK.

J660. Solve in positive integers the equation

$$x^2 - xy + y^2 = \left(\frac{x+y}{2} + 6\right)^2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution 1 by the author

The given equation is equivalent to

$$4(x^2 - xy + y^2) = (x+y)^2 + 24(x+y) + 144,$$

which reduces to $(x-y)^2 = 8(x+y) + 48$. We treat this as a quadratic equation in x:

$$x^2 - 2x(y+4) + (y^2 - 8y - 48) = 0.$$

Its discriminant is $4(y^2+8y+16-y^2+8y+48) = 64(y+4)$. Then $y+4 = n^2$ for some positive integer $n \geq 3$, yielding the solutions $(x,y) = (n^2+4n, n^2-4)$, $n = 3,4,5,\ldots$ and $(x,y) = (n^2-4n, n^2-4)$, $n = 5,6,7,\ldots$ (the permutations do not bring new solutions).

Solution 2 by Corneliu Mănescu-Avram, Ploieşti, Romania

Denote $s = x+y$, $p = xy$. Squaring, we see that $s^2/4$ is an integer, so $s/2$ is an integer, $s = 2u$, with u positive integer. The equation becomes

$$4u^2 - 3p = (u+6)^2,$$

whence $p = u^2 - 4u - 12$. Then x, y are solutions of the quadratic equation

$$z^2 - 2uz + u^2 - 4u - 12 = 0.$$

From $(z-u)^2 = 4(u+3)$, it follows that $u+3$ must be a square, $u+3 = t^2$, with t positive integer and we get infinitely many solutions $\{x,y\} = \{t^2+2t-3, t^2-2t-3\}$. The solutions are positive, if $t^2-2t-3 = (t-1)^2 - 4 > 0$, so $t \geq 4$.

Editor's Comment. Notice that both solutions produce the same pairs of integers. Indeed, if $t = n-1$ where $n \geq 5$, then $t^2+2t-3 = n^2-4$ and $t^2-2t-3 = n^2-4n$; if $t = n+1$ where $n \geq 3$, then $t^2+2t-3 = n^2+4n$ and $t^2-2t-3 = n^2-4$.

Also solved by G. C. Greubel, Newport News, VA; Matthew Too, University of Illinois Urbana-Champaign, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Nihad Hashimov; Sundaresh. H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; William Han, Horace Mann School, Bronx, NY, USA; Anderson Torres, Sao Paulo, Brazil; Ganghun Kim, The Fessenden School, Newton, MA, USA; Srijan Sundar, Oxford, UK.

Senior problems

S655. Evaluate the product

$$\sin\frac{\pi}{42}\sin\frac{5\pi}{42}\sin\frac{13\pi}{42}\sin\frac{17\pi}{42}\sin\frac{19\pi}{42}\sin\frac{31\pi}{42}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution 1 by Kousik Sett, India

Assume that

$$X=\sin\frac{\pi}{42}\sin\frac{5\pi}{42}\sin\frac{13\pi}{42}\sin\frac{17\pi}{42}\sin\frac{19\pi}{42}\sin\frac{31\pi}{42}.$$

Since

$$\sin\frac{\pi}{42}=\sin\left(\frac{\pi}{2}-\frac{20\pi}{42}\right)=\cos\frac{10\pi}{21},\qquad \sin\frac{5\pi}{42}=\cos\frac{8\pi}{21},\qquad \sin\frac{13\pi}{42}=\cos\frac{4\pi}{21},$$

$$\sin\frac{17\pi}{42}=\cos\frac{2\pi}{21},\qquad \sin\frac{19\pi}{42}=\cos\frac{\pi}{21},\qquad \text{and}\qquad \sin\frac{31\pi}{42}=\sin\left(\frac{\pi}{2}+\frac{10\pi}{42}\right)=\cos\frac{5\pi}{21},$$

we can write

$$X=\cos\frac{\pi}{21}\cos\frac{2\pi}{21}\cos\frac{4\pi}{21}\cos\frac{5\pi}{21}\cos\frac{8\pi}{21}\cos\frac{10\pi}{21}.$$

Let us assume that

$$Y=\sin\frac{\pi}{21}\sin\frac{2\pi}{21}\sin\frac{4\pi}{21}\sin\frac{5\pi}{21}\sin\frac{8\pi}{21}\sin\frac{10\pi}{21}.$$

Multiplying the above two expressions and using the formula $2\sin\theta\cos\theta=\sin 2\theta$, we can write

$$2^6\cdot XY=\sin\frac{2\pi}{21}\sin\frac{4\pi}{21}\sin\frac{8\pi}{21}\sin\frac{10\pi}{21}\sin\frac{16\pi}{21}\sin\frac{20\pi}{21}.$$

Since

$$\sin\frac{16\pi}{21}=\sin\left(\pi-\frac{5\pi}{21}\right)=\sin\frac{5\pi}{21},\qquad \text{and}\qquad \sin\frac{20\pi}{21}=\sin\left(\pi-\frac{\pi}{21}\right)=\sin\frac{\pi}{21},$$

we obtain

$$2^6\cdot XY=Y.$$

Since $Y\neq 0$, finally, we have

$$X=\frac{1}{2^6}=\frac{1}{64}.$$

Solution 2 by Srijan Sundar, Oxford, UK

Let P be the given product and let $z=e^{i\pi/42}$. Noticing that

$$\sin\frac{\pi}{42}=\sin\frac{41\pi}{42},\qquad \sin\frac{5\pi}{42}=\sin\frac{37\pi}{42},\ldots$$

then

$$\begin{aligned}P^2 &= \frac{z-z^{-1}}{2i}\cdot\frac{z^5-z^{-5}}{2i}\cdot\frac{z^{11}-z^{-11}}{2i}\cdot\ldots\cdot\frac{z^{41}-z^{-41}}{2i}\\ &= \frac{(z^2-1)(z^{10}-1)(z^{22}-1)\cdot\ldots\cdot(z^{82}-1)}{64^2}\end{aligned}$$

But note that $z^2 = e^{2i\pi/42}$. So, the product above is $\dfrac{\Phi_{42}(1)}{64^2}$, Φ_{42} being the 42nd cyclotomic polynomial. So we can express

$$\begin{aligned}(x-z^2)(x-z^{10})\cdot\ldots\cdot(x-z^{82}) &= \frac{x^{42}-1}{(x^{21}-1)(x+1)(x^2-x+1)(x^6-x^5+x^4-x^3+x^2-x+1)} \\ &= \frac{x^{21}+1}{(x+1)(x^2-x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}\end{aligned}$$

At $x = 1$ this evaluates to 1. So, $P = \dfrac{1}{64}$.

Also solved by G. C. Greubel, Newport News, VA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Sundaresh. H. R., Shivamogga, Karnataka, India; Farmonov Sukhrobjon, Uzbekistan; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Teawoo Kim, The Beekman School, New York, NY, USA.

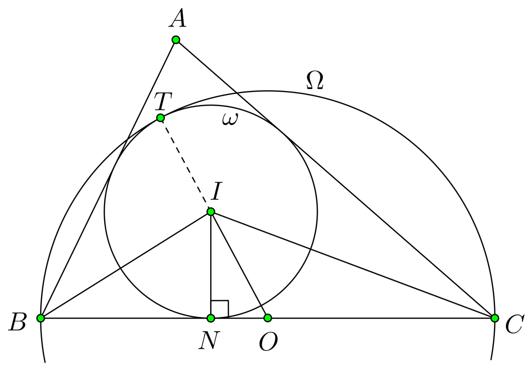
S656. Let ABC be a triangle with semi-perimeter s and area K. Prove that the circle of diameter BC is tangent to the incircle if and only if

$$\frac{1}{s-b}+\frac{1}{s-c}=\frac{s}{K}$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Kousik Sett, India

Denote the circle of diameter BC by Ω and the incircle of triangle ABC by ω. Assume that $T=\Omega\cap\omega$. Let O and I be the centers of Ω and ω, respectively; and N be the orthogonal projection of I on BC. Therefore, $OB=OC=OT=a/2$. If r is the inradius of triangle ABC, then $IO=a/2-r$. It is well-known that $BN=s-a$, so $NO=|BO-BN|=|a/2-(s-b)|$.



From the right triangle INO, we get

$$\begin{aligned} IO^2 =& IN^2+NO^2,\\ \Rightarrow \left(\frac{a}{2}-r\right)^2 =& r^2+\left\{\frac{a}{2}-(s-b)\right\}^2. \end{aligned}$$

Simplifying, we get $\quad ar=a(s-b)-(s-b)^2=(s-b)(s-c).$ $\qquad [\because a=s-b+s-c]$ Therefore,

$$\frac{a}{(s-b)(s-c)}=\frac{1}{r}.$$

Using the area formula $K=sr$; and since $a=s-b+s-c$, the above relationship can be written as

$$\frac{1}{s-b}+\frac{1}{s-c}=\frac{s}{K}.$$

For the converse part, assume that

$$\frac{1}{s-b}+\frac{1}{s-c}=\frac{s}{K}.$$

If Ω and ω become tangent at T then $IT=r$, so $OI=a/2-r$. Therefore, for tangency of Ω and ω, it is enough to prove $OI=a/2-r$. Since $K=sr$ and $a=s-b+s-c$, the given condition can be written as $a(s-b)-(s-b)^2=ar$. Using this condition, from the right triangle INO, we get

$$\begin{aligned} IO^2 =& IN^2+NO^2\\ =& r^2+\left\{\frac{a}{2}-(s-b)\right\}^2 = r^2+\frac{a^2}{4}-a(s-b)+(s-b)^2\\ =& r^2+\frac{a^2}{4}-ar=\left(\frac{a}{2}-r\right)^2. \end{aligned}$$

Therefore, $IO=\dfrac{a}{2}-r$. Hence Ω and ω become tangent.

Also solved by Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Farmonov Sukhrobjon, Uzbekistan; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Teawoo Kim, The Beekman School, New York, NY, USA; Srijan Sundar, Oxford, UK.

S657. Solve in prime numbers the equation

$$p^6 + q^6 + r^6 + 2 = s^3 - t^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

A perfect cube is 0 or ± 1 modulo 9, so $n^6 \equiv 0, 1 \pmod 9$ for all integers n.
Because $p^6 + q^6 + r^6 + 2 \equiv 2, 3, 4, 5 \pmod 9$ and $s^3 - t^3 \equiv 0, \pm 1, \pm 2 \pmod 9$, it follows that

$$p^6 + q^6 + r^6 + 2 \equiv s^3 - t^3 \equiv 2 \pmod 9.$$

It follows that $p^6 \equiv q^6 \equiv r^6 \equiv 0 \pmod 9$, so $p \equiv q \equiv r \equiv 0 \pmod 3$. Hence $p = q = r = 3$, implying $s^3 - t^3 = 2189$. Since $s > t$, then by parity we get $t = 2$ and then $s = 13$.
Thus $(p, q, r, s, t) = (3, 3, 3, 13, 2)$.

Also solved by Farmonov Sukhrobjon, Uzbekistan; Theo Koupelis, Clark College, Washington, USA; Ganghun Kim, The Fessenden School, Newton, MA, USA; Srijan Sundar, Oxford, UK.

S658. Let ABC be a triangle whose side lengths satisfy the condition $a+b+c=2$. Prove that

$$\sqrt{\frac{a}{a^2+bc}}+\sqrt{\frac{b}{b^2+ca}}+\sqrt{\frac{c}{c^2+ab}}\geq 2.$$

Proposed by Vasile Cîrtoaje, Petroleum-Oil University, Ploieşti, Romania

Solution 1 by the author

Assume that $a\geq b\geq c$ and write the inequality in the homogeneous form

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geq 2,$$

where

$$x=\frac{a(a+b+c)}{2(a^2+bc)},\qquad y=\frac{b(a+b+c)}{2(b^2+ca)},\qquad z=\frac{c(a+b+c)}{2(c^2+ab)}.$$

Since

$$1-x=\frac{(a-b)(a-c)+bc}{2(a^2+bc}\geq 0,$$

$$1-y=\frac{c(2a-b)-b(a-b)}{2(b^2+ca)}\geq\frac{(a-b)(2a-b)-b(a-b)}{2(b^2+ca)}=\frac{2(a-b)^2}{2(b^2+ca)}\geq 0,$$

$$1-z=\frac{(a-c)(b-c)+ab}{2(c^2+ab)}>0,$$

we have $\sqrt{x}\geq x$, $\sqrt{y}\geq y$, $\sqrt{z}\geq z$, therefore it suffices to show that

$$x+y+z\geq 2,$$

that is

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\geq\frac{4}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\geq\frac{(a+b+c)^2}{a(a^2+bc)+b(b^2+ca)+c(c^2+ab)}.$$

Thus, it is enough to prove that

$$(a+b+c)^3\geq 4(a^3+b^3+c^3)+12abc.$$

Substituting $a=y+z$, $b=z+x$ and $c=x+y$, where $x,y,z\geq 0$, we get

$$(a+b+c)^3-4(a^3+b^3+c^3)-12abc=24xyz\geq 0.$$

The equality occurs for a degenerate triangle with $a=b=1$ and $c=0$ (or any cyclic permutation).

Solution 2 by Mohamed Amine, Algiers, Algeria

Since ABC is a triangle, then $a\leq b+c$ and so $a+b+c\leq 2(b+c)$, which implies $b+c\geq 1$. Similarly, $a+b\geq 1$ and $c+a\geq 1$. Using this fact and the AM-GM Inequality we have

$$\begin{aligned}
\sqrt{\frac{a}{a^2+bc}}+\sqrt{\frac{b}{b^2+ca}}+\sqrt{\frac{c}{c^2+ab}} &= \frac{a\sqrt{b+c}}{\sqrt{a(b+c)}\cdot\sqrt{a^2+bc}}+\frac{b\sqrt{c+a}}{\sqrt{b(c+a)}\cdot\sqrt{b^2+ca}}+\frac{c\sqrt{a+b}}{\sqrt{c(a+b)}\cdot\sqrt{c^2+ab}}\\
&\geq \frac{2a}{a^2+ab+bc+ca}+\frac{2b}{b^2+ab+bc+ca}+\frac{2c}{c^2+ab+bc+ca}\\
&= 3-\left(\frac{bc}{a^2+ab+bc+ca}+\frac{ca}{b^2+ab+bc+ca}+\frac{ab}{c^2+ab+bc+ca}\right)\\
&\geq 3-\left(\frac{bc}{ab+bc+ca}+\frac{ca}{ab+bc+ca}+\frac{ab}{ab+bc+ca}\right)\\
&= 2.
\end{aligned}$$

Also solved by Nicuşor Zlota, "Traian Vuia" Technical College; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

S659. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Find the greatest constant λ such that the following inequality holds

(a) $a^2+b^2+c^2+\lambda abc \leq 1$.

(b) $a^3+b^3+c^3+\lambda abc \leq 1$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by Srijan Sundar, Oxford, UK

(a) Note if that $a=b=c=\frac{1}{3}$, then $a^2+b^2+c^2+18abc=1$, so $\lambda \leq 18$. But by Cauchy-Schwarz Inequality

$$1=(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)(c+b+a) \geq a^2+b^2+c^2+18abc.$$

So, $\lambda=18$.

(b) Again, note that if $a=b=c=\frac{1}{3}$, then $a^3+b^3+c^3+24abc=1$, so $\lambda \leq 24$. But by the AM-GM Inequality

$$1=(a+b+c)^3=a^3+b^3+c^3+6abc+3(a^2b+b^2c+c^2a)+3(a^2c+c^2b+b^2a) \geq a^3+b^3+c^3+24abc.$$

So, $\lambda=24$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $p=a+b+c=1$, $q=ab+bc+ca$, and $r=abc$. From the given condition we get $a, b, c \in [0,1]$, with at most two of the variables a, b, c being equal to zero and at most one of them being equal to 1. If one or two of the variables is zero, then the given inequalities hold for any λ because $a^3+b^3+c^3 \leq a^2+b^2+c^2 \leq a+b+c=1$. Let $r \neq 0$. Using AM-GM we get $p \geq 3r^{1/3}$, and thus $0<r \leq 1/27$. Also, $q \geq 3r^{2/3}$, and thus $\frac{q}{r} \geq \frac{3}{r^{1/3}} \geq 9$.

(a) We have $a^2+b^2+c^2=p^2-2q=1-2q$, and thus

$$\lambda \leq \frac{1-(1-2q)}{r}=2 \cdot \frac{q}{r}.$$

Therefore, the greatest constant λ is equal to the least value of $2q/r$, namely $\lambda_{gr}=18$. Equality occurs when $a=b=c=1/3$.

(b) We have $a^3+b^3+c^3=p^3-3pq+3r=1-3q+3r$, and thus

$$\lambda \leq \frac{1-(1-3q+3r)}{r}=3 \cdot \frac{q}{r}-3.$$

Therefore, the greatest constant λ is obtained when q/r obtains its least value, namely $\lambda_{gr}=3 \cdot 9-3=24$. Equality occurs when $a=b=c=1/3$.

Also solved by Arkady Alt, San Jose, California, USA; Nicuşor Zlota, "Traian Vuia" Technical College; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Anderson Torres, Sao Paulo, Brazil; Daniel Pascuas, Barcelona, Spain; Teawoo Kim, The Beekman School, New York, NY, USA.

S660. Let a, b, c be positive real numbers such that

$$1-2(a+b+c)+3(ab+bc+ca)-4abc=0.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+56(a+b+c)\geq 54.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

If one of the numbers is equal to 1, let's say $c=1$, then the condition from the hypothesis leads us to the following relationship

$$(a-1)(b-1)=0 \Longrightarrow a=1 \text{ or } b=1$$

so the inequality is clearly true.

We assume that all numbers are different from 1. The condition in the hypothesis is equivalent to

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=4.$$

If one of the numbers a, b, c is greater than 1, the inequality is clearly true. Therefore, there remains the case where $a, b, c \in (0,1)$. with the following substitutions

$$x=\frac{1}{1-a}-1,\ y=\frac{1}{1-b}-1,\ z=\frac{1}{1-c}-1,\ x,y,z>0,\ x+y+z=1.$$

Without loss of generality, assume that $z=\max\{x,y,z\}$ and the inequality becomes

$$\sum_{cyc}\frac{x+1}{x}+56\sum_{cyc}\frac{x}{x+1}\geq 54,$$

or

$$\sum_{cyc}\frac{2x+y+z}{x}+56\sum_{cyc}\frac{x}{2x+y+z}\geq 54,$$

or

$$\sum_{cyc}\frac{y+z}{x}-6\geq 28\left(\sum_{cyc}\frac{y+z}{2x+y+z}-\frac{3}{2}\right),$$

or

$$\frac{2(x-y)^2}{xy}+\frac{(x+y)(z-x)(z-y)}{xyz}\geq$$
$$\frac{28(x-y)^2}{(2x+y+z)(x+2y+z)}+\frac{28(3x+3y+2z)(z-x)(z-y)}{2(2x+y+z)(x+2y+z)(x+y+2z)}.$$

But this is true because we have

$$(2x+y+z)(x+2y+z)\geq(2x+2y)(2x+2y)\geq 16xy>14xy,$$

and

$$\begin{aligned}
&(x+y)(2x+y+z)(2y+z+x)(2z+x+y)-14xyz(3x+3y+2z)=\\
=&2(x+y)z^3+7(x-y)^2z^2+(7x^3-19x^2y-19xy^2+7y^3)z\\
&+2x^4+9x^3y+14x^2y^2+9xy^3+2y^4\\
=&2(x+y)z^3+7(x-y)^2z^2+7(x+y)(x-y)^2z-12xyz(x+y)\\
&+2(x^4+y^4)+9xy(x^2+y^2)+14x^2y^2\\
\geq&(x+y)^2z^2-12xyz(x+y)+36x^2y^2\\
=&[(x+y)z-6xy]^2\geq 0.
\end{aligned}$$

The equality holds when $x = y = z = \dfrac{1}{3}$ which implies $a = b = c = \dfrac{1}{4}$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $p = a+b+c$, $q = ab+bc+ca$, and $r = abc$. From AM-GM we get $\dfrac{q}{r} \geq \dfrac{3}{r^{1/3}} \geq \dfrac{9}{p}$. The given condition is equivalent to $4r = 1 - 2p + 3q$. Thus, if $p \leq \frac{1}{2}$, we get $4r \geq 3q \geq 9r^{2/3}$, and therefore $r \geq (\frac{9}{4})^3$, which leads to a contradiction because $r \leq (\frac{p}{3})^3 \leq (\frac{1}{6})^3$. Thus, $p > 1/2$.

The desired inequality is equivalent to $\dfrac{q}{r} + 56p \geq 54$, which holds true for any $p \geq \frac{3}{4}$; indeed, because $pq \geq 9r$ we get

$$\frac{q}{r} + 56p \geq \frac{9}{p} + 56p \geq 54 \Longleftrightarrow (14p-3)(4p-3) \geq 0.$$

When $p = \frac{3}{4}$, equality occurs when $a = b = c = \frac{1}{4}$.

We now examine the case $\frac{1}{2} < p < \frac{3}{4}$. From the desired inequality we get

$$\frac{q}{r} + 56p \geq 54 \Longleftrightarrow \frac{4q}{1-2p+3q} + 56p \geq 54 \Longleftrightarrow q \leq \frac{(2p-1)(54-56p)}{2(79-84p)}. \quad (*)$$

From the obvious inequality $(a-b)^2(b-c)^2(c-a)^2 \geq 0$ we get $p^2q^2 - 4q^3 + 2p(9q-2p^2)r - 27r^2 \geq 0$. Substituting $r = (1-2p+3q)/4$, we get

$$(2p-q-3)\left[64q^2 - q(16p^2+88p-51) + 16p^3 + 16p^2 - 30p + 9\right] \geq 0.$$

But $2p - q - 3 < \frac{3}{2} - q - 3 < 0$, and thus $q_- \leq q \leq q_+$, where

$$q_{\pm} = \frac{16p^2 + 88p - 51 \pm (3-4p)\sqrt{(3-4p)(11-4p)}}{128}. \quad (**)$$

From $(*)$ and $(**)$ we get that it is sufficient to show that

$$q_+ \leq \frac{(2p-1)(54-56p)}{2(79-84p)},$$

or

$$(3-4p)(79-84p)\sqrt{(3-4p)(11-4p)} \leq (3-4p)(191+8p-336p^2).$$

Both sides of the above inequality are non-negative for $\frac{1}{2} < p < \frac{3}{4}$. Squaring and simplifying we get

$$(2p-1)(588p^2 - 882p + 331) \geq 0,$$

which is obvious because $p > \frac{1}{2}$ and $588p^2 - 882p + 331 > 0$.

Also solved by Nicuşor Zlota, "Traian Vuia" Technical College; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy;

Undergraduate problems

U655. Let n be a positive integer non congruent to 1 (mod 3) and let z be a root of unity of order $2^{n+1}-1$. Evaluate

$$(\mathrm{Re}(z)-1/2)\left(\mathrm{Re}(z^2)-1/2\right)\left(\mathrm{Re}(z^4)-1/2\right)\cdot\ldots\cdot\left(\mathrm{Re}(z^{2^n})-1/2\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Let P be the product. If $z=1$, then $P=\left(\frac{1}{2}\right)^{n+1}$. Let's now suppose that $z\neq 1$. We have

$$\mathrm{Re}\left(z^{2^k}\right)-\frac{1}{2}=\frac{1}{2}\left(z+\frac{1}{z}-1\right),\qquad k=0,1,2,\ldots,n,$$

hence

$$\begin{aligned}
2^{n+1}\left(z+\frac{1}{z}+1\right)P &= \left(z^2+\frac{1}{z^2}+1\right)\left(z^2+\frac{1}{z^2}-1\right)\left(z^4+\frac{1}{z^4}-1\right)\cdot\ldots\cdot\left(z^{2^n}+\frac{1}{z^{2^n}}-1\right)\\
&= \left(z^4+\frac{1}{z^4}+1\right)\left(z^4+\frac{1}{z^4}-1\right)\cdot\ldots\cdot\left(z^{2^n}+\frac{1}{z^{2^n}}-1\right)\\
&\vdots\\
&= \left(z^{2^n}+\frac{1}{z^{2^n}}+1\right)\left(z^{2^n}+\frac{1}{z^{2^n}}-1\right)\\
&= z^{2^{n+1}}+\frac{1}{z^{2^{n+1}}}+1\\
&= z+\frac{1}{z}+1.
\end{aligned}$$

We cannot have $z+\frac{1}{z}+1=0$, as this would imply $z^3=1$, which, combined with $z^{2^{n+1}-1}=1$, would yield $z=1$, a contradiction. So $P=\frac{1}{2^{n+1}}$.

Also solved by Theo Koupelis, Clark College, Washington, USA; Teawoo Kim, The Beekman School, New York, NY, USA; Srijan Sundar, Oxford, UK.

U656. Let $K \subset L$ be fields such that the equation $z^2 - z + 1 = 0$ has no roots in K.Assume that $x, y \in L \setminus K$ such that $x^2 + 2y, y^2 + 2x, xy \in K$. Prove that $K(x) = K(y)$.

Proposed by Mircea Becheanu, Canada and Mihaela Berindeanu, Romania

Solution by the author

We denote $x^2 + 2y = a,\ y^2 + 2x = b,\ xy = c$ where $a, b, c \in K$. Clearly, $c \neq 0$.

If characteristic of K is 2 the conditions are $x^2 = a, y^2 = b, xy = c \in K$. Then $x^2 y = ay = cx$ which gives $x = c^{-1}ay$. Similarly one obtains $y = c^{-1}bx$. These show that $K(x) = K(y)$.

Assume now that $\text{char}(K) \neq 2$. From the hypothesis we have:

$$x^2 y + 2y^2 = ay \Rightarrow cx + 2y^2 = ay \tag{1}$$

and

$$2y^2 + 4x = 2b. \tag{2}$$

After subtracting (1)-(2) one obtains

$$(c-4)x = ay - 2b.$$

In a similar way one obtains

$$(c-4)y = bx - 2a.$$

These last equalities are a linear system of equations

$$(4-c)x + ay = 2b$$

$$bx + (4-c)y = 2a,$$

with coefficients in K and solution (x, y). The matrix of the system is

$$A = \begin{pmatrix} 4-c & a \\ b & 4-c \end{pmatrix}$$

If $\det(A) = (4-c)^2 - ab \neq 0$ the system has a solution unique $(x, y) \in K \times K$. This is a contradiction. Hence, $\det(A) = (4-c)^2 - ab = 0$ and the rank of A is 1. The extended matrix of the system is

$$B = \begin{pmatrix} 4-c & a & 2b \\ b & 4-c & 2a \end{pmatrix}$$

Because the system has a solution it follows that $\text{rank}(B) = 1$. Therefore the following determinants are zero:

$$\begin{vmatrix} 4-c & 2b \\ b & 2a \end{vmatrix} = 0 \text{ and} \begin{vmatrix} a & 2b \\ 4-c & 2a \end{vmatrix} = 0.$$

Observe that $4 - c \neq 0$. From equalities:

$$a(4-c) = b^2 \text{ and } b(4-c) = a^2$$

we obtain

$$\frac{a}{b} = \frac{b^2}{a^2} \Leftrightarrow a^3 = b^3.$$

Therefore we have $(a-b)(a^2 + ab + b^2) = 0$. If $a - b \neq 0$ it follows that $a^2 + ab + b^2 = 0$ and then $(a/b)^2 + (a/b) + 1 = 0$. This contradicts the hypothesis about K. In conclusion we have $a = b$. This gives

$$x^2 + 2y = y^2 + 2x \Leftrightarrow (x-y)(x+y) = 2(x-y).$$

If $x = y$ the conclusion follows. If $x + y = 2$ it follows that x and y are the roots in L of the quadratic equation $t^2 - 2t + c = 0$. We can take
$x = 1 + \sqrt{1-c}$ and $y = 1 - \sqrt{1-c}$. This shows that $K(x) = K(y) = K(\sqrt{1-c})$.

U657. Evaluate

$$\lim_{n\to\infty} \ln^2(n+k) \cdot \frac{\displaystyle\sum_{1\le i<j\le n} 3^{-(i+j)}}{\displaystyle\sum_{1\le i<j\le n} (ij)^{-1}}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by G. C. Greubel, Newport News, VA

Using

$$\sum_{1\le i<j\le n} a_i\, a_j = \frac{1}{2}\left[\left(\sum_{j=1}^{n} a_j\right)^2 - \sum_{j=1}^{n} a_j^2\right]$$

then

$$\sum_{1\le i<j\le n} \frac{1}{3^{i+j}} = \frac{1}{16}\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{3^{n-1}}\right)$$

and

$$\sum_{1\le i<j\le n} \frac{1}{i\,j} = \frac{1}{2}\left((H_n)^2 - H_n^{(2)}\right),$$

where H_n is the harmonic number and $H_n^{(r)}$ is the generalized harmonic number. Now using

$$\begin{aligned} \ln(n+k) &= \ln\left(n\left(1+\frac{k}{n}\right)\right) = \ln n + \frac{k}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \\ H_n &\approx \ln n + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right) \\ H_n^{(2)} &\approx \zeta(2) - \gamma_1(n+1), \end{aligned}$$

where γ is the Euler-Mascheroni constant. Then the limit takes the form

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{\ln^2(n+k) \sum\limits_{1\le i<j\le n} 3^{-i-j}}{\sum\limits_{1\le i<j\le n} \frac{1}{i\,j}} \\ &= \frac{1}{8}\lim_{n\to\infty} \frac{\left(\ln n + \frac{k}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)^2}{\ln^2 n + \mathcal{O}\left(\frac{1}{n}\right)} \left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{3^{n-1}}\right) \\ &= \frac{1}{8}\lim_{n\to\infty}\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{3^{n-1}}\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\ &= \frac{1}{8}. \end{aligned}$$

Also solved by Matthew Too, University of Illinois Urbana-Champaign, USA; Nicuşor Zlota, "Traian Vuia" Technical College; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Srijan Sundar, Oxford, UK.

U658. Find all the quadruples (a, b, c, d) of positive integers that simultaneously verify the equations

$$3a! + b = 19^c \quad \text{and} \quad 3a + 4b! = 19^d.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution 1 by Anderson Torres, Sao Paulo, Brazil

If $3a+4b! = 19^d$, then $b! \equiv 1 \pmod 3$. It follows that $b = 1$, since with $b \geq 3$ we would have $b! \equiv 0 \pmod 3$, and $b = 2$ obviously does not work. Now we have

$$\begin{aligned} 3a! + 1 &= 19^c \\ 3a + 4 &= 19^d. \end{aligned}$$

Now, $3a+4 = 19^d$ implies $3a+4 \equiv 0 \pmod{19}$, or $a \equiv 5 \pmod{19}$. Since $3a!+1 = 19^d$ implies $3a!+1 \equiv 0 \pmod{19}$, or $a! \equiv 6 \pmod{19}$, then $a < 19$ (or else $a! \equiv 0 \pmod{19}$).

Therefore the only possibility is $a = 5$. Plugging $a = 5, b = 1$ in the previous equations, we verify that $d = 1$ and $c = 2$ work.

The quadruple is $(a, b, c, d) = (5, 1, 2, 1)$.

Solution 2 by the author

We show that $(a, b, c, d) = (5, 1, 2, 1)$ is the only such quadruple. We will denote by $v_p(x)$ the exponent of the prime p in the prime factorization of the positive integer x (thus, $v_2(36) = v_3(36) = 2$, while $v_p(36) = 0$ for any other prime except for 2 and 3). By a well-known theorem of Legendre, we can see that

$$v_p(t!) \geq \frac{p^{v_p(t)} - 1}{p - 1}$$

and

$$v_p((p^m)!) = \frac{p^m - 1}{p - 1}$$

for any positive integers t and m, and any prime p.

Suppose first that $a \leq b$, which implies that a divides $b!$, hence that a divides $3a+4b! = 19^d$. This means that $a = 19^e$ for some nonnegative integer e, and implies, by the reminded theorem of Legendre, that

$$v_{19}(a!) = v_{19}((19^e)!) = \frac{19^e - 1}{18}.$$

But we have

$$v_{19}(b) = v_{19}(19^c - 3a!) \geq \min\{c, v_{19}(3a!)\} = v_{19}(3a!) = v_{19}(a!) = \frac{19^e - 1}{18}$$

(the minimum is $v_{19}(3a!)$ because $19^c > 3a! > 19^{v_{19}(3a!)}$), and

$$v_{19}(b!) = v_{19}(4b!) = v_{19}(19^d - 3a) = v_{19}(19^d - 3 \cdot 19^e) = e.$$

Since, as we saw in the beginning,

$$e = v_{19}(b!) \geq \frac{19^{v_{19}(b)} - 1}{18} \geq \frac{19^{\frac{19^e - 1}{18}} - 1}{18}$$

we get

$$e \geq \frac{19^{\frac{19^e - 1}{18}} - 1}{18} \geq \frac{19^e - 1}{18} \geq e,$$

where we used

$$\frac{19^e - 1}{18} \geq e \Leftrightarrow 19^e \geq 1 + 18e$$

(so, basically, Bernoulli's inequality) for nonnegative integer e, with equality for $e \in \{0,1\}$. The sequence of inequalities from above shows that we *must* have equality in $19^e \geq 1 + 18e$, therefore $e \in \{0,1\}$ follows.

For $e = 0$ the equations are $3 + b = 19^c$ and $3 + 4b! = 19^d$ and, clearly, the second does not hold modulo 19 for $b \geq 19$. Checking b from $\{1, \ldots, 18\}$ we see that only $b = 16$ verifies the first equation — however, this is not good for the second equation because $3 + 4 \cdot 16! \equiv 3 + 4 \cdot 9 \equiv 1 \pmod{19}$ ($16! \equiv 9 \pmod{19}$ is easily obtained from Wilson's theorem). When $e = 1$ we must have $3 \cdot 19! + b = 19^c$ and $3 \cdot 19 + 4b! = 19^d$. The first equation shows that b is a multiple of 19, but if it is at least 38, the second equation does not hold modulo 19^2. So, only $b = 19$ remains, for which the second equation becomes $3 + 4 \cdot 18! = 19^{d-1}$, and, again, it is not verified modulo 19.

Thus the case $a \leq b$ leads to no solution, hence it remains to consider $a > b$. Exactly as in the first case, now we get that b is a divisor of 19^d, hence it is a power of 19, say $b = 19^f$, with f a nonnegative integer. Still as in the first case, we get

$$v_{19}(a!) = f$$

and

$$v_{19}(a) \geq \frac{19^f - 1}{18}.$$

Then

$$f = v_{19}(a!) \geq \frac{19^{v_{19}(a)} - 1}{18} \geq \frac{19^{\frac{19^f - 1}{18}} - 1}{18} \geq \frac{19^f - 1}{18} \geq f$$

implies equality in

$$\frac{19^f - 1}{18} \geq f,$$

therefore implies $f \in \{0,1\}$.

For $f = 0$ the equations are $3a! + 1 = 19^c$ and $3a + 4 = 19^d$, and the first one cannot be satisfied (modulo 19) for $a \geq 19$. From the set $\{1, \ldots, 18\}$ only $a = 5$ is a solution for the second equation (with $d = 1$), and it turns out that it verifies the first equation, too (with $c = 2$) — so we get the announced solution $(5, 1, 2, 1)$. For $f = 1$ we need have $3a! + 19 = 19^c$ together with $3a + 4 \cdot 19! = 19^d$. The second equation yields a a multiple of 19, and the first one (taken modulo 19^2) shows that this multiple can only be 19. But with $a = 19$, after dividing by 19, none of the equations is valid modulo 19, and we conclude that the only solution is $(a, b, c, d) = (5, 1, 2, 1)$.

Also solved by Sundaresh. H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Teawoo Kim, The Beekman School, New York, NY, USA; Srijan Sundar, Oxford, UK.

U659. Evaluate

$$\lim_{n\to\infty} n\sin\left(\left((2\pi n)^p+2^p\pi^p n^{p-1}p\right)^{1/p}\right)-n\pi(1-p).$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution by Daniel Pascuas, Barcelona, Spain

Since

$$\begin{aligned}\left((2\pi n)^p+2^p\pi^p n^{p-1}p\right)^{1/p} &= 2\pi n(1+\tfrac{p}{n})^{1/p} = 2\pi n\big(1+\tfrac{1}{p}\tfrac{p}{n}+\tfrac{1}{2}\tfrac{1}{p}(\tfrac{1}{p}-1)(\tfrac{p}{n})^2+o(\tfrac{1}{n^2})\big)\\ &= 2\pi n\big(1+\tfrac{1}{n}+\tfrac{1-p}{2}\tfrac{1}{n^2}+o(\tfrac{1}{n^2})\big)\\ &= 2\pi(n+1)+\tfrac{\pi(1-p)}{n}+o(\tfrac{1}{n}), \qquad \text{as } n\to\infty,\end{aligned}$$

and $\sin x = x+o(x^2)$, as $x\to 0$, we have that

$$n\sin\Big(\left((2\pi n)^p+2^p\pi^p n^{p-1}p\right)^{1/p}\Big)=n\sin\Big(\tfrac{\pi(1-p)}{n}+o(\tfrac{1}{n})\Big)=\pi(1-p)+o(1), \quad \text{as } n\to\infty.$$

Therefore

$$\lim_{n\to\infty} n\sin\Big(\left((2\pi n)^p+2^p\pi^p n^{p-1}p\right)^{1/p}\Big)-n\pi(1-p)=\begin{cases}\infty, & \text{if } p>1,\\ 0, & \text{if } p=1,\\ -\infty, & \text{if } p<1.\end{cases}$$

Also solved by G. C. Greubel, Newport News, VA; Joshua Pité, Cambridge Rindge and Latin School, MA; Theo Koupelis, Clark College, Washington, USA.

U660. Let $n \geq 2$ be an integer and $a \in (-2, 2)$ a real number.

(1) Prove that the matrix equation $X^2 + aX + I_n = O_n$ has a solution in the ring $M_n(\mathbb{C})$.

(2) Prove that the equation has a solution in $M_n(\mathbb{R})$ if and only if n is even.

(3) Prove that in each of the above situations the equation has infinitely many solutions.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Solution by the author

Denote $a = 2\cos t, t \in \mathbb{R}$, and the equation is equivalent to $(X + (\cos t)I_n)^2 + (\sin^2 t)I_n = O_n$.

(1) The matrix $X = (-\cos t + i \sin t)I_n \in M_n(\mathbb{C})$ satisfies the equation.

(2) "$\Rightarrow$" For a solution $X \in M_n(\mathbb{R})$, we have $(X + (\cos t)I_n)^2 = -(\sin^2 t)I_n$ and $\sin t \neq 0$, hence

$$\det{}^2(X + (\cos t)I_n) = (-1)^n \sin^{2n} t > 0,$$

therefore n must be even.

"$\Leftarrow$" We will show that solutions exist for every even positive integer n. Consider the matrix

$$Y = \frac{1}{\sin t}(X + (\cos t)I_n) \in M_n(\mathbb{R}).$$

For $n = 2$, the equation $Y^2 = -I_2$ has solutions in $M_2(\mathbb{R})$, for example

$$Y_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If $n = 2m$ the equation $Y^2 = -I_{2m}$ has the solution defined by the diagonal 2×2 blocks

$$Y_m = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & 0 \\ & & \ddots & \\ & 0 & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{pmatrix}$$

and the conclusion follows.

(3) If X is a solution, then for every invertible matrix U, we have

$$(UXU^{-1})^2 + a(UXU^{-1}) + I_n = U(X^2 + aX + I_n)U^{-1} = UO_nU^{-1} = O_n,$$

hence UXU^{-1} is also a solution.

Also solved by Daniel Pascuas, Barcelona, Spain.

O655. Let a, b, and d_n, $n = 1, 2, \ldots$ be integers such that a is nonzero and relatively prime to every d_n, $n \geq 1$, and, also, d_n and d_{n+1} are relatively prime for every $n \geq 2$. Prove that there exists a permutation $c_1, c_2, \ldots$ of the numbers $an + b$, $n = 1, 2, \ldots$ such that d_n divides $c_1 + \cdots + c_n$ for any positive integer n.

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution by the authors

We build $c_1, c_2, \ldots$ inductively. First, since $(a, d_1) = 1$ (we denote by (u, v) the greatest common divisor of the integers u and v), there exist solutions to the congruence $ax + b \equiv 0 \pmod{d_1}$, and these solutions are the terms of an arithmetic progression infinite in both directions, with common difference d_1. Consequently we can pick such a solution x which is a positive integer, and we can consider $c_1 = ax + b$ for this x (clearly, d_1 divides c_1).

Assume that $c_1, \ldots, c_j$ were built to be distinct terms of the set $\{an + b \mid n = 1, 2, \ldots\}$ and such that $c_1 + \cdots + c_i$ is divisible by d_i for any $i = 1, \ldots, j$; we show how we can construct c_{j+1}, and c_{j+2}. If among $c_1, \ldots, c_j$ we can find all numbers $an+b$ for $n = 1, \ldots, k$, but not $a(k+1)+b$, then we put $c_{j+2} = a(k+1)+b$ ($k = 0$ is allowed, meaning that $a \cdot 1 + b$ is not among $c_1, \ldots, c_j$; in that case we consider $c_{j+2} = a + b$). Once c_{j+2} was fixed, we consider the system of congruences

$$x \equiv -(c_1 + \cdots + c_j) \pmod{d_{j+1}}, \quad x \equiv -(c_1 + \cdots + c_j + c_{j+2}) \pmod{d_{j+2}}.$$

Since $j + 1 \geq 2$, we have $(d_{j+1}, d_{j+2}) = 1$ by hypothesis, hence the system has solutions according to the Chinese remainder theorem. Moreover, all the solutions form an infinite (in both directions) arithmetic progression with common difference $d_{j+1}d_{j+2}$. Since $(a, d_{j+1}d_{j+2}) = 1$, the congruence $ay + b \equiv x \pmod{d_{j+1}d_{j+2}}$ has integer solutions y for any integer x (but, of course, we will be interested only in the case when x is a solution of the above system), and the solutions form an infinite (two-sided) arithmetic progression with common difference $d_{j+1}d_{j+2}$. All these show that we can find an $m \geq 1$ such that

$$am + b \equiv -(c_1 + \cdots + c_j) \pmod{d_{j+1}}, \quad am + b \equiv -(c_1 + \cdots + c_j + c_{j+2}) \pmod{d_{j+2}},$$

and, moreover, $am + b$ does not appear among $c_1, \ldots, c_j, c_{j+2}$. If we now consider $c_{j+1} = am + b$, we clearly have that d_i divides $c_1 + \cdots + c_i$ for every $i = 1, \ldots, j+2$, and that $c_1, \ldots, c_{j+2}$ are distinct elements of the set $\{an + b \mid n = 1, 2, \ldots\}$. Moreover, it is clear that $c_1, c_2, \ldots$ that we construct in this manner are all distinct (since every c_{j+1} differs from any of $c_1, \ldots, c_j$), and contain all the elements of the set $\{an+b \mid n = 1, 2, \ldots\}$, due to the way we choose c_{j+2} for every j — meaning that $c_1, c_2, \ldots$ is, indeed, a permutation of the numbers $an + b$, $n = 1, 2, \ldots$. The proof is complete.

Remarks. 1) This is merely an adaptation (a slight generalization) of a well-known contest problem, which asks to prove that there is a permutation $c_1, c_2, \ldots$ of the positive integers such that $c_1 + \cdots + c_n$ is divisible by n for all $n \geq 1$. Thus the conditions imposed to a and $d_1, d_2 \ldots$ are such that the proof from the particular case can be generalized. It would be interesting to see if other conditions would also assure the validity of the result.

2) We need not necessarily assume in the statement that $a \neq 0$, since if each d_n is from the set $\{1, -1\}$ then there is nothing to prove, while otherwise $a = 0$ would not be relatively prime to at least one of the d_ns — so, anyway, we must only consider that a is nonzero.

3) Here is a pretty nice looking particular case: if k is a given positive integer, prove that there exists a permutation $c_1, c_2, \ldots$ of the positive odd integers such that $c_1 + \cdots + c_n$ is divisible by $(2n - 1)^k$ for any positive integer n.

Also solved by Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil.

O656. Let $n \geq 2$ be an integer. Find the largest integer k such that

$$\left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}$$

for all nonnegative real numbers a_i which satisfy $a_1 \geq a_2 \geq \cdots \geq a_n$.

Proposed by Vasile Cîrtoaje, Petroleum-Oil University, Ploieşti, Romania

Solution by Theo Koupelis, Clark College, Washington, USA

When all variables a_i, $i = 1, 2, \ldots, n$, are equal, the given inequality is valid for all positive integers $k \in [1, n]$. For $a_1 > 0$ and $a_2 = a_3 = \cdots = a_n = 0$ we get $k \leq \sqrt{n}$. It is now sufficient to show that the given inequality holds for $k = k_{\max} = \lfloor \sqrt{n} \rfloor$ for any real, nonnegative numbers a_i.

(i) Let $n = m^2$, where m is a positive integer. Then $k_{\max} = m$, and thus

$$m^2 \left(\sum_{i=1}^{m} a_i\right)^2 \geq m^2 \sum_{j=1}^{m^2} a_j^2 \Longleftrightarrow \sum_{1 \leq i < j \leq m} 2a_i a_j \geq \sum_{\ell=m+1}^{m^2} a_\ell^2.$$

The above is obvious because on both sides of the inequality we have $2 \cdot \frac{(m-1)m}{2} = m(m-1) = m^2 - m$ terms, and $a_1 \geq a_2 \geq \cdots \geq a_n$.

(ii) Let $n = m^2 + \lambda$, where λ is a positive integer in $[1, 2m]$. Then $k_{\max} = m$, and thus

$$(m^2 + \lambda)\left(\sum_{i=1}^{m} a_i\right)^2 \geq m^2 \sum_{j=1}^{m^2+\lambda} a_j^2 \Longleftrightarrow m^2 \left[\left(\sum_{i=1}^{m} a_i\right)^2 - \sum_{j=1}^{m^2} a_j^2\right] + \left[\lambda \left(\sum_{i=1}^{m} a_i\right)^2 - m^2 \sum_{\ell=m^2+1}^{m^2+\lambda} a_\ell^2\right] \geq 0.$$

The above is obvious because both brackets are nonnegative, based on the result in (i), the fact that $\sum_{\ell=m^2+1}^{m^2+\lambda} a_\ell^2 \leq \lambda \cdot a_{m^2+1}^2$, and $\sum_{i=1}^{m} a_i \geq m \cdot a_m \geq m \cdot a_{m^2+1}$.

Also solved by Srijan Sundar, Oxford, UK.

O657. Let $p_1 = 2$, $p_2 = 3, \ldots$ be the increasing sequence of all primes. Prove that a sequence $q_1, q_2, \ldots$ of distinct primes can be found such that $q_1 + \cdots + q_n$ is divisible by p_n for every $n \geq 1$.

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution by the authors

We build the sequence $q_1, q_2, \ldots$ inductively, starting with $q_1 = p_1 = 2$ (so, q_1 is divisible by p_1, but not by p_2). Assume that we have already chosen $q_1, \ldots, q_i$ such that $q_1 + \cdots + q_i$ is divisible by p_i, but not by p_{i+1}, and we further show how we can define q_{i+1} such that $q_1 + \cdots + q_{i+1}$ is divisible by p_{i+1}, but not by p_{i+2}.

Let r be that one of $1 - (q_1 + \cdots + q_i)$ and $2 - (q_1 + \cdots + q_i)$ which is *not* divisible by p_{i+2} (if both of them have this property, then we just choose one of them; but, clearly, they cannot be both divisible by p_{i+2}).

By the Chinese remainder theorem the system of congruences

$$x \equiv -(q_1 + \cdots + q_i) \pmod{p_{i+1}}, \quad x \equiv r \pmod{p_{i+2}}$$

has infinitely many solutions which are the terms of an arithmetic progression $(x_0 + tp_{i+1}p_{i+2})_{t\in\mathbb{Z}}$, where x_0 is a fixed solution. Since $-(q_1 + \cdots + q_i)$ is not divisible by p_{i+1} by the inductive hypothesis, and r is not divisible by p_{i+2} according to the choice we made, any solution x (including, of course, x_0) is relatively prime to each of p_{i+1} and p_{i+2}, hence to the common difference $p_{i+1}p_{i+2}$ of the above arithmetic progression. Consequently, by Dirichlet's theorem, the progression contains infinitely many primes — that is, there are infinitely many primes satisfying both congruences above. So, we can consider q_{i+1} to be such a prime, big enough in order to be different from any of $q_1, \ldots, q_i$. Thus q_{i+1} verifies

$$q_{i+1} \equiv -(q_1 + \cdots + q_i) \pmod{p_{i+1}} \Leftrightarrow q_1 + \cdots + q_{i+1} \equiv 0 \pmod{p_{i+1}}$$

(that is, $q_1 + \cdots + q_{i+1}$ is divisible by p_{i+1}), and

$$q_{i+1} \equiv r \pmod{p_{i+2}},$$

which actually says that we have either

$$q_1 + \cdots + q_{i+1} \equiv 1 \pmod{p_{i+2}},$$

or

$$q_1 + \cdots + q_{i+1} \equiv 2 \pmod{p_{i+2}}.$$

Since $i + 2 \geq 3$ (hence p_{i+2} is at least 5), no matter which of these congruences holds, it follows that $q_1 + \cdots + q_{i+1}$ is not divisible by p_{i+2}, as we need. This finishes the inductive construction of the sequence $q_1, q_2, \ldots$, and the proof.

Also solved by Nihad Hashimov; Theo Koupelis, Clark College, Washington, USA.

O658. Prove that there are infinitely many positive integers a such that

$$a! + (a+2)! \mid (2a+2)!$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution 1 by the author

We shall prove that $a = x^2(x^2+2)$ works.

Indeed, if $x \in \{1, 2, \ldots, 9\}$ we are done. Assume $x \geq 10$. We want to prove

$$1 + (a+1)(a+2) \mid (a+1)(a+2) \cdot \ldots \cdot (2a+2).$$

The left-hand side is factored as $(x^2+x+1)(x^2+x+1)(x^4+3x^2+3)$ while the right-hand side is

$$B = (a+1) \cdot \ldots \cdot (2a+2) = (x^4+2x^2+1)(x^4+2x^2+2) \cdot \ldots \cdot (2x^4+4x^2+3).$$

Notice that x^4+3x^2+3 is a term in the product above. We claim that $\gcd(x^4+3x^2+3, (x^2-x+1)(x^2+x+1)) = 1$. Indeed,

$$(x+1)(x^4+3x^2+3) - (x^2+2x+1)(x^2+x+1) = 2.$$

Since x^4+3x^2+3 is a factor of $(a+1) \cdot \ldots \cdot (2a+2)$, it suffices to prove that $A = (x^2-x+1)(x^2+x+1)$ divides B for all x. Let p be a prime dividing A. Clearly $p \leq x^2+x+1$. Notice that there are at least

$$\left\lfloor \frac{2a+2}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor \geq \frac{a+2}{p} - 2 \geq \frac{x^2(x^2+2)+2}{x^2+x+1} - 2 > x^2 - x - 1$$

multiples of p in B. That is, $\nu_p(B) \geq x^2 - x + 1$. On the other hand, for all $x \geq 10$:

$$\nu_p(A) \leq \log_p A \leq \log_2(x^4+x^2+1) < \log_2(2x^2) < 8\log x + 1 < 8x + 1 < x^2 - x + 1.$$

Solution 2 by Theo Koupelis, Clark College, Washington, USA

We show that all $a = n^2 - 1$, where $n \geq 2$ is a positive integer, satisfy the desired property. We have

$$a! + (a+2)! \mid (2a+2)! \Longleftrightarrow 1 + (a+1)(a+2) \mid (a+1)(a+2)\cdots(2a+2),$$

or

$$P(n): \qquad n^4 + n^2 + 1 \mid n^2(n^2+1)\cdots(2n^2).$$

We use induction to show that the statement $P(n)$ holds for all $n \geq 2$. When $n = 2$, we have $21 \mid 4 \cdot 5 \cdots 8$, which is true because $6 \cdot 7 = 2 \cdot 21$. When $n = 3$, we have $91 \mid 9 \cdot 10 \cdots 18$, which is true because $13 \cdot 14 = 2 \cdot 91$. Let the statement $P(k)$ hold for some positive integer $k \geq 2$. Then $(k+1)^4 + (k+1)^2 + 1 = (k^2+k+1)(k^2+3k+3)$, and thus

$$P(k+1): \qquad (k^2+k+1)(k^2+3k+3) \mid (k^2+2k+1)(k^2+2k+2)\cdots(2k^2+4k+2).$$

This is obvious because the terms $2(k^2+k+1)$ and k^2+3k+3 are clearly different factors of the dividend for all $k \geq 2$.

Also solved by Parsia Tajallaei, AE High School, Tehran, Iran; Srijan Sundar, Oxford, UK.

O659. Let ABC be a triangle and let M be any point inside the triangle. Denote by x, y, z the distance from M to sides BC, CA, AB, respectively. Prove that

$$\min\{MA, MB, MC\} \le \frac{\sqrt{abc(ayz+bzx+cxy)}}{2S} \le \max\{MA, MB, MC\}$$

where S is the area of triangle ABC.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

Denote by S_a, S_b, S_c the area of triangles BMC, CMA, AMB respectively. We have known that

$$S_a\overrightarrow{MA} + S_b\overrightarrow{MB} + S_c\overrightarrow{MC} = \overrightarrow{0}.$$

By squaring the both sides we get

$$S_a^2MA^2 + S_b^2MB^2 + S_c^2MC^2 + \sum_{\text{cyc}} 2S_bS_c\overrightarrow{MB}\overrightarrow{MC} = 0,$$

or

$$S_a^2MA^2 + S_b^2MB^2 + S_c^2MC^2 + \sum_{\text{cyc}} S_bS_c(MB^2 + MC^2 - a^2) = 0,$$

$$\sum_{\text{cyc}}(S_a + S_b + S_c)S_aM_a^2 = a^2S_bS_c + b^2S_cS_a + c^2S_aS_b,$$

$$S_aMA^2 + S_bMB^2 + S_cMC^2 = \frac{a^2S_bS_c + b^2S_cS_a + c^2S_aS_b}{S}.$$

On the other hand

$$(S_a+S_b+S_c)\min\{MA^2, MB^2, MC^2\} \le S_aMA^2+S_bMB^2+S_cMC^2 \le (S_a+S_b+S_c)\max\{MA^2, MB^2, MC^2\}$$

i.e.

$$S \cdot \min\{MA^2, MB^2, MC^2\} \le S_aMA^2 + S_bMB^2 + S_cMC^2 \le S \cdot \max\{MA^2, MB^2, MC^2\}.$$

Combining these we obtain

$$\min\{MA^2, MB^2, MC^2\} \le \frac{a^2S_bS_c + b^2S_cS_a + c^2S_aS_b}{S^2} \le \max\{MA^2, MB^2, MC^2\}.$$

This is equivalent to

$$\min\{MA, MB, MC\} \le \frac{\sqrt{a^2S_bS_c + b^2S_cS_a + c^2S_aS_b}}{S} \le \max\{MA, MB, MC\}.$$

Now we note that

$$S_a = \frac{ax}{2}, S_b = \frac{by}{2}, S_c = \frac{cz}{2}$$

and the conclusion follows.

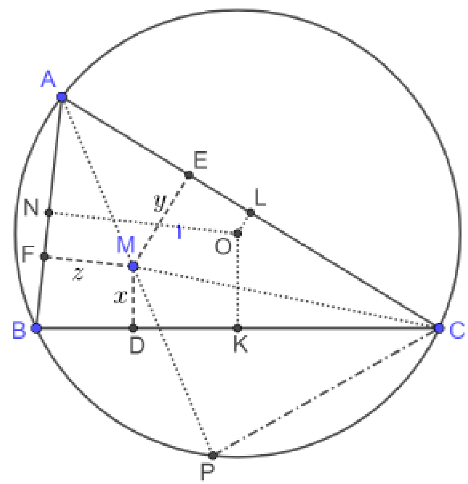
Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let D, E, F be the projections of M on the lines BC, AC, AB, respectively. Then $MD = x$, $ME = y$, and $MF = z$. The points B, F, D, M are on the circle with diameter MB, because $\angle MDB = \angle MFB = 90°$. Thus $\sin\angle FMD = \sin B$ and $FD = MB \cdot \sin B$. Similarly we find $FE = MA \cdot \sin A$, and $DE = MC \cdot \sin C$. Thus, $bzx = 2R\sin B \cdot MF \cdot MD = 4R[MFD]$, where R is the circumradius of $\triangle ABC$, and $[MFD]$ is the area of $\triangle MFD$. Similarly we find $ayz = 4R[MFE]$, and $cxy = 4R[MDE]$. Therefore,

$$ayz + bzx + cxy = 4R[DEF]. \quad (*)$$

Let P be the second point of intersection of the line AM with the circumcircle (O) of $\triangle ABC$. Points A, F, M, E are concyclic because $\angle MEA = \angle MFA = 90°$; similarly, points E, M, D, C are concyclic because $\angle MDC = \angle MEC = 90°$. Thus, $\angle FED = \angle FEM + \angle MED = \angle FAM + \angle MCD = \angle BAP + \angle MCD = \angle BCP + \angle MCD = \angle MCP$. Also, $\angle MPC = \angle APC = \angle ABC$. Using the law of sines in $\triangle MPC$ we get $\dfrac{\sin B}{MC} = \dfrac{\sin \angle FED}{MP}$. Thus,

$$[DEF] = \frac{1}{2} \cdot FE \cdot DE \cdot \sin \angle FED = \frac{1}{2} \cdot MA \sin A \cdot MC \sin C \cdot \frac{MP}{MC} \sin B$$
$$= \frac{1}{2}(R^2 - OM^2) \cdot \sin A \cdot \sin B \cdot \sin C = \frac{abc(R^2 - OM^2)}{16R^3}. \quad (**)$$



From $(*)$ and $(**)$ we get

$$\frac{\sqrt{abc(ayz + bzx + cxy)}}{2S} = \frac{abc\sqrt{R^2 - OM^2}}{4RS} = \sqrt{R^2 - OM^2}.$$

Let K, L, N be the midpoints of BC, AC, AB, respectively. Without loss of generality, let M be inside the quadrilateral $ONBK$. Then $\min\{MA, MB, MC\} = MB$ and $90° \leq \angle OMB \leq 180°$. Thus, using the law of cosines in $\triangle OMB$ we get $OB^2 = R^2 \geq OM^2 + MB^2$, and therefore $MB \leq \sqrt{R^2 - OM^2}$. Also, $\angle AMC \leq 180°$, and thus at least one of the angles $\angle AMO$ and $\angle CMO$ is not obtuse. Without loss of generality, let $\angle OMC \leq OMA$ and thus $MC \geq MA$; but then $\cos \angle OMC = (OM^2 + MC^2 - R^2)/(2OM \cdot MC) \geq 0$, and therefore $\sqrt{R^2 - OM^2} \leq MC$. Thus, $MB \leq \sqrt{R^2 - OM^2} \leq MC$.

O660. Let a, b, c be positive real numbers such that

$$1-2(a+b+c)+3(ab+bc+ca)-4abc=0.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+32 \geq \frac{33}{a+b+c}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

If one of the numbers is equal to 1, let's say $c=1$, then the condition from the hypothesis leads us to the following relationship

$$(a-1)(b-1)=0 \Longrightarrow a=1 \text{ or } b=1$$

so the inequality is clearly true.

We assume that all numbers are different from 1. The condition in the hypothesis is equivalent to

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=4.$$

If one of the numbers a, b, c is greater than 1, the inequality is true because

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c},$$

$$32 \geq \frac{32}{a+b+c}.$$

Therefore, there remains the case where $a, b, c \in (0,1)$. with the following substitutions

$$x=\frac{1}{1-a}-1,\ y=\frac{1}{1-b}-1,\ z=\frac{1}{1-c}-1,\ x, y, z>0,\ x+y+z=1.$$

The inequality becomes

$$\frac{x+1}{x}+\frac{y+1}{y}+\frac{z+1}{z}+32 \geq \frac{33}{\frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1}},$$

or, after homogenizing

$$\frac{2x+y+z}{x}+\frac{x+2y+z}{y}+\frac{x+y+2z}{z}+32 \geq \frac{33}{\frac{x}{2x+y+z}+\frac{y}{x+2y+z}+\frac{z}{x+y+2z}}.$$

Without loss of generality, assume that $x+y+z=3$ and let $q=xy+yz+zx=3(1-t^2)$, $t \in (0,1)$, $r=xyz <= 1$. After some simple computations this inequality reduces to

$$35+\frac{3q}{r} \geq \frac{11(3q+r+54)}{2q+r+9},$$

or, after expanding and rearranging terms

$$24r^2-(279-40q)r+6q^2+27q \geq 0,$$

or

$$24r^2-(159+120t^2)r+54(1-t^2)^2+81(1-t^2) \geq 0 \Longleftrightarrow f(r) \geq 0.$$

Since

$$\frac{159+120t^2}{48}>1$$

and from a known result $r \leq (1-t)^2(1+2t)$ it suffice to prove that

$$f\left((1-t)^2(1+2t)\right) \geq 0,$$

which, after performing the calculations, becomes

$$6t^2(4-t)(1-t)(4t-1)^2 \geq 0,$$

clearly true. The equality holds when $t=0$ which implies $x=y=z$ so $a=b=c=\frac{1}{4}$ or for $t=\frac{1}{4}$ which implies $x=y=\frac{3}{4}$, $z=\frac{3}{2}$ so $a=b=\frac{1}{5}$, $c=\frac{1}{3}$.

Also solved by Nicuşor Zlota, "Traian Vuia" Technical College; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.