Four Circles Tangent to an Ellipse

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Abstract. We present a geometric proof of Iwata's Theorem — an old Sangaku problem about four circles tangent to an ellipse and inscribed in an angle. Our approach helps to discover another interesting property in this configuration, which seems not to be observed before. We also prove the correctness of Taxia Limneou's construction of the circles in this configuration.

The following problem has been recently posted by Stanley Rabinowitz in the *Romantics of Geometry Facebook Group* [4].

Problem 1.

An ellipse is inscribed in an angle. Four circles o_1, o_2, o_3, o_4 with radii r_1, r_2, r_3, r_4, respectively, are also inscribed in the given angle and are tangent to the ellipse, as show in Figure 1. Prove that

$$r_1 \cdot r_4 = r_2 \cdot r_3 .$$

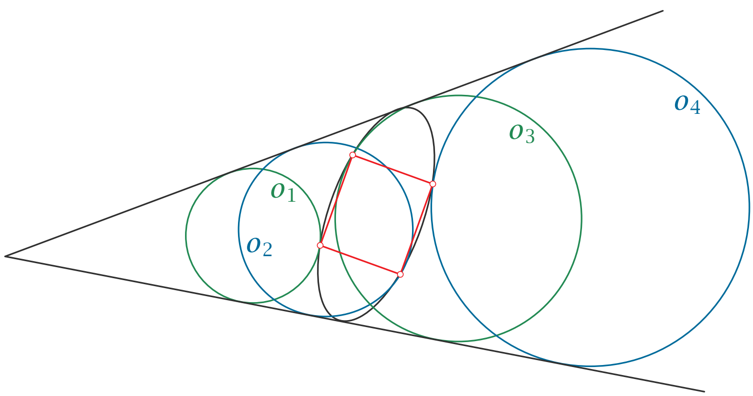

Fig. 1

This result is one of the Sangaku problems and is known as Iwata's Theorem. According to [5] the first proof, presented by Iwata in 1866, was 52 pages long. Much shorter solution, but still quite long and computational is given in [5].

In this note I give a short, geometric proof based on the properties of homotheties and perspective transformations. Due to this approach I was able to discover another nice property hidden in this configuration, which is not mentioned in [5].

Problem 2.

The tangency points of the ellipse with the circles are vertices of a rectangle.

After discovering this result, I announced it in the *Romantics of Geometry Facebook Group* [3]. In response, Chi Nguyen Chuong posted a proof using, among other things, Brianchon's Theorem and the properties of the cross-ratio [1].

A little bit later Taxia Limneou [2] provided the following, elegant construction of circles tangent to a given ellipse inscribed in an angle.

Let e be an ellipse with center O, tangent at points B and C to the arms of an angle with vertex A (see Figure 2). Let ω be the circle tangent to the ellipse e at point C and inscribed in the given angle. Besides point C, the ellipse e intersects the circle ω at two other points, K and N. Point M is the midpoint of segment CK. The line OM intersects the ellipse e at points T and U. Then points T and U are the points of tangency of the ellipse with two of the sought four circles.

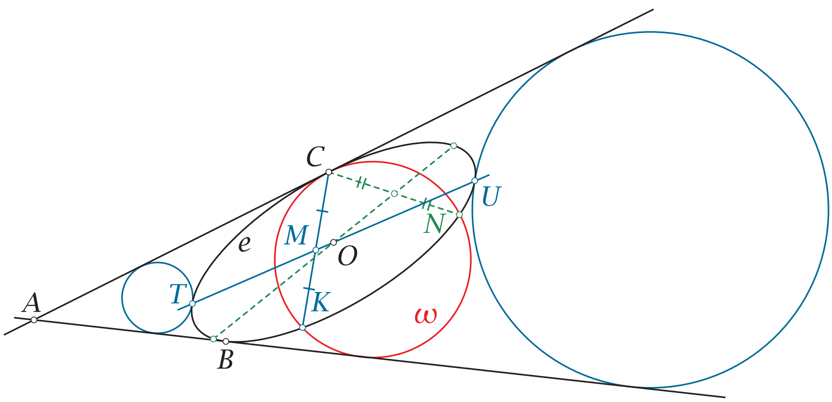

Fig. 2

The remaining two points of tangency are obtained by an analogous construction with point N instead of point K.

In this paper we also present a geometric proof of the correctness of this construction.

1. The Tool: Perspective Transformations

Let P be a point and let k be a line not containing P, both lying on a projective plane. Assume f is a projective mapping from the plane to itself such that P and all points of k are fixed under f. Such mappings f are called *perspective transformations*; point P is called the *center* and line k — the *axis* of f.

The existence of a perspective transformation for a given center P and axis k can be justified with aid of three-dimensional space.

Let α be a given projective plane, containing point P and line k, immersed in three dimensions. Let r be a rotation about a line k by a certain angle and let $\alpha' = r(\alpha)$. On the line connecting points P and $r(P)$, we choose any point S different from P and $r(P)$. Let $g\colon \alpha' \to \alpha$ be a projection with center S. Then the transformation $f = g \circ r$ is a projective mapping of the plane α onto itself, where point P and every point on line k are fixed points.

In order to understand how a perspective transformation maps individual points we don't have to use three dimensions. Let x be an arbitrary line passing through P and let $A = x \cap k$ (see Figure 3). Since A and P are fixed points of f, line x is mapped to itself. It implies that for any point X, points X, $X' = f(X)$, and P are collinear. A similar argument can be used to prove a dual property: for any line y, lines y, $y' = f(y)$, and k are concurrent.

If k is the infinity line, then the perspective transformations are just homotheties with center P. Also for other lines k there are many perspective transformations with a given center P. To define the mapping uniquely, it suffices to set the image T' of a sample point T in such a way that P, T, and T' are collinear. Then the construction of an image X' of an arbitrary point X can be done using above properties.

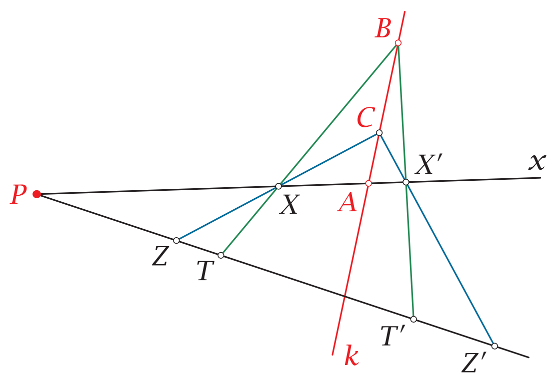

Fig. 3

Namely, if X does not lie on line TT', then point $X' = f(X)$ lies on line PX, but since $B = TX \cap k$ is a fixed point, X' must also lie on line BT'. Thus $X' = PX \cap BT'$. Now, knowing the image X' of X we may in the same way construct the image Z' of any point Z lying on line PT.

Let c and d be two tangent conics inscribed in an angle with vertex P (eee Figure 4). Let A be a touching point of the conics and let k be the common tangent line at A to both conics. Then there exists a perspective transformation with center P, axis k that maps c onto d.

Indeed, assume T and T' are the touching points of the conics c and d, respectively, with one arm of the angle. Consider a perspective transformation f with center P, axis k and such that T maps to T'. Then the image c' of c under f is a conic tangent to k at A, tangent to line PT at T' and tangent to the line containing the second arm of the angle. But these conditions determine a conic uniquely, so $c' = d$.

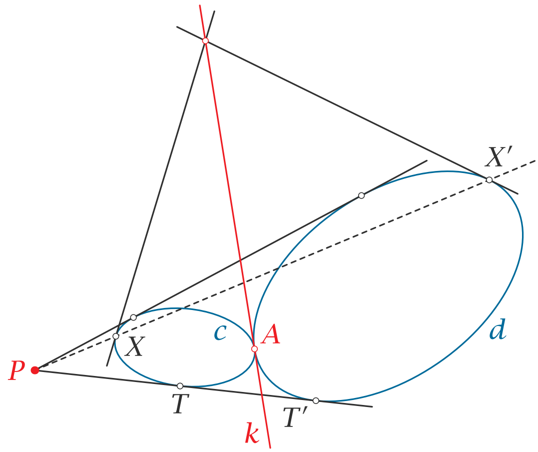

Fig. 4

Finally, observe that for any point X lying on c the transformation f maps the tangent line at X (to c) to the tangent line at $X' = f(X)$ (to d) and hence these two tangents intersect at a point lying on k.

2. Two Simple Lemmas

Lemma 2.1

Circles o_1 and o_2 are inscribed in an angle with vertex P (see Figure 5). A line passing through P intersects circles o_1 and o_2 at points A, B, respectively. Assume the tangent lines at A and B to circles o_1 and o_2 meet at C. Then $AC = BC$.

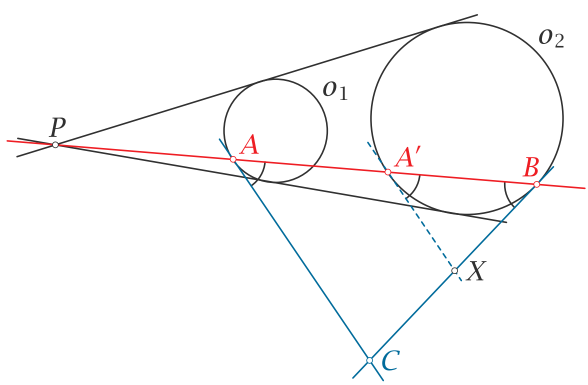

Fig. 5

Proof

Consider a homothety with center P mapping circle o_1 to o_2. Let A' be the image of A under this homothety. Assume the tangent line at A' to o_2 intersects line BC at point X. Because of the homothety lines AC and $A'X$ are parallel, so we obtain $\angle CAB = \angle XA'B = \angle A'BX$, implying $AC = BC$. This completes the proof.

Lemma 2.2

Assume segments SA, SB are tangent to an ellipse at points A, B, respectively, such that $SA = SB$ (see Figure 6). Let c and d be lines parallel to SB and SA, respectively, and tangent to the ellipse at points C, D, respectively. Then A, B, C, and D are vertices of a rectangle.

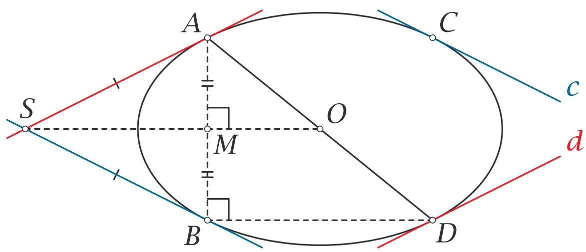

Fig. 6

Proof

Let M and O be the midpoints of segments AB and AD, respectively. Since $SA \parallel d$, O is the center of the ellipse. Let V be a direction of line AB. Then points S, M, O lie on v — the polar of V with respect to the ellipse, so they are collinear. Since $SA = SB$, lines OM and AB are perpendicular. Hence also AB and BD are perpendicular. Consequently, $ABDC$ is a parallelogram with $\angle ABD = 90^\circ$, so it is a rectangle.

3. Solutions to Problems 1 and 2

Denote by A and B the tangency points of the ellipse with circles o_1 and o_2, respectively (see Figure 7). Draw the tangent lines at A, B to the ellipse and let them intersect at S. Let f be a perspective transformation with center P and axis AS that maps the ellipse to circle o_1. Then the tangent line to o_1 at $Z = f(B)$ passes through S. By Lemma 2.1 it follows that $SB = SZ = SA$.

Assume line AB intersects circles o_1 and o_2 for the second time at points M and N, respectively. Let n be the tangent line to o_2 at N. Since $\angle SAB = \angle SBA$, line n is parallel to SA. Similarly, m is parallel to SB.

A homothety with center P that maps o_2 to o_1 takes point N to some point Y and line n to line y. Then $y \parallel n$ and y is tangent to o_1 at Y. The inverse homothety, however, maps point M to some point X and line m to line x. Then $x \parallel m$ and x is tangent to o_2 at X.

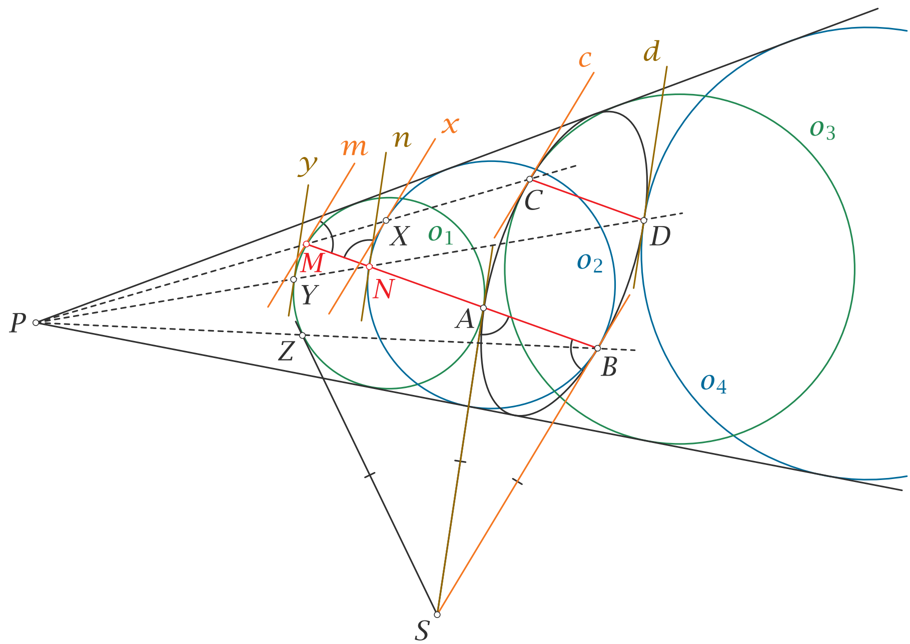

Fig. 7

Let $D = f^{-1}(Y)$. The tangent line d to the ellipse at D, line y, and line AS are concurrent, so $d \parallel y$. Hence a homothety with center P that maps Y to D, takes circle o_1 to a circle o_1' tangent to d at D. Hence $o_1' = o_4$.

Similarly, let g be a perspective transformation with center P and axis BS mapping the ellipse to circle o_2. Let $C = g^{-1}(X)$ and let c be the tangent line to the ellipse at point C. Since lines x, c, BS are concurrent and $x \parallel SB$, it follows that $x \parallel c$. Hence a homothety with center P that maps X to C, takes circle o_2 to a circle o_2' tangent to the ellipse at point C. Therefore, $o_2' = o_3$.

Since $SA = SB$, $c \parallel SB$ and $d \parallel SA$, it follows from Lemma 2.2 that $ABDC$ is a rectangle. This completes the proof of Problem 2.

For Problem 1 observe that since $ABDC$ is a rectangle, $MN \parallel CD$. So there is a homothety with center P that maps points M and N to points C and D, respectively. This homothety maps circles o_1, o_2 to circles o_3, o_4, respectively. Thus $r_1/r_2 = r_3/r_4$, and consequently $r_1 r_4 = r_2 r_3$. This completes the proof of Problem 1.

4. Proof of the correctness of Taxia Limneou's construction

The correctness of the construction immediately follows from the following statement.

Assume the ellipse with center O is tangent at points B and C to the arms of the angle with vertex A. Without loss of generality, assume that $AB < AC$. The circle ω is inscribed in the given angle, tangent to the ellipse at point C and to the second arm of the angle at point F. Let K be the point of intersection of the shorter arc CF of circle ω with the longer arc BC of the ellipse. Moreover, let o be the circle inscribed in the given angle externally tangent to the given ellipse at point T. Then, the points O, T, and the midpoint of segment CK are collinear.

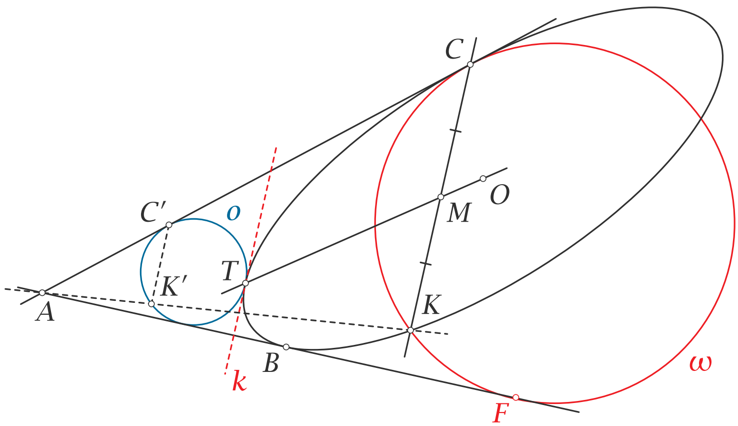

Fig. 8

Indeed: Let k be the common tangent of the ellipse and the circle o at point T, and consider a perspective transformation centered at A with axis k mapping the ellipse to the circle o. Let C' and K' be the images of points C and K, respectively, under this transformation. Thus, we conclude that the lines CK and $C'K'$ intersect on the line k. But points C' and K' are also images of points C and K, respectively, under a homothety centered at A. Therefore, lines CK, $C'K'$, and k are parallel.

Let V be the direction of these lines (a point at infinity). Then, points O, M, and T lie on the polar line v of point V with respect to the ellipse, hence they are collinear. This completes the proof.

Using this reasoning, one can establish a more general property where the circle ω and the ellipse are not tangent.

Specifically, let ω be a circle tangent to the arms of the angle at points D and F such that $AB < AF = AD < AC$. Point K is the intersection point of the shorter arc DF of circle ω with the longer arc BC of the ellipse, and point L is the intersection point of the longer arc DF of circle ω with the shorter arc BC of the ellipse. Let o be the circle inscribed in the given angle, tangent to one of its arms at point E, such that $AE < AB$ or $AE > AC$. Let P and Q be the intersection points of circle o and the ellipse. Then, the lines KL and PQ are parallel, and consequently, the point O and the midpoints of segments KL and PQ are collinear.

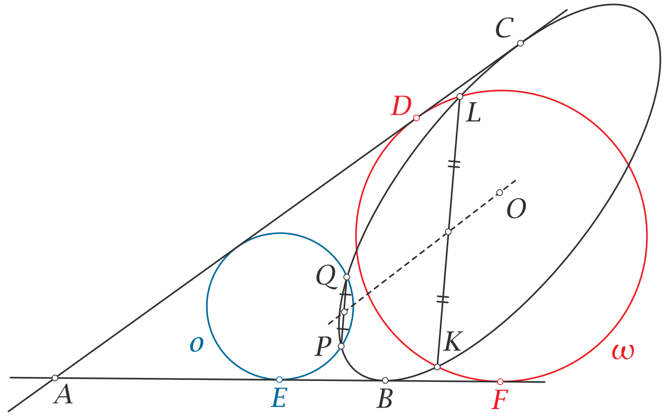

Fig. 9

5. Final remarks

Our proofs use only projective properties of an ellipse and can be easily adapted for any conic. Therefore, Problems 1 and 2 remain valid for hyperbolas as well, as illustrated in Figure 10. In particular, the equality $r_1 \cdot r_4 = r_2 \cdot r_3$ holds for hyperbolas, as well.

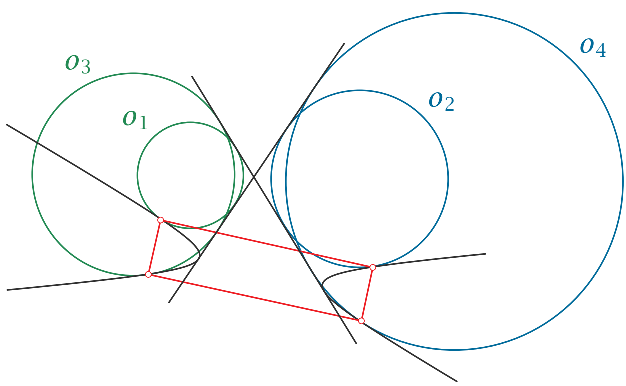

Fig. 10

For parabolas, the situation is slightly different because in that case, only two circles inscribed in the angle and tangent to the parabola can be constructed. In this case, the conclusion of Problem 2 can be formulated as follows: the points of tangency of the parabola with the circles are symmetric with respect to the axis of symmetry of the parabola (see Figure 11).

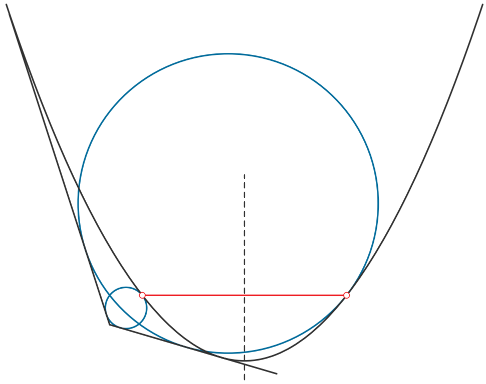

Fig. 11

References

[1] Chi Nguyen Chuong, *Solution to Problem 11222*, Romantics of Geometry Facebook Group (November 27, 2022).

[2] Taxia Limneou, *Problem 14464*, Romantics of Geometry Facebook Group (December 18, 2023).

[3] Waldemar Pompe, *Problem 11222*, Romantics of Geometry Facebook Group (November 26, 2022).

[4] Stanley Rabinowitz, *Problem 11069*, Romantics of Geometry Facebook Group (October 29, 2022).

[5] J. Marshall Unger, *Solutions of Two Japanese Ellipse Problems*, Forum Geometricorum **16** (2016), p. 85–94.