

## Junior Problems

**J679.** Let  $n$  be a positive integer. Find a representation of  $N = 6^{8n} + 6^{4n+1} + 1$  as the sum of the squares of three distinct positive integers.

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

**J680.** Let  $ABC$  be a triangle with incenter  $I$  and inradius  $r$ . On altitude  $AD$  point  $F$  is taken such that  $AF = r$ . Let  $E$  be the midpoint of  $BC$ . Prove that  $E, I, F$  are colinear.

*Proposed by Mihaela Berindeanu, Bucharest, România*

**J681.** Solve in positive real numbers the system of equations

$$\begin{aligned}\frac{5xy + y + 5}{xy + 1} + \frac{xy + 1}{xy + y + 1} &= \frac{37}{6} \\ \frac{3yz + z + 3}{yz + 1} + \frac{yz + z + 1}{yz + 2z + 1} &= \frac{17}{4} \\ \frac{2zx + x + 2}{zx + 1} + \frac{zx + 3x + 1}{zx + 4x + 1} &= \frac{10}{3}.\end{aligned}$$

*Proposed by Alessandro Ventullo, Milan, Italy*

**J682.** Let  $a$  and  $b$  be positive real numbers. Prove that

$$\left(\frac{a}{b} + \frac{b^2}{a^2}\right) \left(\frac{a^2}{b^2} + \frac{b}{a}\right) + \frac{9ab}{a^2 - ab + b^2} \geq 13.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

**J683.** Let  $a_0$  be a real number and let  $a_{n+1} = a_n^2 + \frac{1}{6}$ . Prove that  $a_{n+2} > a_{n-1}^2$  for all positive integers  $n$ .

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**J684.** Prove that there are no rational numbers  $x, y$  such that

$$2x^2 + 12y^2 = 1.$$

*Proposed by Dorin Andrica and George Cătălin Țurcaș, Cluj-Napoca, România*

# Senior Problems

**S679.** Find all positive integers  $n$  for which there is a positive integer  $k$  such that  $(n-1)! + 1 = (k-16)^2$  and  $(n+1)! + 9 = (k+16)^2$ .

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

**S680.** Let  $x, y, z$  be positive real numbers. Prove that

$$\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \leq \sqrt{\frac{16(x+y+z)}{3} + \frac{2(xy+yz+zx)}{x+y+z}}.$$

*Proposed by Marius Stănean, Zalău, România*

**S681.** Show that there are no integers  $x$  and  $y$  satisfying

$$x^2 + y^{1013} = 2024^{2025}.$$

*Proposed by Vlad Crişan, Switzerland*

**S682.** Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{a^2}{2bc} + \frac{b}{a+b} + \frac{c}{a+c} \geq \frac{a}{b+c} + 1.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**S683.** Let  $OA, OB, OC$  be rays in the plane such that  $OB^2 = 2OA \cdot OC$  and

$$\frac{AB^2}{OA} + \frac{BC^2}{OC} = 3(OA + OC).$$

Find  $\angle AOC$ .

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**S684.** Let  $O$  and  $P$  be points in the plane. Prove that there is a circle  $\Gamma$  of center  $O$ , whose radius is an integer and there are points  $A, B, C$  on  $\Gamma$  such that

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

*Proposed by Dorin Andrica and George Cătălin Ţurcaş, Cluj-Napoca, România*

# Undergraduate Problems

**U679.** Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1 - \ln \cos x}{x^2 + (1 - \ln \cos x)^2} dx.$$

*Proposed by Paolo Perfetti, Università degli studi “Tor Vergata”, Roma, Italy*

**U680.** Prove that 18 is the largest positive constant  $k$  such that

$$\frac{1}{ab^2 + k} + \frac{1}{bc^2 + k} + \frac{1}{ca^2 + k} \geq \frac{3}{1 + k}$$

for all  $a \geq b \geq c \geq 0$  such that  $a + b + c = 3$ .

*Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România*

**U681.** Let  $a$  and  $b$  be integers. Prove that

$$\prod_{k=1}^{n-1} \left( a^2 + b^2 + 1 - 2a \cos \frac{2k\pi}{n} - 2b \sin \frac{2k\pi}{n} \right)$$

is an integer which is a sum of two perfect squares.

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România*

**U682.** Evaluate

$$\int \frac{\sin x \cos x}{\sqrt{5 + \sin 4x}} dx.$$

*Proposed by Mihaela Berindeanu, Bucharest, România*

**U683.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function that has a primitive and let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be such a primitive. Prove that the following are equivalent:

- (i)  $(x - y)(f(x) - f(y)) \leq (x - y)^2$ , for all  $x, y \in \mathbb{R}$ .
- (ii)  $F\left(x + \frac{1}{n}\right) - 2F(x) + F\left(x - \frac{1}{n}\right) \leq \frac{1}{n^2}$ , for all  $x \in \mathbb{R}$  and all  $n \in \mathbb{N}$ .

*Proposed by Dan-Ştefan Marinescu, România*

**U684.** Find all pairs  $(a, b)$  of relatively prime positive integers with  $a > b$  and such that  $n^2$  divides  $a^n - b^n$  for only finitely many positive integers  $n$ .

*Proposed by Titu Andreescu, USA and Marian Tetiva, România*

# Olympiad Problems

**O679.** Let  $P$  be a point inside a circle  $\Gamma$  of center  $O$ . Line  $OP$  intersects circle  $\Gamma$  in points  $A$  and  $Q$ . A variable chord through  $P$  intersects  $\Gamma$  in  $B$  and  $C$ . Prove that if  $OP \leq PQ$  then  $\angle BAC \geq \pi/3$ .

*Proposed by Mircea Becheanu, Canada*

**O680.** Let  $a \geq b \geq 1 \geq c \geq 0$  such that  $ab + bc + ca = 3$ . Prove that

$$\frac{1}{3a^2 + 5} + \frac{1}{3b^2 + 5} + \frac{1}{3c^2 + 5} \leq \frac{3}{8}.$$

*Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România*

**O681.** Let  $x, y, z$  be positive real numbers. Prove that

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} \leq \sqrt{\frac{9(x+y+z)}{2(xy+yz+zx)}}.$$

*Proposed by Marius Stănean, Zalău, România*

**O682.** Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{a + b + c}{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**O683.** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove that

$$\frac{b+c}{\sqrt{bc+2a^2+2a}} + \frac{c+a}{\sqrt{ca+2b^2+2b}} + \frac{a+b}{\sqrt{ab+2c^2+2c}} \geq 2.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

**O684.** A sequence  $(a_n)_{n \geq 1}$  of positive integers is called *admissible* if for all  $n > 2$  the divisibility

$$1 + a_1 + \cdots + a_{n-1} \mid (1 + a_1)(1 + a_2) \cdots (1 + a_{n-1}) - 1$$

holds precisely when  $n$  is a prime number. Let  $f$  be a nonconstant polynomial with integer coefficients.

- (a) Can the sequence  $(a_n)_{n \geq 1}$  defined by  $a_n = f(n)$  be admissible?
- (b) Can the sequence  $(a_n)_{n \geq 1}$  defined by  $a_n = f(2^n)$  be admissible?

*Proposed by Gabriel Dospinescu, Lyon, France*