

Junior Problems

J667. Prove that the sum of squares of two twin primes can be written as the sum of squares of three positive integers.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J668. Let ABC be a triangle with incenter I and semiperimeter s . A line through I intersects AB and AC in P and Q , respectively, such that

$$\frac{PB}{PA} = \frac{AC}{BC} \quad \text{and} \quad \frac{QC}{QA} = \frac{AB}{BC}.$$

Prove that $bc = s\sqrt{2} \cdot AI$.

Proposed by Mihaela Berindeanu, Bucharest, România

J669. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ca}{abc} \geq \frac{3(a+b+c)^2}{a^3+b^3+c^3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J670. Find all n -tuples $(x_1, x_2, \dots, x_n)$ of real numbers in the interval $(0, 2)$ such that

$$2x_1 - \frac{x_2}{2-x_2} = 2x_2 - \frac{x_3}{2-x_3} = \dots = 2x_n - \frac{x_1}{2-x_1} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J671. Let $a > 1$ be a real number. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) = f(x)f(y) - a \sin x \sin y,$$

for all $x, y \in \mathbb{R}$.

Proposed by Mihaela Berindeanu, Bucharest, România

J672. Let $ABCD$ be a convex quadrilateral such that

$$(AB - BC)^2 + (CD - DA)^2 = 2(AB \cdot AD - AC \cdot BD + BC \cdot CD).$$

Prove that the statements

- (a) $ABCD$ is cyclic
- (b) $ABCD$ is circumscribable

are equivalent.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior Problems

S667. Prove that the positive difference of the cubes of two twin primes can be written as the sum of squares of three positive integers.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

S668. Let $a \geq b \geq c \geq d \geq 0$ such that $ab + bc + cd + da = 4$. Prove that

$$\frac{4}{3} \leq \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} + \frac{1}{d+2} \leq \frac{3}{2}.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România

S669. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sum_{cyc} bc\sqrt{2a+b+c} \geq \sqrt{\frac{9abc}{4} + \frac{3(a+b+c)}{4} + \frac{1}{a+b+c}}.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

S670. Find the least positive integer z for which the equation

$$x^4 - 7(x^2y^2 + z^2 + 1) + y^4 = 2024$$

has solutions in positive integers and solve it for that value of z .

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S671. Let ABC be a triangle with circumradius R and let Ω be its Brocard point such that $\angle\Omega AB = \angle\Omega BC = \angle\Omega CA = \alpha$. Prove that if $\cot\alpha = 5\cot A$ and $BC = a$ then

$$\cos A \geq \frac{1}{3} \quad \text{and} \quad a \leq 3R\sqrt{\frac{2}{5}}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

S672. Let $P(x)$ be a polynomial of degree 10 with integer coefficients such that it has only real roots and $P(x)$ divides the polynomial $P(P(x) + 2x - 4)$. Find the value of

$$\frac{P(2024)}{P(206)}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Undergraduate Problems

U667. Solve in complex numbers the system of equations

$$\begin{aligned}x^2 + xy + xz + xt &= -5 \\xy + y + xz + t &= 1 \\xz + y + z + xt &= -2 \\xt + xy + z + t &= 1.\end{aligned}$$

Proposed by Mircea Becheanu, Canada

U668. Evaluate

$$\int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U669. Evaluate

$$\lim_{n \rightarrow \infty} \sin \left(\frac{n!(1+n)^{n^2} \sqrt{e}}{n^{n+n^2}} \right)^2.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

U670. Let $k > 1$ be an integer. Evaluate

$$\lim_{n \rightarrow \infty} n^2 \int_0^1 \frac{\frac{x}{n} \sin \frac{x}{n} + \cos \frac{x}{n}}{\left(\frac{x}{n}\right)^k + 1} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, România

U671. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function such that

$$\int_0^1 x(1-x)^2 f(x) dx = 0.$$

Prove that there exists $\alpha \in (0, 1)$ such that

$$\int_0^\alpha x^2 f(x) dx = \alpha \int_0^\alpha x f(x) dx.$$

Proposed by Florin Stănescu, Găești, România

U672. Let $p > 1$ be an integer. Evaluate

$$\lim_{n \rightarrow \infty} \left[n^2 \sin \left(2\pi n \left(1 + \frac{p}{n} \right)^{\frac{1}{p}} \right) - n\pi(1-p) \right].$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma

Olympiad Problems

O667. Prove that for any positive real numbers a, b, c ,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{25}{6}.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

O668. How many ordered 4-tuples (a, b, c, d) of pairwise distinct numbers $a, b, c, d \in \{1, 2, \dots, 17\}$ are there such that $a - b + c - d$ is divisible by 17 ?

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O669. Let $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$. Prove that

$$a_1 + a_2 + \dots + a_n + \frac{k(a_1 - a_2)^2}{a_1 + a_2} \leq \sqrt{n(a_1^2 + a_2^2 + \dots + a_n^2)},$$

where $k = \frac{\sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}}.$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România

O670. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} + \frac{45}{a+b+c} \geq \frac{7}{a} + \frac{7}{b} + \frac{7}{c}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O671. On chord AB of circle ω with radius r , consider the points K and M in the order A, K, M, B . Chord CD is drawn through point K such that $\angle AKD$ is acute. Chord EF is drawn through point M such that $\angle BME$ is acute. The chords CD and EF meet inside ω at point L . In the curvilinear regions ADK , BEM , CFL , circles of radii r_2, r_3, r_4 , respectively, are inscribed. The inradius of triangle KLM is equal to r_1 . Prove that $2r_1 + r_2 + r_3 + r_4 \leq r$.

Proposed by Waldemar Pompe, Warsaw, Poland

O672. Let $a \geq b \geq c \geq 1 \geq d \geq e > 0$ such that $ab + bc + cd + de + ea = 5$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + 10 \geq 3(a + b + c + d + e).$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România