

Junior Problems

J673. Let a be a positive real number. Prove that $(a - 1)\sqrt{a} > 1$ if and only if $a^2(\sqrt{a} - 1) > 1$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J674. Let Γ be a circle of center O , radius 2, and P a point on it. On the tangent to Γ at P , point Q is taken such that $PQ = 2r$, $1 < r < 2$, and a circle γ is drawn with diameter PQ . The circles Γ and γ intersect again at M . We denote $\angle MOP = \alpha$. It is given that $r \sin \alpha = 1$. Without using calculus, find the common area inside these two circles.

Proposed by Mircea Becheanu, Canada

J675. A table of integers is a rectangular table with k rows and l columns in which all numbers $1, 2, \dots, kl$ are written in order starting from the upper-left corner. Let n be a positive integer and let p be a prime. We say that n is p -tabular if there is a table of integers which contains n in the middle row and $n + p + 1$ in the last row, just to the right of the column of n . Find all 2027-tabular integers.

Proposed by Alessandro Ventulo, Milan, Italy

J676. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + y) + f(f(f(x + y))) \leq 2(f(x) - y),$$

for all $x, y \in \mathbb{R}$.

Proposed by Florin Stănescu, Găești, România

J677. Let a, b, c be nonzero real numbers, not all equal, satisfying the system of equations

$$a^3 + b^3 - 2abc = b - c, \quad b^3 + c^3 - 2abc = c - a, \quad c^3 + a^3 - 2abc = a - b.$$

Evaluate

$$\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J678. Let ABC be a triangle with circumcenter O , incenter I , and such that $2AC = AB + BC$. Let M be the midpoint of AB and S the midpoint of OB . Prove that $SI = SM$.

Proposed by Mihaela Berindeanu, Bucharest, România

Senior Problems

S673. Points D and E lie on the sides BC and CA , respectively, of an equilateral triangle ABC with segments BD and AE of equal lengths. Inside triangle ABC , a point P is chosen such that the sum of the angles APE and BPD is 180° . Prove that $\angle PAB = \angle DPC$.

Proposed by Waldemar Pompe, Warsaw, Poland

S674. Solve in rational numbers the system of equations

$$(x-2)^2 + (y+1)^3 = 2024, \quad (x+2)^2 + y^3 = 2025.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S675. Find all pairs of nonnegative integers (m, n) such that $2^m \cdot 3^n + 1$ is a perfect square.

Proposed by Alessandro Ventulo, Milan, Italy

S676. Let ABC be a triangle with orthocenter H and center of gravity G . Let S and M be the midpoints of the segments AH and AG , respectively. Prove that if $\sin C \sin(B-A) = \sin B \sin(A-C)$ then SM is perpendicular on AG .

Proposed by Mihaela Berindeanu, Bucharest, România

S677. Let x, y, z be real numbers such that $xy + yz + zx = 1$. Prove that

$$\frac{xy}{(x+y)\sqrt{z^2+1}} + \frac{yz}{(y+z)\sqrt{x^2+1}} + \frac{zx}{(z+x)\sqrt{y^2+1}} \leq \frac{3}{4}.$$

Proposed by Iancu-Ioan Sandea, Cluj-Napoca, România

S678. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{(b+c)^2} + \frac{b}{(c+d)^2} + \frac{c}{(d+a)^2} + \frac{d}{(a+b)^2} \geq \frac{4}{a+b+c+d}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Undergraduate Problems

U673. Let $A_1A_2 \dots A_n$ be a cyclic polygon inscribed in a circle of center O and radius R , and let G be its centroid. Prove that the inequality

$$MA_1 + MA_2 + \dots + MA_n \geq nR$$

holds for every point M in the plane of the polygon if and only if $O = G$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ştefan Marinescu, Hunedoara, România

U674. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{32a^2 - 48a + 25} + \frac{1}{32b^2 - 48b + 25} + \frac{1}{32c^2 - 48c + 25} \leq \frac{1}{3}.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploieşti, România

U675. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \ln \left(1 + \frac{i}{n} \right) \ln \left(1 + \frac{j}{n} \right).$$

Proposed by Mihaela Berindeanu, Bucharest, România

U676. Let $a > 1$ be an integer nondivisible by 4. Prove that there are infinitely many primes p of the form $4k - 1$ such that $p|a^d - 1$, for some $d < \frac{p-1}{2}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U677. Evaluate the integral:

$$\int_0^{\infty} \frac{\ln^2 x}{(x+1)(x^2+1)} dx.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, România

U678. Evaluate

$$\int_0^{\infty} \frac{(\cos(ax) - \cos(bx))^2}{x^3} dx.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Olympiad Problems

O673. Let $p_1 < p_2 < \dots < p_s$ be the prime numbers, $P = \prod_{i=1}^s p_i$ and for every $1 \leq i \leq s$, denote $P_i = P/p_i$.

For every $i, 1 \leq i \leq s$, let P'_i be an integer such that $P_i P'_i \equiv 1 \pmod{p_i}$. Let $x_1, \dots, x_n$ be arbitrary integers, where $n \geq p_s$. For every $j, 1 \leq j \leq n$, let $M^{(j)}$ be the number of the sums of j among the integers $x_1, \dots, x_n$ which are not divisible by any of $p_1, \dots, p_s$ and let

$$M = M^{(1)} - M^{(2)} + \dots + (-1)^{n-1} M^{(n)}.$$

For every $i, 1 \leq i \leq s$, and every $j, 1 \leq j \leq n$, let $N_i^{(j)}$ be the number of the sums of j among $x_1, \dots, x_n$ which are not divisible by p_i , but are divisible by at least one $p_k, 1 \leq k \leq s$ and $k \neq i$ and let

$$N_i = N_i^{(1)} - N_i^{(2)} + \dots + (-1)^{n-1} N_i^{(n)}.$$

Prove that

$$M + P_1 P'_1 N_1 + \dots + P_s P'_s N_s \equiv 0 \pmod{p_1 \dots p_s}.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O674. Prove that 8 is the greatest value of the positive constant k for which

$$a^2 + b^2 + c^2 + d^2 - 4 \geq k(a + b + c + d - 4)$$

whenever $a = b \geq c \geq d \geq 0$ such that $ab + bc + cd + da = 4$.

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România

O675. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{2b^2 + ca}{a^2 + bc + ca}} + \sqrt{\frac{2c^2 + ab}{b^2 + ca + ab}} + \sqrt{\frac{2a^2 + bc}{c^2 + ab + bc}} \geq 3.$$

Proposed by An Zhenping, Xianyang Normal University, China

O676. Let x, y, z be distinct non-zero integers such that

$$\frac{(x-1)^2}{z} + \frac{(y-1)^2}{x} + \frac{(z-1)^2}{y} = \frac{(x-1)^2}{y} + \frac{(y-1)^2}{z} + \frac{(z-1)^2}{x}.$$

Find the minimum possible value of $|64x + 19y + 4z|$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O677. Prove that for any positive real numbers a, b, c

$$\frac{1}{4}(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \sqrt{\frac{2ab}{a^2 + b^2}} + \sqrt{\frac{2bc}{b^2 + c^2}} + \sqrt{\frac{2ca}{c^2 + a^2}} \geq \frac{21}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O678. Let $a_1, \dots, a_k$ be positive integers and let $f(x)$ be a nonconstant polynomial with integer coefficients. Prove that there exists an infinite set P of primes such that for any $p \in P$, $f(n) + a_1^n + \dots + a_k^n$ is divisible by p for infinitely many n .

Proposed by Titu Andreescu, USA, and Marian Tetiva, România