Junior Problems

J661. Find all positive integers n for which

$$\frac{(n^2+1)!+(n^2)!+(n^2-1)!}{(n+1)!}=(n+3)^2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J662. Let ABC be a triangle and let I be its incenter. Line AI intersects the circumcircle at D. An arbitray line through D intersects BC at X and the circumcircle at Y. The perpendicular bisector of the segment IX intersects ID at Z. Prove that YZ is a symmedian in triangle IXY.

Proposed by Mihaela Berindeanu, Bucharest, România

J663. Let $f:(0,\infty)\longrightarrow\mathbb{R}$ be a function such that

$$\frac{x}{y}f\left(\frac{y}{x}\right)+\left(\frac{x}{y}-\frac{y}{x}\right)f\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{y}{x}f\left(\frac{x}{y}\right)=\frac{x^3}{y^3}-\frac{y^3}{x^3}$$

for all positive real numbers x and y. Evaluate $f(2024)$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J664. Let a,b,c be nonnegative real numbers such that $ab+bc+ca\geq 3$. Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J665. Let ABC be a triangle and let O be its circumcenter. Line AO intersects side BC at X and the circumcenter at Y. Prove that if $AX=3AY$ then $\angle A\geq\pi/3$.

Proposed by Mihaela Berindeanu, Bucharest, România

J666. Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$f(xy)=f(x)f(y+1)+f(x-1)f(y),$$

for all $x,y\in\mathbb{R}$.

Proposed by Mircea Becheanu, Canada

Senior Problems

S661. Find all n for which there is a positive integer k such that

$$(n-1)!+1=(k-30)^2 \quad \text{and} \quad (n+1)!+1=(k+30)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S662. Prove that in any triangle ABC,

(a) $\dfrac{1}{r}+\dfrac{9}{4R+r} \geq \dfrac{4}{r_a+r_b}+\dfrac{4}{r_b+r_c}+\dfrac{4}{r_c+r_a}$.

(b) $2R+5r \geq h_a+h_b+h_c$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S663. Let ABC be a triangle with $\angle A=\angle B+60°$. Prove that

$$(\cos 3A+i\sin 3B)^4+(\cos 3B+i\sin 3A)^4=2(\cos 3C+i\sin 3C)^2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

S664. Let x,y,z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{1}{x^3+y^3+z^3}+\frac{19}{xy+yz+zx} \geq 66.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

S665. Let $0<a_1<a_2<\cdots<a_n<\cdots$ be a sequence of integers such that, for all n, a_n can not be represented in the form

$$a_n=\varepsilon_1 a_1+\cdots+\varepsilon_{n-1}a_{n-1}$$

where $\varepsilon_i \in \{0,1\}$. Let N_M be the number of elements of the sequence which do not exceed M. Prove that for each k

$$N_M \leq a_k+\frac{M}{k+1}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

S666. Let $0 \leq a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$ such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$a_1a_2+a_2a_3+\cdots+a_na_1 \leq n.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploieşti, România

Undergraduate Problems

U661. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) = f(x)f(y+1) - (2y+1)f(x),$$

for all $x, y \in \mathbb{R}$.

Proposed by Mircea Becheanu, Canada

U662. Evaluate

$$\int_{-1}^{1} \frac{\arccos x}{\sqrt{x^4 - 3x^2 + 2}}\, dx.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, România

U663. Let $(x_n)_{n\geq 2}$ be the sequence defined by

$$x_n = n^2 \left(\frac{\cos\frac{1}{n} - 1}{\cos\frac{1}{n^2} - 1} - n^2 + \frac{1}{12} \right).$$

Evaluate $\lim\limits_{n\to\infty} n^2 \left(x_n - \dfrac{31}{360} \right)$.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U664. For every integer $n \geq 0$ define

$$I_n = \int \frac{x^n e^{\arctan x}}{\sqrt{1+x^2}} dx.$$

Prove that

$$(n+1)I_{n+1} + I_n + nI_{n-1} = x^n \sqrt{1+x^2}\, e^{\arctan x}$$

for all $n > 0$ and then evaluate I_2.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

U665. Let K be a field of characteristic 2 and let d be a positive integer. Prove that there are at most $2d-1$ pairs $(P(x),\ Q(x))$ of nonconstant polynomials in $K[x]$, of degree at most d and such that

$$xP(x)^2 + Q(x)^2 + xP(x)Q(x) = 1.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U666. Prove that

$$\frac{\sin x}{x} + \frac{\sin^4 x}{x^4} \geq 2\cos x + \sin^4 x \cos^4 x,$$

for all $0 < x \leq \frac{\pi}{2}$.

Proposed by Paolo Perfetti, Universita degli Studi di Tor Vergata Roma, Italy

Olympiad Problems

O661. Let $a, b, c > 0$ such that

$$\sum_{cyc} \frac{a+b}{\sqrt{a^2-ab+b^2}} \leq \frac{2}{3}(a+b+c)^2.$$

Prove that $a^4+b^4+c^4 \geq 3$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O662. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\frac{1}{8} \leq a^4+b^4+c^4+26abc \leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O663. Prove that $k = \frac{3}{4}$ is the smallest positive constant such that

$$\left(\frac{ka+b+c+d}{k+3}\right)^2 \geq \frac{ab+bc+cd+de+ea}{5}$$

whenever $a \geq b \geq c \geq d \geq e \geq 0$.

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploieşti, România

O664. Let ABC be an acute triangle with circumcircle Γ. Let A_1 be the midpoint of the segment BC, A_2 the reflexion of A in the line BC and let A_3 be the intersection of A_1A_2 with Γ. Similarly, the points B_1, B_2, B_3 and C_1, C_2, C_3 are defined. Prove that the lines AA_3, BB_3, CC_3 are concurent.

Proposed by Mihaela Berindeanu, Bucharest, România

O665. Let $a_1, a_2, \ldots, a_{n+1}$ bepositive numbers. Prove that

$$\sum_{i=1}^{n+1} \frac{1}{(1+na_i)^{n+1}} \geq \frac{1}{(n+1)^{n-1}(1+na_1a_2\cdots a_{n+1})}.$$

Proposed by An Zhenping, Xianyang Normal University, China

O666. Let ABC be a triangle with incenter I. Prove that

(a) $AI^2+BI^2+CI^2 \geq \dfrac{a^2+b^2+c^2}{3}$.

(b) $AI^2+BI^2+CI^2 \geq \dfrac{a^2}{2+\cos B+\cos C} + \dfrac{b^2}{2+\cos C+\cos A} + \dfrac{c^2}{2+\cos A+\cos B}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam