

Junior problems

J673. Let a be a positive real number. Prove that $(a-1)\sqrt{a} > 1$ if and only if $a^2(\sqrt{a}-1) > 1$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Theo Koupelis, Clark College, Washington, USA

The conclusion follows from the identity $(x^3 - x - 1)(x^2 - x + 1) = x^5 - x^4 - 1$ for $x = \sqrt{a}$. Indeed, $x^3 - x - 1 > 0$ if and only if $x^5 - x^4 - 1 > 0$, as $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$ for all real numbers x .

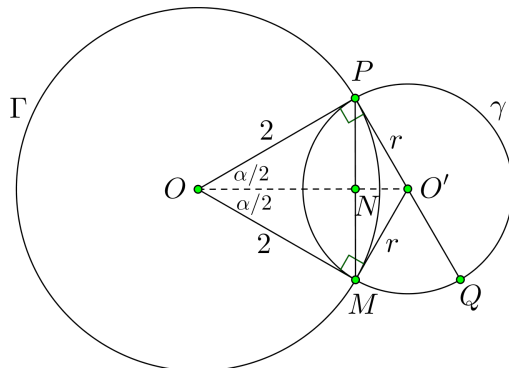
Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, California, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundares H. R., Shivamogga, Karnataka, India; Zhu Tongle, Hebei Province, China; Daniel Pascuas, Barcelona, Spain; Diar Gashi, Unversity of Primorska, Slovenia; Prajnanaswaroop S, Bangalore, Karnataka, India; Prasanna Pawar, AMMOC, Bengaluru, India; Srijan Sundar, Texas, USA; William Han, Horace Mann School, Bronx, NY, USA.

J674. Let Γ be a circle of center O , radius 2, and P a point on it. On the tangent to Γ at P , point Q is taken such that $PQ = 2r$, $1 < r < 2$, and a circle γ is drawn with diameter PQ . The circles Γ and γ intersect again at M . We denote $\angle MOP = \alpha$. It is given that $r \sin \alpha = 1$. Without using calculus, find the common area inside these two circles.

Proposed by Mircea Becheanu, Canada

Solution by Kousik Sett, India

Let O' be the center of circle γ . Join O', M . Since $PO = MO$ and $PO' = MO'$, $POMO'$ is a kite. Then OO' is perpendicularly bisects PM . Let $N = OO' \cap PM$. So $\angle O'OP = \angle O'OM = \alpha/2$. Since $OP \perp O'P$ we have $\angle OO'P = \pi/2 - \alpha/2$.



Considering the length PN from two right triangles PON and $PO'N$, we obtain

$$\begin{aligned} 2 \sin \frac{\alpha}{2} &= r \cos \frac{\alpha}{2} \\ \Rightarrow 4 \sin^2 \frac{\alpha}{2} &= r \cdot 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = r \sin \alpha = 1. \\ \Rightarrow 2(1 - \cos \alpha) &= 1 \Rightarrow \cos \alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}. \end{aligned}$$

Since $r = 2 \tan \frac{\alpha}{2}$ and $\alpha = \frac{\pi}{3}$, we have $r = \frac{2}{\sqrt{3}}$.

The common area of inside circles Γ and γ

= (Sector area of POM - area of triangle POM) + (Sector area of $PO'M$ - area of triangle $PO'M$).

So the required area

$$\begin{aligned} &= \left(\frac{1}{2} \cdot 2^2 \cdot \alpha - \frac{1}{2} \cdot 2^2 \cdot \sin \alpha \right) + \left(\frac{1}{2} \cdot r^2 \cdot (\pi - \alpha) - \frac{1}{2} \cdot r^2 \cdot \sin(\pi - \alpha) \right) \\ &= \frac{1}{2} [4\alpha - 4 \sin \alpha + (\pi - \alpha)r^2 - r^2 \sin \alpha] \\ &= \frac{1}{2} \left[\frac{4\pi}{3} - 4 \cdot \frac{\sqrt{3}}{2} + \frac{2\pi}{3} \cdot \frac{4}{3} - \frac{2}{\sqrt{3}} \right] = 2 \left[\frac{5\pi}{9} - \frac{2}{\sqrt{3}} \right]. \end{aligned}$$

Also solved by Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Srijan Sundar, Texas, USA.

J675. A table of integers is a rectangular table with k rows and ℓ columns in which all numbers $1, 2, \dots, k\ell$ are written in order starting from the upper-left corner. Let n be a positive integer and let p be a prime. We say that n is p -tabular if there is a table of integers which contains n in the middle row and $n + p + 1$ in the last row, just to the right of the column of n . Find all 2027-tabular integers.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

Since n is in the middle row, there are an odd number of rows. Let $k = 2j + 1$ be the number of rows, where $j \in \mathbb{N}^*$ and let ℓ be the number of columns, where $\ell \in \mathbb{N}^*$. Since between n and $n + p + 1$ there are $j\ell$ numbers, then $(n + p + 1) - j\ell - 1 = n$, i.e. $j\ell = p$. Since the definition requires $\ell \geq 2$, then $j = 1$ and $\ell = p$. So, we have a $3 \times p$ table and n is in the second row and it cannot be in the last box, which means that $p + 1 \leq n \leq 2p - 1$. If $p = 2027$, we get $n \in \{2028, 2029, \dots, 4053\}$.

Also solved by Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Srijan Sundar, Texas, USA.

J676. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) + f(f(f(x+y))) \leq 2(f(x) - y),$$

for all $x, y \in \mathbb{R}$.

Proposed by Florin Stănescu, Găești, Romania

Solution by Diar Gashi, University of Primorska, Slovenia

Let $P(x, y)$ represent the substitutions we take for x and y . Let $f^n(x) := \underbrace{f(\dots(f(x)\dots))}_n$.

Notice that

$$P(x, 0) : f(x) + f^3(x) \leq 2f(x) \implies f^3(x) \leq f(x). \quad (1)$$

So,

$$\begin{aligned} 2f^3(x+y) &= f^3(x+y) + f^3(x+y) \stackrel{(1)}{\leq} f(x+y) + f^3(x+y) \stackrel{P(x,y)}{\leq} 2(f(x) - y) \\ &\implies f^3(x+y) \leq f(x) - y \end{aligned} \quad (2)$$

$$f^3(x) = f^3(0+x) \stackrel{(2)}{\leq} f(0) - x \implies f^3(x) \leq f(0) - x \quad (3)$$

Also taking $x \rightarrow 0$ and $y \rightarrow f(0)$ in (2) we get that $f^4(0) \leq 0$. So,

$$f^3(0) = f^3(x + (-x)) \stackrel{(2)}{\leq} f(x) + x \implies f^3(0) - x \leq f(x). \quad (4)$$

Moreover, $x \rightarrow f^3(0)$ in (4) gives $0 \leq f^4(0)$. So, $f^4(0) = 0$ and using this we get that

$$P(0, f(0)) : f^2(0) \leq 0$$

and $x \rightarrow f^2(0)$ in (4) gives that $0 \leq f^2(0)$. Hence, $f^2(0) = 0$ and therefore $f^3(0) = f(0)$.

So we have that

$$f(0) - f^2(x) = f^3(0) - f^2(x) \stackrel{(4)}{\leq} f^3(x) \stackrel{(3)}{\leq} f(0) - x \implies x \leq f^2(x) \quad (5)$$

But, (5) implies that $f(x) \leq f^3(x)$ and this combined with (1) implies that $f(x) = f^3(x)$.

So,

$$f(0) - x = f^3(0) - x \stackrel{(4)}{\leq} f(x) = f^3(x) \stackrel{(3)}{\leq} f(0) - x.$$

Therefore $f(x) = f(0) - x$ for all $x \in \mathbb{R}$ and it is easy to see that $f(x) = c - x$, where $c \in \mathbb{R}$, satisfies the given condition.

Also solved by Theo Koupelis, Clark College, Washington, USA.

J677. Let a, b, c be nonzero real numbers, not all equal, satisfying the system of equations

$$a^3 + b^3 - 2abc = b - c, \quad b^3 + c^3 - 2abc = c - a, \quad c^3 + a^3 - 2abc = a - b.$$

Evaluate

$$\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Adding the three equations yields

$$2(a^3 + b^3 + c^3 - 3abc) = 0,$$

hence

$$0 = c^3 - abc + b - c = a^3 - abc + c - a = b^3 - abc + a - b.$$

It follows that

$$\frac{c^2}{b} - a + \frac{1}{c} - \frac{1}{b} = \frac{a^2}{c} - b + \frac{1}{a} - \frac{1}{c} = \frac{b^2}{a} - c + \frac{1}{b} - \frac{1}{a} = 0.$$

Summing up, and taking into account that $a^3 + b^3 + c^3 - 3abc = 0$ with a, b, c not all equal implies $a + b + c = 0$, we get

$$\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} = 0.$$

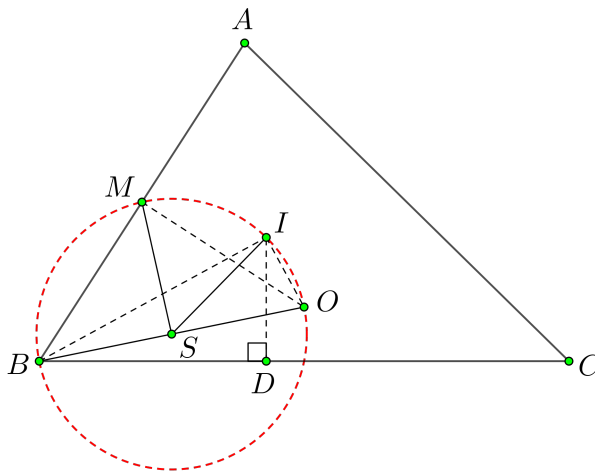
Also solved by Kousik Sett, India; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Srijan Sundar, Texas, USA.

J678. Let ABC be a triangle with circumcenter O , incenter I , and such that $2AC = AB + BC$. Let M be the midpoint of AB and S the midpoint of OB . Prove that $SI = SM$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Kousik Sett, India

Since $2AC = AB + BC$, we have $2b = c + a$. So $3b = 2s$ where s is the semiperimeter of triangle ABC . Let D be the projection of I on BC . Therefore, $ID = r$ which is the inradius of triangle ABC . Also, it is known that $BD = s - b$. Let R be the circumradius and Δ be the area of triangle ABC .



Therefore,

$$\begin{aligned} BI^2 &= r^2 + (s - b)^2 = \frac{\Delta^2}{s^2} = \frac{s(s - a)(s - b)(s - c)}{s^2} + (s - b)^2 \\ &= (s - b) \left[\frac{(s - a)(s - c)}{s} + (s - b) \right] \\ &= (s - b) \left[s - (a + c) + \frac{ac}{s} + (s - b) \right] \\ &= (s - b) \cdot \frac{ac}{s} \\ &= (2s - 2b) \cdot \frac{ac}{2s} = (3b - 2b) \cdot \frac{ac}{3b} = \frac{ac}{3}. \end{aligned}$$

Since M is the midpoint of AB and O is the circumcenter of triangle ABC , we infer that MO is perpendicular to AB , i.e., $\angle BMO = 90^\circ$. So there exists a circle passing through B , O , and M and OB is the diameter of the circle. Since S is the midpoint of OB , the center of the circle is S . We will prove I lies on this circle. So it is enough to prove $\angle BIO = 90^\circ$.

Since $BO = R$ and from the well-known fact $IO = \sqrt{R^2 - 2Rr}$, we have

$$BO^2 - IO^2 = R^2 - (R^2 - 2Rr) = 2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{2s} = \frac{abc}{3b} = \frac{ac}{3} = BI^2.$$

So $BO^2 = BI^2 + IO^2$ and hence $\angle BIO = 90^\circ$. Therefore, I lies on the circle. Thus $SI = SM$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Miquel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Thanh Nam Nguyen, Tu Son Secondary School, Vietnam; Theo Koupelis, Clark College, Washington, USA; William Han, Horace Mann School, Bronx, NY, USA.

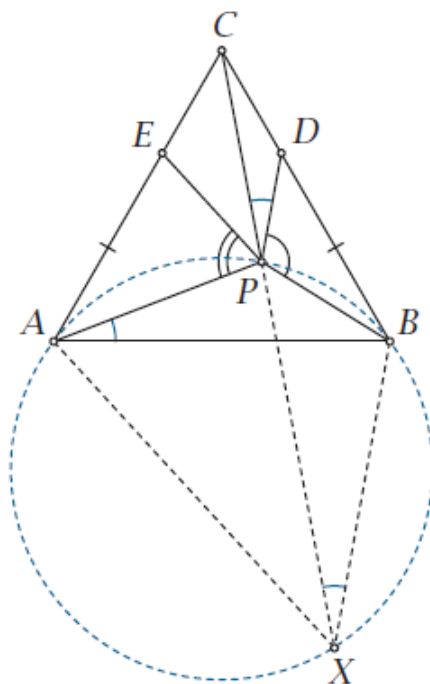
Senior problems

S673. Points D and E lie on the sides BC and CA , respectively, of an equilateral triangle ABC with segments BD and AE of equal lengths. Inside triangle ABC , a point P is chosen such that the sum of the angles APE and BPD is 180° . Prove that $\angle PAB = \angle DPC$.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by the author

Consider a homothety centered at C with scale factor $k = AC/EC = BC/DC$. This homothety maps points D and E to points B and A , respectively, and also maps point P to a certain point X . Therefore, point X lies on the ray CP .



The homothety maps line k to a line k' parallel to k . Thus, $AX \parallel EP$ and $BX \parallel DP$, implying $\angle PXB = \angle CPD$ and $\angle AXB = \angle EPD$. Since

$$\angle AXB + \angle APB = \angle EPD + \angle APB = 180^\circ,$$

quadrilateral $AXBP$ is cyclic. Hence $\angle PAB = \angle PXB = \angle CPD$.

Also solved by Kousik Sett, India; Theo Koupelis, Clark College, Washington, USA.

S674. Solve in rational numbers the system of equations

$$(x - 2)^2 + (y + 1)^3 = 2024, \quad (x + 2)^2 + y^3 = 2025.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Subtracting the first equation from the second yields $8x = 3y^2 + 3y + 2$ and substituting into $(8x + 16)^2 + 64y^3 = 64 \cdot 2025$ leads to the equation

$$9y^4 + 82y^3 + 117y^2 + 108y - 129276 = 0.$$

We have $129276 = 2^2 \cdot 3^5 \cdot 7 \cdot 19$ and one solution is $y = 9$. With synthetic division, the other rational solutions must be solutions of the equation

$$9y^3 + 163y^2 + 1584y + 14364 = 0.$$

Then from $14364 = 2^2 \cdot 3^3 \cdot 7 \cdot 19$ and the rational root theorem, it follows that there are no other rational solutions. Hence the only solution to the system is $(x, y) = (34, 9)$.

Also solved by Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Prajnanaswaroop S, Bangalore, Karnataka, India.

S675. Find all pairs of nonnegative integers (m, n) such that $2^m \cdot 3^n + 1$ is a perfect square.

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

If $m = 0$, then $3^n + 1 = t^2$, where $t \in \mathbb{N}$. We get $3^n = (t-1)(t+1)$. It follows that $t-1$ and $t+1$ are both odd, so $(t-1, t+1) = 1$ and we get $t-1 = 1$ and $t+1 = 3^n$, which gives $t = 2$ and $n = 1$. If $n = 0$, then $2^m + 1 = t^2$, where $t \in \mathbb{N}$. We get $2^m = (t-1)(t+1)$. It follows that $t-1$ and $t+1$ are both even and we get $t-1 = 2a$ and $t+1 = 2(a+1)$, where $a \in \mathbb{N}$. Hence, $2^m = 2^2 a(a+1)$. If $a > 1$, then $a(a+1)$ contains an odd factor greater than 1, contradiction. So, $a = 1$ and $2^m = 2^3$, which gives $m = 3$. Now, assume that $m \geq 1$ and $n \geq 1$. Let $t \in \mathbb{N}$ such that $2^m \cdot 3^n + 1 = t^2$, i.e.

$$2^m \cdot 3^n = (t-1)(t+1).$$

Then, $t-1$ and $t+1$ are both even and one of them is divisible by 4, which gives $m \geq 3$. Moreover, $t-1 = 2a$ and $t+1 = 2(a+1)$, where $a \in \mathbb{N}$, so

$$2^{m-2} \cdot 3^n = a(a+1).$$

As $(a, a+1) = 1$, then $a = 1, a+1 = 2^{m-2} \cdot 3^n$ or $a = 2^{m-2}, a+1 = 3^n$ or $a = 3^n, a+1 = 2^{m-2}$. The first case is impossible for $m, n \geq 1$.

- (i) Let $a = 2^{m-2}$ and $a+1 = 3^n$. Then, $3^n - 2^{m-2} = 1$. If $m = 3$, then $3^n = 3$, which gives $n = 1$. If $m \geq 4$, then $3^n \equiv 1 \pmod{4}$, which implies that n is even. So, $n = 2k$, where $k \in \mathbb{N}$ and $3^{2k} - 2^{m-2} = 1$, i.e. $(3^k - 1)(3^k + 1) = 2^{m-2}$. We get

$$\begin{aligned} 3^k - 1 &= 2^\alpha \\ 3^k + 1 &= 2^\beta, \end{aligned}$$

where $\alpha + \beta = m - 2$. Clearly, $\beta \geq \alpha$ and subtracting these two equations, we obtain $2 = 2^\beta - 2^\alpha = 2^\alpha(2^{\beta-\alpha} - 1)$, which gives $\alpha = 1$ and $\beta = 2$, i.e. $m = 5$ and $n = 2$. So, $(m, n) \in \{(3, 1), (5, 2)\}$.

- (ii) Let $a = 3^n$ and $a+1 = 2^{m-2}$. Then, $2^{m-2} - 3^n = 1$. If $n = 1$, then $2^{m-2} = 4$, which gives $m = 4$. If $n \geq 2$, then $2^{m-2} \equiv 1 \pmod{9}$, which implies that $m - 2 = 6k$, where $k \in \mathbb{N}^*$. Hence, $2^{6k} - 3^n = 1$, i.e. $(2^{3k} - 1)(2^{3k} + 1) = 3^n$. Since $(2^{3k} - 1, 2^{3k} + 1) = 1$, then $2^{3k} - 1 = 1$ and $2^{3k} + 1 = 3^n$. We get $2^{3k} = 2$, contradiction. So, $(m, n) = (4, 1)$.

In conclusion, $(m, n) \in \{(0, 1), (3, 0), (3, 1), (5, 2), (4, 1)\}$.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Let m, n be nonnegative integers satisfying that $2^m \cdot 3^n + 1$ is a perfect square, i.e. $2^m \cdot 3^n + 1 = a^2$, where a is a positive integer. It follows that $2^m \cdot 3^n = a^2 - 1 = (a-1)(a+1)$. Now there are two possibilities in terms of the parity of a :

Case 1. a is even. Then $a = 2k$, for some positive integer k , so $2^m \cdot 3^n = (2k-1)(2k+1)$ is odd, and therefore $m = 0$, $2k-1 = 3^{n_0}$, and $2k+1 = 3^{n_1}$, where the n_j 's are integers such that $0 \leq n_0 < n_1$. It follows that $3^{n_1} - 3^{n_0} = 2$, and so $n_0 = 0$ and $n_1 = 1$, i.e. $a = 2$ and $(m, n) = (0, 1)$.

Case 2. a is odd. Then $a = 2k-1$, where k is a positive integer, so $2^m \cdot 3^n = 4(k-1)k$. It follows that $m, k \geq 2$ and $2^{m-2} \cdot 3^n = k(k-1)$. Since k and $k-1$ are coprime, we have that either $k-1 = 3^n$ and $k = 2^{m-2}$, or $k-1 = 2^{m-2}$ and $k = 3^n$. Thus there are two subcases:

Subcase 2.1. $2^{m-2} = 3^n + 1$.

Since $3^2 = 81 \equiv 1 \pmod{4}$, we have that if n is even then $3^n \equiv 1 \pmod{4}$, so $2^{m-2} \equiv 2 \pmod{4}$, which means that $m-2 = 1$, and hence $(m, n) = (3, 0)$ and $a = 3$.

Since $3^2 = 81 \equiv 1 \pmod{8}$, we have that if n is odd then $3^n \equiv 3 \pmod{8}$, so $2^{m-2} \equiv 4 \pmod{8}$, which means that $m-2 = 2$, and hence $(m, n) = (4, 1)$ and $a = 7$.

Subcase 2.2. $2^{m-2} = 3^n - 1$. Since $3^n = 2^{m-2} + 1 \geq 2$, n must be positive.

As above, if n is odd then $2^{m-2} \equiv 2 \pmod{8}$, which means that $m-2 = 1$, and hence $(m, n) = (3, 1)$

and $a = 5$. On the other hand, if n is even, then $n = 2\ell$, for some positive integer ℓ , so $2^{m-2} = 3^n - 1 = (3^\ell + 1)(3^\ell - 1)$. It follows that $3^\ell + 1$ and $3^\ell - 1$ are powers of 2 whose difference is equal to 2. Therefore $3^\ell + 1 = 4$ and $3^\ell - 1 = 2$, and so $\ell = 1$. Hence $(m, n) = (5, 2)$ and $a = 17$.

Summarizing, the only pairs of nonnegative integers (m, n) such that $2^m \cdot 3^n + 1$ is a perfect square are $(0, 1)$, $(3, 0)$, $(3, 1)$, $(4, 1)$, and $(5, 2)$.

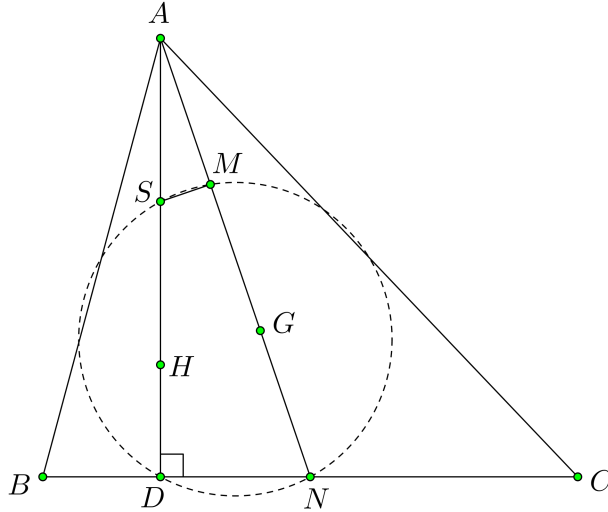
Also solved by Daniel Văcaru, National Economic College “Maria Teiuleanu”, Pitești, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Diar Gashi, University of Primorska, Slovenia; Nihad Hashimov, AMMOC, Azerbaijan; Rama Lakshmi Chunduri - University of California, San Diego; Srijan Sundar, Texas, USA; William Han, Horace Mann School, Bronx, NY, USA.

S676. Let ABC be a triangle with orthocenter H and center of gravity G . Let S and M be the midpoints of the segments AH and AG , respectively. Prove that if $\sin C \sin(B - A) = \sin B \sin(A - C)$ then SM is perpendicular on AG .

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Kousik Sett, India

Let $D = AH \cap BC$ and $N = AG \cap BC$. Then $AD \perp BC$. It is known that the nine-point circle of triangle ABC passes through S , D , and N . Let R be the circumradius of triangle ABC .



By Apollonius' Theorem, we have

$$AB^2 + AC^2 = 2(BN^2 + AN^2) \Rightarrow AN = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}.$$

Since M is the midpoint of AG and S is the midpoint of AH , we have

$$AM = \frac{1}{3}AN \text{ and } AS = \frac{1}{2}AH = R \cos A. \text{ Also, } AD = \frac{bc}{2R}.$$

Since $\sin C = \sin(B + A)$ and $\sin B = \sin(A + C)$, the given condition

$$\sin C \sin(B - A) = \sin B \sin(A - C) \sin B,$$

is equivalent to

$$\begin{aligned} \sin(B + A) \sin(B - A) &= \sin(A + C) \sin(A - C) \\ \Rightarrow \sin^2 B - \sin^2 A &= \sin^2 A - \sin^2 C \\ \Rightarrow b^2 - a^2 &= a^2 - c^2 \\ \Rightarrow b^2 + c^2 &= 2a^2. \end{aligned} \tag{1}$$

Now, using (1), we have

$$AS \cdot AD = \frac{bc \cos A}{2} = \frac{b^2 + c^2 - a^2}{4} = \frac{a^2}{4},$$

and

$$AM \cdot AN = \frac{1}{3} \cdot AN^2 = \frac{1}{3} \times \frac{1}{4} (2b^2 + 2c^2 - a^2) = \frac{a^2}{4}.$$

So, we get

$$AS \cdot AD = AM \cdot AN.$$

Therefore, by the converse of the power of point theorem, we infer that S, D, N, M lie on a circle. Since S, D, N lie on the nine-point circle, M also lies on the nine-point circle. Thus $DNMS$ is a cyclic quadrilateral and since $\angle SDN = 90^\circ$, we must have $\angle SMN = 90^\circ$. Therefore, SM is perpendicular to AG and we are done.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA.

S677. Let x, y, z be real numbers such that $xy + yz + zx = 1$. Prove that

$$\frac{xy}{(x+y)\sqrt{z^2+1}} + \frac{yz}{(y+z)\sqrt{x^2+1}} + \frac{zx}{(z+x)\sqrt{y^2+1}} \leq \frac{3}{4}.$$

Proposed by Iancu-Ioan Sandea, Cluj-Napoca, Romania

Solution 1 by Titu Zvonaru, Comănești, Romania

Since $z^2 + 1 = z^2 + xy + yz + zx = (z+x)(z+y)$, by the AM-GM inequality we obtain

$$\frac{xy}{(x+y)\sqrt{(z+x)(z+y)}} = \frac{xy\sqrt{(z+x)(z+y)}}{(x+y)(z+x)(z+y)} \leq \frac{xy(2z+x+y)}{2(x+y)(y+z)(z+x)}.$$

It suffices to prove that

$$\frac{xy(2z+x+y)}{(x+y)(y+z)(z+x)} + \frac{yz(2x+y+z)}{(y+z)(z+x)(x+y)} + \frac{zx(2y+z+x)}{(z+x)(x+y)(y+z)} \leq \frac{3}{2}.$$

After some simple calculations, it remains to show that

$$x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 \geq 6xyz,$$

which is true by the AM-GM inequality.

Equality holds if and only if $x = y = z = \frac{1}{\sqrt{3}}$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

(i) From the condition $xy + z(x+y) = 1$ we get that $x+y \neq 0$ since otherwise we would have $x^2 = -1$. Similarly we get $(x+y)(y+z)(z+x) \neq 0$.

(ii) If $z = 0$, from the given condition we get $xy = 1$, and the desired inequality becomes equivalent to

$$\frac{x}{x^2+1} \leq \frac{3}{4} \iff 3x^2 - 4x + 3 \geq 0 \iff 3\left(x - \frac{2}{3}\right)^2 + \frac{5}{3} \geq 0,$$

which is obvious. Similarly if $x = 0$ or $y = 0$. Let $xyz \neq 0$.

(iii) If $x, y, z < 0$, then the desired inequality is obvious because the left-hand-side is negative. If $x, y > 0$ and $z < 0$, the given condition becomes $xy - |z|(x+y) = 1$. (*) If $x, y < 0$ and $z > 0$, the given condition becomes $|x||y| - z(|x| + |y|) = 1$, which is similar to (*). Therefore, we can set $x = 1/\tan A$, $y = 1/\tan B$, and $z = 1/\tan C$, where A, B, C are the angles of a triangle, because in every triangle the condition $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ holds true. With this substitution we get

$$\frac{xy}{(x+y)\sqrt{z^2+1}} = \frac{\sin C}{\tan A + \tan B} = \cos A \cos B,$$

and thus the desired inequality becomes equivalent to

$$S := \cos A \cos B + \cos B \cos C + \cos C \cos A \leq \frac{3}{4}.$$

Setting $t = \sin(C/2) > 0$, we get

$$\begin{aligned} S &= \frac{1}{2} [\cos(A-B) + \cos(A+B)] + \cos C \cdot 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \\ &\leq \frac{1}{2} - \frac{1}{2} \cos C + 2 \cos C \sin \frac{C}{2} = \frac{1}{2} - \frac{1}{2} (1 - 2t^2) + 2(1 - 2t^2)t = -4t^3 + t^2 + 2t, \end{aligned}$$

and therefore, it suffices to show that

$$-4t^3 + t^2 + 2t \leq \frac{3}{4} \iff 16t^3 - 4t^2 - 8t + 3 \geq 0 \iff (2t - 1)^2(4t + 3) \geq 0,$$

which is obvious. Equality occurs when $x = y = z = \sqrt{3}/3$.

Solution 3 by Marius Stănean, Zalău, Romania

The inequality can be rewritten as

$$\frac{xy}{(x+y)\sqrt{(z+x)(z+y)}} + \frac{yz}{(y+z)\sqrt{(x+y)(x+z)}} + \frac{zx}{(z+x)\sqrt{(y+z)(y+x)}} \leq \frac{3}{4}. \quad (1)$$

We have two cases:

1. $x + y, y + z, z + x < 0$, the inequality is not true. A counterexample of the problem statement is $x = y = -4, z = \frac{15}{8}$, where

$$LHS - \frac{3}{4} \approx 0.0208 > 0.$$

2. $x + y, y + z, z + x > 0$, the inequality is true. It is obvious that only one of the numbers x, y, z can be negative. Also, we have

$$\begin{aligned} \sqrt{x+z} + \sqrt{y+z} &= \sqrt{x+z+y+z+2\sqrt{(x+z)(y+z)}} \\ &= \sqrt{x+y+2z+2\sqrt{z^2+1}} \\ &> \sqrt{x+y}. \end{aligned}$$

Therefore, there is a triangle ABC with the sides $a = \sqrt{y+z}, b = \sqrt{z+x}, c = \sqrt{x+y}$. The inequality becomes

$$\sum_{cyc} ab(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \leq 3a^2b^2c^2,$$

or

$$\sum_{cyc} 4 \cos A \cos B \leq 3 \iff \frac{s^2 + r^2}{R^2} - 4 \leq 3.$$

But by Gerretsen's Inequality, i.e. $s^2 \leq 4R^2 + 4Rr + 3r^2$, and Euler's Inequality, i.e. $R \geq 2r$, we have

$$\frac{s^2 + r^2}{R^2} - 4 \leq \frac{4R^2 + 4Rr + 3r^2 + r^2}{R^2} - 4 = 4 \left(\frac{r}{R} + \frac{r^2}{R^2} \right) \leq 4 \left(\frac{1}{2} + \frac{1}{4} \right) = 3.$$

The equality holds when $x = y = z$.

Also solved by Arkady Alt, San Jose, California, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

S678. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{(b+c)^2} + \frac{b}{(c+d)^2} + \frac{c}{(d+a)^2} + \frac{d}{(a+b)^2} \geq \frac{4}{a+b+c+d}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

The inequality is

$$\sum_{\text{cyc}} a \sum_{\text{cyc}} \frac{a}{(b+c)^2} \geq 4$$

Cauchy–Schwarz yields

$$\sum_{\text{cyc}} a \sum_{\text{cyc}} \frac{a}{(b+c)^2} \geq \left(\sum_{\text{cyc}} \sqrt{a} \frac{\sqrt{a}}{b+c} \right)^2,$$

so it is sufficient to prove the inequality

$$\sum_{\text{cyc}} \frac{a}{b+c} \geq 2 \iff \sum_{\text{cyc}} \frac{a^2}{a(b+c)} \geq 2.$$

By Titu's Lemma

$$\sum_{\text{cyc}} \frac{a^2}{a(b+c)} \geq \frac{(a+b+c+d)^2}{a(b+c) + b(c+d) + c(d+a) + d(a+b)},$$

so we need to prove

$$\frac{(a+b+c+d)^2}{a(b+c) + b(c+d) + c(d+a) + d(a+b)} \geq 2$$

which simplifies to

$$a^2 + b^2 + c^2 + d^2 \geq 2(ac + bd) \iff (a-c)^2 + (b-d)^2 \geq 0,$$

clearly true. Equality holds if and only if $a = c$ and $b = d$.

Also solved by Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA.

Undergraduate problems

U673. Let $A_1A_2 \dots A_n$ be a cyclic polygon inscribed in a circle of center O and radius R , and let G be its centroid. Prove that the inequality

$$MA_1 + MA_2 + \dots + MA_n \geq nR$$

holds for every point M in the plane of the polygon if and only if $O \equiv G$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ştefan Marinescu, Hunedoara, Romania

Solution by the authors

We consider the polygon $A_1A_2 \dots A_n$ in the complex plane and let $a_1, a_2, \dots, a_n$ be the complex coordinates of its vertices. Without loss of generality, we may assume that the circumcenter O is the origin of the complex plane, that is we have $|a_1| = |a_2| = \dots = |a_n| = R$.

If $M(m)$, then the inequality $MA_1 + MA_2 + \dots + MA_n \geq nR$ becomes

$$\sum_{k=1}^n |m - a_k| \geq nR,$$

which holds for every complex number m . This inequality is equivalent to

$$\sum_{k=1}^n (|m - a_k| - |a_k|) \geq 0,$$

that is

$$\sum_{k=1}^n \frac{|m - a_k|^2 - |a_k|^2}{|m - a_k| + |a_k|} \geq 0.$$

Considering $a_k = a'_k + a''_k i, k = 1, 2, \dots, n, m = x + yi$, the last inequality becomes

$$\sum_{k=1}^n \frac{x^2 - 2a'_k x + y^2 - 2a''_k y}{\sqrt{(x - a'_k)^2 + (y - a''_k)^2} + \sqrt{a'^2_k + a''^2_k}} \geq 0,$$

which holds for every real numbers x, y .

For $y = 0$ and $x > 0$, it follows

$$\sum_{k=1}^n \frac{x - 2a'_k}{\sqrt{(x - a'_k)^2 + a''^2_k} + \sqrt{a'^2_k + a''^2_k}} \geq 0.$$

Taking $x \rightarrow 0$, one obtains $\sum_{k=1}^n a'_k \leq 0$. For $y = 0$ and $x < 0$, with a similar argument we get $\sum_{k=1}^n a'_k \geq 0$, hence necessarily we have $\sum_{k=1}^n a'_k = 0$.

Similarly, for $x = 0$ and $y > 0$ (respectively $y < 0$), it follows $\sum_{k=1}^n a''_k = 0$.

Finally, we obtain $\sum_{k=1}^n a_k = 0$, that is the centroid G of $A_1A_2 \dots A_n$ coincides with the circumcenter O .

For the converse, let assume that $O = G$, that is we have $\sum_{k=1}^n a_k = 0$, hence $\sum_{k=1}^n \overline{a_k} = 0$. Using the modulus inequality, for every point M in the plane we have

$$\begin{aligned} \sum_{k=1}^n R \cdot MA_k &= \sum_{k=1}^n |\overline{a_k}| |m - a_k| = \sum_{k=1}^n |m \overline{a_k} - a_k \overline{a_k}| = \\ &= \sum_{k=1}^n |m \overline{a_k} - R^2| \geq \left| \sum_{k=1}^n (m \overline{a_k} - R^2) \right| = \left| m \sum_{k=1}^n \overline{a_k} - nR^2 \right| = |0 - nR^2| = nR^2, \end{aligned}$$

and the inequality is proved.

Remarks. 1) When $n = 3$, the inequality $MA_1 + MA_2 + MA_3 \geq 3R$ holds for every point M in the plane of the triangle $A_1A_2A_3$ if and only if $O = G$, that is if and only if $\triangle A_1A_2A_3$ is equilateral.

In this case a different argument of this property can be given by using the Fermat-Toricelli point T of the triangle. Indeed, if $\triangle A_1A_2A_3$ is equilateral, then its Fermat-Toricelli point T is the circumcenter O , hence for every point M in the plane we have

$$MA_1 + MA_2 + MA_3 \geq MA_1 + MA_2 + MA_3 = OA_1 + OA_2 + OA_3 = 3R.$$

Conversely, if the Fermat-Toricelli point and the circumcenter agree then it is well-known that the three angles at the T are 120 degrees. So by the property of the circumcenter, and side-angle-side, it follows immediately that the triangle is equilateral.

2) When $n = 4$, the inequality $MA_1 + MA_2 + MA_3 + MA_4 \geq 4R$ holds for every point M in the plane of the cyclic quadrilateral $A_1A_2A_3A_4$ if and only if $O = G$, that is if and only if $A_1A_2A_3A_4$ is a rectangle. One implication is obvious. For the second, from our problem it follows that if the inequality holds for every point M in the plane, then we have $a_1 + a_2 + a_3 + a_4 = 0$. That is $a_1 + a_2 + a_3 = -a_4$, hence the orthocenter of $\triangle A_1A_2A_3$ is on the circumcircle, that is $\triangle A_1A_2A_3$ is right-angled at A_2 and we have $a_2 + a_4 = 0$. Finally, we get $a_2 + a_4 = a_1 + a_3 = 0$, that is $A_1A_2A_3A_4$ is a cyclic parallelogram, hence it is a rectangle.

3) We can reformulate the problem as follows. Define the Fermat-Toricelli point of the polygon $A_1A_2 \dots A_n$ as the unique point T in the plane that minimizes the function $f(M) = MA_1 + MA_2 + \dots + MA_n$. If the polygon is cyclic and inscribed in the circle of center O , then we have $T = O$ if and only if $O = G$.

Also solved by Joshua Pit  , Cambridge Rindge and Latin School, MA, USA; Kousik Sett, India; Nicu  or Zlota, "Traian Vuia" Technical College, Foc  ani, Romania; Theo Koupelis, Clark College, Washington, USA;

U674. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{32a^2 - 48a + 25} + \frac{1}{32b^2 - 48b + 25} + \frac{1}{32c^2 - 48c + 25} \leq \frac{1}{3}.$$

Proposed by Vasile Cîrtoaje, Oil-Gas Universty of Ploiești, Romania

Solution by the author

We see that $32x^2 - 48x + 25 = 2(4x - 3)^2 + 7 > 0$ for any real x . Assume that $a \geq b \geq c$ and write the inequality as $F \leq \frac{1}{3}$.

Case 1: $c > \frac{3}{4}$. For fixed a , assume that b is a function of c , i.e. $b = 3 - a - c$. We have

$$\frac{F'(c)}{16} = \frac{-(4b-3)b'}{(32b^2 - 48b + 25)^2} - \frac{4c-3}{(32c^2 - 48c + 25)^2} = \frac{4b-3}{(32b^2 - 48b + 25)^2} - \frac{4c-3}{(32c^2 - 48c + 25)^2}.$$

We will show that $F'(c) \geq 0$, i.e.

$$\left(\frac{32c^2 - 48c + 25}{32b^2 - 48b + 25} \right)^2 \geq \frac{4c-3}{4b-3}.$$

Since

$$\left(\frac{32c^2 - 48c + 25}{32b^2 - 48b + 25} \right)^2 \geq \frac{2(32c^2 - 48c + 25)}{32b^2 - 48b + 25} - 1,$$

it suffices to show that

$$\frac{2(32c^2 - 48c + 25)}{32b^2 - 48b + 25} - 1 \geq \frac{4c-3}{4b-3},$$

i.e.

$$1 - \frac{4c-3}{4b-3} \geq 2 \left(1 - \frac{32c^2 - 48c + 25}{32b^2 - 48b + 25} \right), \quad \frac{4(b-c)}{4b-3} \geq \frac{32(b-c)(2b+2c-3)}{32b^2 - 48b + 25}.$$

This is true if

$$\frac{1}{4b-3} \geq \frac{8(2b+2c-3)}{32b^2 - 48b + 25}.$$

Since $2b + 2c - 3 = 3 - 2a \leq 3 - 2b$, it suffices to prove that

$$\frac{1}{4b-3} \geq \frac{8(3-2b)}{32b^2 - 48b + 25}.$$

Indeed,

$$32b^2 - 48b + 25 - 8(3-2b)(4b-3) = 96b^2 - 192b + 97 > 96b^2 - 192b + 96 = 96(b-1)^2 \geq 0.$$

Since $F'(c) \geq 0$, $F(c)$ is increasing and has the maximum value when c is maximum, hence when $c = b$. So, we only need to prove the original inequality for $c = b$, i.e. to show that $a + 2b = 3$ involves

$$\frac{1}{32a^2 - 48a + 25} + \frac{2}{32b^2 - 48b + 25} \leq \frac{1}{3}.$$

Write the inequality as follows

$$\frac{1}{128b^2 - 288b + 119} \leq \frac{32b^2 - 48b + 19}{3(32b^2 - 48b + 25)},$$

$$64b^4 - 240b^3 + 337b^2 - 210b + 49 \geq 0, \quad (b-1)^2(8b-7)^2 \geq 0.$$

Case 2: $0 \leq c \leq \frac{3}{4}$. For fixed b , assume that a is a decreasing function of c , i.e. $a = 3 - b - c$. We have

$$\begin{aligned} \frac{F'(c)}{16} &= \frac{-(4a-3)a'}{(32a^2-48a+25)^2} - \frac{4c-3}{(32c^2-48c+25)^2} = \frac{4a-3}{(32a^2-48a+25)^2} - \frac{4c-3}{(32c^2-48c+25)^2} \\ &\geq \frac{4a-3}{(32a^2-48a+25)^2} > 0. \end{aligned}$$

Since $F'(c) > 0$, $F(c)$ is increasing and has the maximum value when c is maximum (a is minimum), hence when $c = b$ or $a = b$. Therefore, we only need to prove the original inequality for these cases. In both cases, the original inequality becomes

$$(b-1)^2(8b-7)^2 \geq 0.$$

So, the proof is completed. The equality occurs when $a = b = c = 1$, and also for $a = \frac{5}{4}$ and $b = c = \frac{7}{8}$ (or any cyclic permutation).

Editor's Comment. Theo Koupelis provided a similar argument and noticed that the condition that a, b, c are nonnegative is not necessary.

Also solved by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Thanh Nam Nguyen, Tu Son Secondary School, Vietnam; Theo Koupelis, Clark College, Washington, USA.

U675. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \ln \left(1 + \frac{i}{n} \right) \ln \left(1 + \frac{j}{n} \right).$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Daniel Pascuas, Barcelona, Spain

Note that

$$\frac{1}{n^2} \sum_{1 \leq i < j \leq n} \ln \left(1 + \frac{i}{n} \right) \ln \left(1 + \frac{j}{n} \right) = \frac{1}{2} \left\{ \left(\frac{1}{n} \sum_{i=1}^n \ln \left(1 + \frac{i}{n} \right) \right)^2 - \frac{1}{n^2} \sum_{i=1}^n \ln^2 \left(1 + \frac{i}{n} \right) \right\}.$$

Moreover, the integrals $\int_0^1 \ln(1+x) dx$ and $\int_0^1 \ln^2(1+x) dx$ are the limits of the sequences of Riemann sums $\{\frac{1}{n} \sum_{i=1}^n \ln(1 + \frac{i}{n})\}_{n \geq 1}$ and $\{\frac{1}{n} \sum_{i=1}^n \ln^2(1 + \frac{i}{n})\}_{n \geq 1}$, respectively, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln \left(1 + \frac{i}{n} \right) = \int_0^1 \ln(1+x) dx \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln^2 \left(1 + \frac{i}{n} \right) = \int_0^1 \ln^2(1+x) dx.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n \ln^2 \left(1 + \frac{i}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \ln^2(1+x) dx = 0.$$

Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \ln \left(1 + \frac{i}{n} \right) \ln \left(1 + \frac{j}{n} \right) &= \frac{1}{2} \left(\int_0^1 \ln(1+x) dx \right)^2 = \frac{1}{2} \left(\int_1^2 \ln x dx \right)^2 \\ &= \frac{1}{2} \left([x \ln x - x]_1^2 \right)^2 = \frac{(2 \ln 2 - 1)^2}{2}. \end{aligned}$$

Also solved by (Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; Joshua Pit , Cambridge Rindge and Latin School, MA, USA; Nicu or Zlota, "Traian Vuia" Technical College, Foc ani, Romania; Paolo Perfetti, Universit  degli studi di Tor Vergata Roma, Italy; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Prajnanaswaroop S, Bangalore, Karnataka, India; Srijan Sundar, Texas, USA; Woojin Kim, Seoul Science High School, Seoul, South Korea.

U676. Let $a > 1$ be an integer nondivisible by 4. Prove that there are infinitely many primes p of the form $4k - 1$ such that $p|a^d - 1$, for some $d < \frac{p-1}{2}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We first prove the following.

Lemma. There are infinitely many primes p of the form $4k - 1$ for which $2p + 1$ is a composite number.

Proof. Let $p_1, \dots, p_t$ be all primes satisfying the statement of our claim. Let p be a prime of the form $4k - 1$ greater than p_t . Consider the following sequence: $q_0 = p$, $q_n = 2q_{n-1} + 1$, $n > 0$. Each term of the sequence is of the form $4k - 1$ and each term is a prime by our assumption. It is easy to show by induction that $q_n = 2^n p + 2^n - 1$ and therefore, q_{p-1} can't be a prime number since it is divisible by p . This completes our proof. $\square$

Let d be a prime of the form $4k - 1$ such that $2d + 1$ is a composite number. There are infinitely many of them by the lemma. Choose $a \equiv 1, 2 \pmod{4}$ and $\gcd(d, a - 1) = 1$. Hence, $\gcd\left(\frac{a^d - 1}{a - 1}, d\right) = 1$. Furthermore, $\frac{a^d - 1}{a - 1}$ is of the form $4k - 1$ and all of its prime factors are of the form $2kd + 1$. Hence, there is a prime p of the form $4k - 1$ such that $p \mid \frac{a^d - 1}{a - 1}$, $\gcd(p, a - 1) = 1$ and then $p = 2kd + 1$. Moreover, $k \geq 2$. It follows that $d < \frac{p-1}{2}$. Furthermore, distinct d will determine distinct p since $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m, n)} - 1$. So, the statement is proved in the case $a \equiv 1, 2 \pmod{4}$.

Indeed, we can do something similar in the case that $a \equiv -1 \pmod{4}$. Let d be a prime number of the form $4k - 1$ such that $\gcd(d, a^2 - 1) = 1$ and $2d + 1$ is composite. Then $\frac{a^{2d} - 1}{a^2 - 1}$ would be of the form $4k - 1$ and hence, we would have a prime divisor p of the form $4k - 1$ such that $\gcd(p, a^2 - 1) = 1$. Moreover, $p = 2sd + 1$ and hence, $s \geq 2$. Therefore, $d < \frac{p-1}{2}$.

U677. Evaluate the integral:

$$\int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania

Solution by Daniel Pascuas, Barcelona, Spain

The substitution $t = \frac{1}{x}$ yields that

$$\int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx = \int_0^\infty \frac{t \ln^2 t}{(1+t)(1+t^2)} dt = \int_0^\infty \frac{x \ln^2 x}{(1+x)(1+x^2)} dx,$$

so, by adding up the first and the third integral, we obtain that

$$2 \int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx = \int_0^\infty \frac{\ln^2 x}{1+x^2} dx,$$

and therefore

$$\int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx = \frac{1}{2} \int_0^\infty \frac{\ln^2 x}{1+x^2} dx = \frac{\pi^3}{16},$$

since

$$\int_0^\infty \frac{\ln^2 x}{1+x^2} dx = \frac{\pi^3}{8}. \quad (*)$$

We give two different proofs of (*):

Real variable proof of (*). The substitution $t = \frac{1}{x}$ yields that

$$\int_1^\infty \frac{\ln^2 x}{1+x^2} dx = \int_0^1 \frac{\ln^2 t}{1+t^2} dt = \int_0^1 \frac{\ln^2 x}{1+x^2} dx, \quad \text{so} \quad \int_0^\infty \frac{\ln^2 x}{1+x^2} dx = 2 \int_0^1 \frac{\ln^2 x}{1+x^2} dx.$$

Since

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{k=0}^{\infty} (x^{4k} - x^{4k+2}) \quad (0 < x < 1),$$

Lebesgue's monotone convergence theorem shows that

$$\int_0^\infty \frac{\ln^2 x}{x^2+1} dx = 2 \sum_{k=0}^{\infty} \int_0^1 (x^{4k} - x^{4k+2}) \ln^2 x dx = 2 \sum_{k=0}^{\infty} (I_{4k} - I_{4k+2}),$$

where $I_n = \int_0^1 x^n \log^2 x dx$, for any nonnegative integer n . By integrating by parts twice we get

$$\begin{aligned} I_n &= \int_0^1 x^n \ln^2 x dx = \frac{1}{n+1} \int_0^1 \ln^2 x dx^{n+1} = \frac{1}{n+1} \left[x^{n+1} \ln^2 x \right]_0^1 - \frac{2}{n+1} \int_0^1 x^n \ln x dx \\ &= -\frac{2}{(n+1)^2} \int_0^1 \ln x dx^{n+1} = -\frac{2}{(n+1)^2} \left[x^{n+1} \ln x \right]_0^1 + \frac{2}{(n+1)^2} \int_0^1 x^n dx = \frac{2}{(n+1)^3}. \end{aligned}$$

Therefore

$$\int_0^\infty \frac{\ln^2 x}{x^2+1} dx = 4 \sum_{n=0}^{\infty} \left(\frac{1}{(4k+1)^3} - \frac{1}{(4k+3)^3} \right) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}.$$

Now let us evaluate the sum of this series by means of Fourier series. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the 2π -periodic function defined by $f(x) = x^2$, for $-\pi \leq x \leq \pi$. The Fourier series of f is

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx),$$

since f is an even function, $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$, and

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} x^2 d \sin(nx) = -\frac{2}{n\pi} \int_{-\pi}^{\pi} x \sin(nx) dx \\ &= \frac{2}{n^2\pi} \int_{-\pi}^{\pi} x d \cos(nx) = \frac{2}{n^2\pi} \left\{ \left[x \cos(nx) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos(nx) dx \right\} \\ &= \frac{2}{n^2\pi} \left[x \cos(nx) \right]_{-\pi}^{\pi} = (-1)^n \frac{4}{n^2}, \quad \text{for every positive integer } n. \end{aligned}$$

Since f is continuous and its Fourier series converges uniformly on $\mathbb{R}$ (by Weierstrass M-test), the Fourier series of f converges to f uniformly on $\mathbb{R}$, so we may integrate term by term this series to obtain that

$$\frac{\pi^3}{6} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(n \frac{\pi}{2}\right) = \frac{\pi^3}{6} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^{\frac{\pi}{2}} \cos(nx) dx = \int_0^{\frac{\pi}{2}} f(x) dx = \frac{\pi^3}{24},$$

and therefore

$$\frac{\pi^3}{6} - 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{24}.$$

Hence

$$\int_0^{\infty} \frac{\ln^2 x}{x^2 + 1} dx = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{8}.$$

Complex variable proof of (*). Note that $\int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx = F''(0)$, where F is the holomorphic function on the strip $-1 < \operatorname{Re} z < 1$ defined by

$$F(z) = \int_0^{\infty} \frac{x^z}{1+x^2} dx \quad (-1 < \operatorname{Re} z < 1).$$

Now we are going to compute the integral $F(z)$ by applying the residue theorem to the meromorphic function $f_z(w) = \frac{w^z}{1+w^2}$ and the keyhole contour $\gamma_{\delta,r,R}$, which we are going to define in a while. Here w^z denotes the branch of the z -power in the domain $\Omega = \mathbb{C} \setminus [0, \infty)$ (where $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$, as usual) whose value at $w = -1$ equals to $e^{i\pi z}$. Thus f_z is a meromorphic function on Ω whose only poles are $\pm i$ (simple poles).

For $-\frac{\pi}{2} < \delta < \frac{\pi}{2}$ and $0 < r < R$, let $\alpha_{\delta,r,R}$ be the line segment which goes from $re^{i\delta}$ to $Re^{i\delta}$.

For $0 < \delta < \frac{\pi}{2}$ and $\varrho > 0$, let $\beta_{\delta,\varrho}$ be the arc of the circle centered at the origin with radius ϱ which goes from $\varrho e^{i\delta}$ to $\varrho e^{-i\delta}$ counterclockwise. Moreover, if α is a path on the complex plane, then α^{-1} denotes the path α traversed in the opposite direction. For $0 < \delta < \frac{\pi}{2}$ and $0 < r < 1 < R$, $\gamma_{\delta,r,R}$ is the concatenation of $\alpha_{\delta,r,R}$, $\beta_{\delta,R}$, $\alpha_{-\delta,r,R}^{-1}$, and $\beta_{\delta,r}^{-1}$. Note that $\gamma_{\delta,r,R}$ is a closed rectifiable curve in $\Omega \setminus \{\pm i\}$. Then the residue theorem shows that

$$\int_{\gamma_{\delta,r,R}} f_z(w) dw = 2\pi i \left\{ \operatorname{Res}(f_z(w); w = i) + \operatorname{Res}(f_z(w); w = -i) \right\}.$$

Since

$$\operatorname{Res}(f_z(w); w = i) = \lim_{w \rightarrow i} (w - i) f_z(w) = \lim_{w \rightarrow i} \frac{w^z}{w + i} = \frac{1}{2i} e^{i\frac{\pi}{2}z} \quad \text{and}$$

$$\operatorname{Res}(f_z(w); w = -i) = \lim_{w \rightarrow -i} (w + i) f_z(w) = \lim_{w \rightarrow -i} \frac{w^z}{w - i} = -\frac{1}{2i} e^{i\frac{3\pi}{2}z},$$

we have that

$$\int_{\gamma_{\delta,r,R}} f_z(w) dw = \pi (e^{i\frac{\pi}{2}z} - e^{i\frac{3\pi}{2}z}) = \pi e^{i\pi z} (e^{-i\frac{\pi}{2}z} - e^{i\frac{\pi}{2}z}).$$

Therefore

$$\pi e^{i\pi z} (e^{-i\frac{\pi}{2}z} - e^{i\frac{\pi}{2}z}) = \int_{\gamma_{\delta,r,R}} f_z(w) dw = \left\{ \int_{\alpha_{\delta,r,R}} + \int_{\beta_{\delta,R}} - \int_{\alpha_{-\delta,r,R}} - \int_{\beta_{\delta,r}} \right\} f_z(w) dw.$$

Then let $\delta \searrow 0$ to get that

$$\pi e^{i\pi z} (e^{-i\frac{\pi}{2}z} - e^{i\frac{\pi}{2}z}) = (1 - e^{2\pi iz}) \int_r^R \frac{x^z}{1+x^2} dx + \int_0^{2\pi} \frac{R^z e^{iz\theta}}{1+R^2 e^{2i\theta}} d\theta - \int_0^{2\pi} \frac{r^z e^{iz\theta}}{1+r^2 e^{2i\theta}} d\theta.$$

But

$$\left| \int_0^{2\pi} \frac{R^z e^{iz\theta}}{1+R^2 e^{2i\theta}} d\theta \right| \leq 2\pi \frac{R^{\operatorname{Re} z} e^{2\pi |\operatorname{Re} z|}}{R^2 - 1} \longrightarrow 0, \quad \text{as } R \nearrow \infty, \quad \text{and}$$

$$\left| \int_0^{2\pi} \frac{r^z e^{iz\theta}}{1+r^2 e^{2i\theta}} d\theta \right| \leq 2\pi \frac{r^{\operatorname{Re} z} e^{2\pi |\operatorname{Re} z|}}{1-r^2} \longrightarrow 0, \quad \text{as } r \searrow 0, \quad \text{for } 0 < \operatorname{Re} z < 1.$$

It follows that

$$F(z) = \int_0^\infty \frac{x^z}{1+x^2} dx = \frac{\pi e^{i\pi z} (e^{-i\frac{\pi}{2}z} - e^{i\frac{\pi}{2}z})}{1 - e^{2\pi iz}} = \frac{\pi \sin(\frac{\pi}{2}z)}{\sin(\pi z)} = \frac{\pi}{2 \cos(\frac{\pi}{2}z)} = \frac{\pi}{2} \sec(\frac{\pi}{2}z),$$

for $0 < \operatorname{Re} z < 1$. By analytic continuation, this identity also holds for $-1 < \operatorname{Re} z < 1$, since $F(z)$ and $\frac{\pi}{2 \cos(\frac{\pi}{2}z)}$ are holomorphic functions on the strip $-1 < \operatorname{Re} z < 1$. As a consequence,

$$F''(z) = \frac{d^2}{dz^2} \left(\frac{\pi}{2} \sec(\frac{\pi}{2}z) \right) = \frac{\pi^2}{4} \frac{d}{dz} \left(\tan(\frac{\pi}{2}z) \sec(\frac{\pi}{2}z) \right) = \frac{\pi^3}{8} \left(\sec^3(\frac{\pi}{2}z) + \tan^2(\frac{\pi}{2}z) \sec(\frac{\pi}{2}z) \right),$$

for $-1 < \operatorname{Re} z < 1$. Hence

$$\int_0^\infty \frac{\ln^2 x}{1+x^2} dx = F''(0) = \frac{\pi^3}{8}.$$

Also solved by Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; Mingcan Fan, Huizhou University, Huizhou, China; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Prajnanaswaroop S, Bangalore, Karnataka, India; Prasanna Pawar, AMMOC, Bengaluru, India; Srijan Sundar, Texas, USA; Sunghoon Kim, Seoul Science High School, South Korea.

U678. Evaluate

$$\int_0^\infty \frac{(\cos(ax) - \cos(bx))^2}{x^3} dx.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution by Mingcan Fan, Huizhou University, Huizhou, China

If $a = b$, the result is obviously 0. Since $\cos(kx) = \cos(-kx)$, without loss of generality, suppose that $b > a > 0$. Using integration by parts, we get

$$\begin{aligned} & \int_0^\infty \frac{(\cos(ax) - \cos(bx))^2}{x^3} dx \\ &= -\frac{(\cos(ax) - \cos(bx))^2}{2x^2} \Big|_0^\infty + \int_0^\infty \frac{(b \sin(bx) - a \sin(ax))(\cos(ax) - \cos(bx))}{x^2} dx \\ &= \int_0^\infty \frac{b \sin(bx) - a \sin(ax)}{x} \int_a^b \sin(xy) dy dx \\ &= \int_a^b \int_0^\infty \frac{b [\cos((b-y)x) - \cos((b+y)x)] - a [\cos((a-y)x) - \cos((a+y)x)]}{2x} dx dy \\ &= \frac{1}{2} \int_a^b \left[b \ln \left(\frac{b+y}{b-y} \right) - a \ln \left(\frac{a+y}{y-a} \right) \right] dy \\ &= \frac{1}{2} [b(b+y)(\ln(b+y) - 1) - b(y-b)(\ln(b-y) - 1) - a(a+y)(\ln(a+y) - 1) + a(y-a)(\ln(y-a) - 1)]_a^b \\ &= b^2 \ln(2b) + a^2 \ln(2a) - \frac{(a+b)^2}{2} \ln(a+b) - \frac{(a-b)^2}{2} \ln(b-a), \end{aligned}$$

where we used Fubini's Theorem to switch the order of integration, the fact that $\cos((a-y)x) = \cos((y-a)x)$, $\int \ln(a \pm x) dx = (x \pm a)(\ln(a \pm x) - 1) + C$ and the Frullani integral:

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x} dx = \ln(b/a).$$

Since $\lim_{x \rightarrow 0^+} x^2 \ln(2x) = 0$, we conclude that

$$\int_0^\infty \frac{(\cos(ax) - \cos(bx))^2}{x^3} dx = \begin{cases} 0, & a = b \\ b^2 \ln 2, & a = 0 \\ a^2 \ln 2, & b = 0 \\ b^2 \ln |2b| + a^2 \ln |2a| - \frac{(a+b)^2}{2} \ln(|a| + |b|) - \frac{(a-b)^2}{2} \ln(|b| - |a|), & \text{otherwise.} \end{cases}$$

Also solved by Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA; Sunghoon Kim, Seoul Science High School, South Korea.

Olympiad problems

O673. Let $p_1, \dots, p_s$ be distinct primes, and, for some $n \geq \max\{p_1, \dots, p_s\}$, let $x_1, \dots, x_n$ be integers. For every $i \in \{1, \dots, s\}$, let P_i be the product of all of $p_1, \dots, p_s$ except for p_i , and let P'_i be an integer such that $P_i P'_i \equiv 1 \pmod{p_i}$.

For every $j \in \{1, \dots, n\}$, let $M^{(j)}$ be the number of the sums of j among $x_1, \dots, x_n$ which are not divisible by any of $p_1, \dots, p_s$, and let

$$M = M^{(1)} - M^{(2)} + \dots + (-1)^{n-1} M^{(n)}.$$

For every $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, n\}$, let $N_i^{(j)}$ be the number of the sums of j among $x_1, \dots, x_n$ which are not divisible by p_i , but are divisible by at least one p_k with $k \in \{1, \dots, s\}$, $k \neq i$, and let

$$N_i = N_i^{(1)} - N_i^{(2)} + \dots + (-1)^{n-1} N_i^{(n)}.$$

Prove that

$$M + P_1 P'_1 N_1 + \dots + P_s P'_s N_s \equiv 0 \pmod{p_1 \cdots p_s}.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution by the authors

The well-known identity

$$\begin{aligned} & (x_1 + \dots + x_n)^l - \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} (x_{i_1} + \dots + x_{i_{n-1}})^l + \\ & + \sum_{1 \leq i_1 < \dots < i_{n-2} \leq n} (x_{i_1} + \dots + x_{i_{n-2}})^l - \dots + (-1)^{n-1} \sum_{1 \leq i_1 \leq n} x_{i_1}^l = 0 \end{aligned}$$

holds for any integers $x_1, \dots, x_n$ whenever l is a positive integer with $l < n$ (more generally, for any $l < n$ and for any elements $x_1, \dots, x_n$ of a commutative ring, but this is not the point here). In particular, for the given integers $x_1, \dots, x_n$, and for any $1 \leq i \leq s$, we have that

$$\begin{aligned} & (x_1 + \dots + x_n)^{p_i-1} - \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} (x_{i_1} + \dots + x_{i_{n-1}})^{p_i-1} + \\ & + \sum_{1 \leq i_1 < \dots < i_{n-2} \leq n} (x_{i_1} + \dots + x_{i_{n-2}})^{p_i-1} - \dots + (-1)^{n-1} \sum_{1 \leq i_1 \leq n} x_{i_1}^{p_i-1} = 0 \end{aligned}$$

(since $p_i - 1 < \max\{p_1, \dots, p_s\} \leq n$). By Fermat's theorem, we have (for any $1 \leq i \leq s$) $x^{p_i-1} \equiv 1 \pmod{p_i}$ whenever x is an integer not divisible by p_i . Moreover, we also have $x^{p_i-1} \equiv 0 \pmod{p_i}$ for x divisible by p_i . By hypothesis, for any $1 \leq i \leq s$, and any $1 \leq j \leq n$, $M^{(j)} + N_i^{(j)}$ represents the number of the sums of j terms among $x_1, \dots, x_n$ which are not divisible by p_i , hence the above equality taken modulo p_i yields

$$(M^{(1)} + N_i^{(1)}) - (M^{(2)} + N_i^{(2)}) + \dots + (-1)^{n-1} (M^{(n)} + N_i^{(n)}) \equiv 0 \pmod{p_i},$$

that is, we have

$$M + N_i \equiv 0 \pmod{p_i}$$

for any $1 \leq i \leq s$. Now, by applying the Chinese Remainder Theorem to this system of congruences, the desired conclusion follows, thus finishing our proof.

O674. Prove that 8 is the greatest value of the positive constant k for which

$$a^2 + b^2 + c^2 + d^2 - 4 \geq k(a + b + c + d - 4)$$

whenever $a = b \geq c \geq d \geq 0$ such that $ab + bc + cd + da = 4$.

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, Romania

Solution by the author

For $a = b = c$, the equality constraint becomes $c^2 + cd = 2$, where $c \in [1, \sqrt{2}]$, while the inequality becomes as follows:

$$3c^2 + d^2 - 4 \geq k(3c + d - 4), \quad 3c^2 + \frac{(2 - c^2)^2}{c^2} - 4 \geq k \left(3c + \frac{2 - c^2}{c} - 4 \right), \quad \frac{4(c^2 - 1)^2}{c^2} \geq \frac{2k(c - 1)^2}{c}.$$

It is true for all $c \in [1, \sqrt{2}]$ if and only if $\frac{2(c+1)^2}{c} \geq k$. Setting $c = 1$, we get the necessary condition $k \leq 8$. To finish the proof, we need to show that

$$2b^2 + c^2 + d^2 - 8(2b + c + d) + 28 \geq 0$$

for

$$(b + c)(b + d) = 4, \quad b \geq c \geq d \geq 0.$$

Denote

$$x = \frac{2b + c}{3}, \quad b \geq x \geq c \geq d \geq 0, \quad x \geq 1.$$

Since $2b^2 + c^2 \geq 3x^2$, it suffices to prove that

$$3x^2 + d^2 - 8(3x + d) + 28 \geq 0,$$

i.e.

$$3x^2 - 24x + 12 + (4 - d)^2 \geq 0.$$

From

$$\begin{aligned} 16 &= (2b + 2c)2b + 2d = (3x + c)(3x + 2d - c) = (3x + d + c - d)(3x + d + d - c) \\ &= (3x + d)^2 - (c - d)^2 \geq (3x + d)^2 - (x - d)^2 = 8(x^2 + dx), \end{aligned}$$

we get

$$d \leq \frac{2 - x^2}{x}, \quad 4 - d \geq 4 - \frac{2 - x^2}{x} = \frac{x^2 + 4x - 2}{x},$$

therefore

$$3x^2 - 24x + 12 + (4 - d)^2 \geq 3x^2 - 24x + 12 + \frac{(x^2 + 4x - 2)^2}{x^2} = \frac{4(x - 1)^4}{x^2} \geq 0.$$

Thus, the proof is completed. For $k = 8$, the equality occurs when $a = b = c = d = 1$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Woojin Kim, Seoul Science High School, Seoul, South Korea.

O675. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{2b^2 + ca}{a^2 + bc + ca}} + \sqrt{\frac{2c^2 + ab}{b^2 + ca + ab}} + \sqrt{\frac{2a^2 + bc}{c^2 + ab + bc}} \geq 3.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Clark College, Washington, USA

Let S denote the left-hand-side of the desired inequality, which is homogeneous. Thus, without loss of generality, we set $a + b + c = 3$ and using Hölder's inequality we get

$$S^2 \cdot \sum_c (a^2 + bc + ca)(2b^2 + ca)^2 \geq (2a^2 + 2b^2 + 2c^2 + ab + bc + ca)^3.$$

Therefore, it suffices to show that

$$(2a^2 + 2b^2 + 2c^2 + ab + bc + ca)^3 \geq 9 \cdot \sum_c (a^2 + bc + ca)(2b^2 + ca)^2. \quad (*)$$

Without loss of generality let $c \leq b \leq a$, and set $b = c + x$, and $a = c + x + y$, where x, y are nonnegative real numbers such that $3c + 2x + y = 3$, and thus $c = (3 - 2x - y)/3$. Substituting, $(*)$ becomes equivalent to

$$\begin{aligned} (x^2 + xy + y^2 + 9)^3 &\geq 3(9 + 3y + y^2 - x^2)(3 + x - y - xy)^2 \\ &\quad + 3(9 - y^2 + 3x - 2xy)(3 + x^2 + xy - 2x - y)^2 \\ &\quad + 3(9 - 3x - 3y + 2xy + x^2)(3 + x + 2y + xy + y^2)^2. \quad (**) \end{aligned}$$

Expanding and simplifying, $(**)$ is equivalent to

$$x^2[x^2(x^2 - 9x + 36) + 27x + 81] + y^2[y^4 + 9y^3 + (36y^2 - 27y + 81)] + xy \cdot A \geq 0,$$

where $A = 9x^4 + 21x^3y - 54x^3 + 7x^2y^2 - 72x^2y + 72x^2 - 9xy^3 - 54xy^2 + 108xy - 3y^4 - 9y^3 + 72y^2 - 81y + 81$. The quantities in brackets above are clearly nonnegative, and thus it suffices to show that $A \geq 0$. But

$$A = 21x^3y + 9x^2(x - 3)^2 + 3y(3 - y)^3 + 9xy(4 + y)(3 - 2x - y) + B,$$

where $B = 25x^2y^2 - 9x^2 - 45xy^2 - 36y^3 + 153y^2 - 162y + 81$. The first four terms in the expression for A above are clearly nonnegative because $3c = 3 - 2x - y \geq 0$, and thus it suffices to show that $B \geq 0$. Squaring the condition $2x + y \leq 3$ we get $9x^2 \leq (81 - 36xy - 9y^2)/4$, and thus it suffices to show that

$$25y^2 \cdot x^2 - (45y^2 - 9y) \cdot x - 36y^3 + \frac{621y^2}{4} - 162y + \frac{243}{4} \geq 0.$$

The discriminant of the above quadratic in x is $\Delta = 18y^2(200y^3 - 750y^2 + 855y - 333)$. Solving the cubic in y in the expression for Δ we get that the discriminant is negative for $0 \leq y \lesssim 2.08$; this is easily seen by writing $200y^3 - 750y^2 + 855y - 333 = 5(y - 2)(40y^2 - 70y + 31) - 23$. Thus, we only need examine the range $2 \leq y \leq 3$. In this case it suffices to show that

$$x \leq \frac{3 - y}{2} \leq \frac{45y^2 - 9y - \sqrt{\Delta}}{50y^2} \iff \sqrt{\Delta} \leq y(25y^2 - 30y - 9).$$

The right-hand-side of the above inequality is positive for $y \geq 2$ because $25y^2 - 30y - 9 = 25y(y - 2) + 20y - 9$. Squaring and simplifying we get

$$y^2(y - 3)(25y^3 - 129y^2 + 171y - 81) \geq 0,$$

which is obvious because

$$25y^3 - 129y^2 + 171y - 81 \leq -54 \iff (y - 3)[(25y - 4)(y - 2) + 1] \leq 0.$$

Equality occurs when $x = y = 0$ and thus when $a = b = c$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

O676. Let x, y, z be distinct non-zero integers such that

$$\frac{(x-1)^2}{z} + \frac{(y-1)^2}{x} + \frac{(z-1)^2}{y} = \frac{(x-1)^2}{y} + \frac{(y-1)^2}{z} + \frac{(z-1)^2}{x}.$$

Find the minimum possible value of $|64x + 19y + 4z|$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Woojin Kim, Seoul Science High School, Seoul, South Korea

Define $a = y - z$ and $b = x - y$. After moving the terms to left,

$$(x-1)^2 \left(\frac{1}{z} - \frac{1}{y} \right) + (y-1)^2 \left(\frac{1}{x} - \frac{1}{z} \right) + (x-1)^2 \left(\frac{1}{y} - \frac{1}{x} \right) = 0.$$

After multiplying by xyz , we get

$$x(x-1)^2(y-z) + y(y-1)^2(z-x) + z(z-1)^2(x-y) = 0.$$

After substituting $y - z$ to a and $x - y$ to b , we have

$$x(x-1)^2a - y(y-1)^2(a+b) + z(z-1)^2b = 0.$$

By moving the terms containing b to the right,

$$a(x(x-1)^2 - y(y-1)^2) = b(y(y-1)^2 - z(z-1)^2).$$

Because $x(x-1)^2 - y(y-1)^2 = (x-y)(x^2 + xy + y^2 - 2x - 2y + 1)$, we have

$$ab(x^2 + xy + y^2 - 2x - 2y + 1) = ab(y^2 + yz + z^2 - 2y - 2z + 1).$$

Since $ab \neq 0$, we can simplify both sides by ab , and after simplifying we obtain

$$(x-z)(x+y+z-2) = 0$$

Because $x - z \neq 0$, we have $x + y + z = 2$. The value that we want to minimize is $|60x + 15y + 8|$ because

$$|64x + 19y + 4z| = |60x + 15y + 4(x + y + z)| = |60x + 15y + 8|$$

Notice that $60x + 15y \equiv 0 \pmod{15}$, so in the expression $|15k + 8|$, the minimum is 7 when $k = -1$. Thus, the minimum is 7 and it is reached for example when $x = 1, y = -5, z = 6$.

Editor's Comment. Theo Koupelis also noticed that the minimum occurs when $(x, y, z) = (t, -4t-1, 3(t+1))$, where t is an integer such that $t \neq 0$ and $t \neq -1$.

Also solved by Theo Koupelis, Clark College, Washington, USA; Hoang Duc Hung, Hanoi University of Science, Vietnam.

O677. Prove that for any positive real numbers a, b, c

$$\frac{1}{4}(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \sqrt{\frac{2ab}{a^2+b^2}} + \sqrt{\frac{2bc}{b^2+c^2}} + \sqrt{\frac{2ca}{c^2+a^2}} \geq \frac{21}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

By the AM-GM inequality we have

$$\sqrt{\frac{2ab}{a^2+b^2}} = \frac{2ab}{\sqrt{(2ab)(a^2+b^2)}} \geq \frac{4ab}{(a+b)^2}.$$

On the other hand

$$\begin{aligned} \frac{1}{4}(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) &= \frac{(a+b+c)(ab+bc+ca)}{4abc} \\ &= \frac{(a+b)(b+c)(c+a)}{4abc} + \frac{1}{4}. \end{aligned}$$

From here and using the AM-GM inequality again we obtain

$$\begin{aligned} &\frac{1}{4}(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \sum_{\text{cyc}} \sqrt{\frac{2ab}{a^2+b^2}} \\ &\geq \frac{1}{4} + \frac{(a+b)(b+c)(c+a)}{4abc} + \sum_{\text{cyc}} \frac{4ab}{(a+b)^2} \\ &= \frac{1}{4} + \frac{(a+b)(b+c)(c+a)}{8abc} + \frac{(a+b)(b+c)(c+a)}{8abc} + \sum_{\text{cyc}} \frac{4ab}{(a+b)^2} \\ &\geq \frac{1}{4} + 5 \sqrt[5]{\frac{(a+b)(b+c)(c+a)}{8abc} \cdot \frac{(a+b)(b+c)(c+a)}{8abc} \cdot \frac{4ab}{(a+b)^2} \cdot \frac{4bc}{(b+c)^2} \cdot \frac{4ca}{(c+a)^2}} \\ &= \frac{1}{4} + 5 \\ &= \frac{21}{4}. \end{aligned}$$

The proof is completed.

Solution 2 by Hoang Duc Hung, Hanoi University of Science, Vietnam

We have

$$\begin{aligned} &\frac{1}{4}(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \sqrt{\frac{2ab}{a^2+b^2}} + \sqrt{\frac{2bc}{b^2+c^2}} + \sqrt{\frac{2ca}{c^2+a^2}} \\ &= \frac{3}{4} + \left(\frac{a^2+b^2}{4ab} + \frac{b^2+c^2}{4bc} + \frac{c^2+a^2}{4ca} \right) + \sqrt{\frac{2ab}{a^2+b^2}} + \sqrt{\frac{2bc}{b^2+c^2}} + \sqrt{\frac{2ca}{c^2+a^2}} \\ &= \frac{3}{4} + \sum_{\text{cyc}} \left[\frac{a^2+b^2}{4ab} + \sqrt{\frac{ab}{2(a^2+b^2)}} + \sqrt{\frac{ab}{2(a^2+b^2)}} \right] \\ &\geq \frac{3}{4} + \sum_{\text{cyc}} \left[3 \cdot \sqrt[3]{\frac{a^2+b^2}{4ab} \cdot \sqrt{\frac{ab}{2(a^2+b^2)}} \cdot \sqrt{\frac{ab}{2(a^2+b^2)}}} \right] \\ &= \frac{3}{4} + \sum_{\text{cyc}} \frac{3}{2} = \frac{3}{4} + 3 \cdot \frac{3}{2} = \frac{21}{4}. \end{aligned}$$

Equality holds if and only if $a = b = c$.

Solution 3 by Woojin Kim, Seoul Science High School, Seoul, South Korea

Since the left side is a cyclic sum of $\frac{1}{4} + \frac{1}{4} \left(\frac{a}{b} + \frac{b}{a} \right) + \sqrt{\frac{2ab}{a^2+b^2}}$, it suffices to show that this is greater or equal to $\frac{7}{4}$. The inequality is

$$\frac{1}{4} \left(\frac{a}{b} + \frac{b}{a} \right) + \sqrt{\frac{2ab}{a^2+b^2}} \geq \frac{3}{2}$$

On the left side, by the AM-GM inequality, this completes the solution because

$$\frac{1}{2} \left(\frac{a^2+b^2}{2ab} \right) + \frac{1}{2} \sqrt{\frac{2ab}{a^2+b^2}} + \frac{1}{2} \sqrt{\frac{2ab}{a^2+b^2}} \geq 3 \sqrt{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{3}{2}.$$

Solution 4 by Theo Koupelis, Clark College, Washington, USA

The desired inequality is equivalent to

$$\left[\frac{a}{b} + \frac{b}{a} + 4\sqrt{\frac{2ab}{a^2+b^2}} \right] + \left[\frac{b}{c} + \frac{c}{b} + 4\sqrt{\frac{2bc}{b^2+c^2}} \right] + \left[\frac{c}{a} + \frac{a}{c} + 4\sqrt{\frac{2ca}{c^2+a^2}} \right] \geq 18,$$

and thus it suffices to show that

$$f(x) := x + \frac{1}{x} + 4\sqrt{\frac{2x}{x^2+1}} \geq 6,$$

when $x > 0$. However,

$$(x-1)^4 \geq 0 \implies (x+1)^4 \geq 8x(x^2+1) \implies \frac{2x}{x^2+1} \geq \frac{16x^2}{(x+1)^4} \implies 4\sqrt{\frac{2x}{x^2+1}} \geq \frac{16x}{(x+1)^2}.$$

Therefore, it suffices to show that

$$g(x) := x + \frac{1}{x} + \frac{16x}{(x+1)^2} \geq 6 \iff x^2(x+1)^2 + (x+1)^2 + 16x^2 \geq 6x(x+1)^2 \iff (x-1)^4 \geq 0,$$

which is obvious. Equality occurs when $a = b = c$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Thanh Nam Nguyen, Tu Son Secondary School, Vietnam; Prajnanaswaroop S, Bangalore, Karnataka, India.

O678. Let $a_1, \dots, a_k$ be positive integers, and let f be a nonconstant polynomial with integer coefficients. Prove that there exists an infinite set P consisting of primes such that, for any $p \in P$, $a_1^n + \dots + a_k^n + f(n)$ is divisible by p for infinitely many positive integers n .

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution by the authors

It is well-known that the set of primes dividing the values on integers of a nonconstant polynomial with integer coefficients is infinite. Applying this to the polynomial $k + f$ we infer that there exists an infinite set P_1 of primes such that, for each $p \in P_1$, we can find an integer n_p for which $k + f(n_p)$ is divisible by p . On the other hand, the set P_2 of primes that divide at least one of $a_1, \dots, a_k$ is finite, therefore $P = P_1 \setminus P_2$ is infinite. We show that P is the set of primes required in the statement of the problem.

Indeed, let p be an arbitrary prime from P ; note again that an integer n_p exists such that $k + f(n_p) \equiv 0 \pmod{p}$, and that p does not divide any of $a_1, \dots, a_k$. According to the Chinese remainder theorem, there are infinitely many positive integers n satisfying both congruences

$$n \equiv 0 \pmod{p-1} \text{ and } n \equiv n_p \pmod{p}.$$

For any such n we have

$$a_1^n \equiv \dots \equiv a_k^n \equiv 1 \pmod{p},$$

since n is divisible by $p-1$, and, by Fermat's little theorem

$$a_1^{p-1} \equiv \dots \equiv a_k^{p-1} \equiv 1 \pmod{p}$$

(because none of $a_1, \dots, a_k$ is divisible by p). Moreover,

$$n \equiv n_p \pmod{p} \Rightarrow f(n) \equiv f(n_p) \pmod{p}.$$

Consequently, for any n satisfying the above system of congruences (and there are infinitely many such n), we have

$$a_1^n + \dots + a_k^n + f(n) \equiv k + f(n_p) \equiv 0 \pmod{p},$$

as desired.

Also solved by Woojin Kim, Seoul Science High School, Seoul, South Korea.