

Junior Problems

J685. Let a be a nonzero real number. Prove that

$$a^3 - \frac{1}{a^3} + a^2 + \frac{1}{a^2} - 3\left(a - \frac{1}{a}\right) = 3$$

if and only if

$$a^5 - \frac{1}{a^5} - 5\left(a^3 - \frac{1}{a^3}\right) + 11\left(a - \frac{1}{a}\right) = 1.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J686. Let a and b be positive real numbers. Prove that

$$\sqrt{x^2 - ax + a^2} + \sqrt{x^2 - bx + b^2} \geq \sqrt{a^2 + ab + b^2},$$

for all real numbers x . When does equality occur?

Proposed by Mircea Becheanu, Canada

J687. Let a, b, c be real numbers such that $ab + bc + ca = 169$ and

$$(a+1)^3 + (b+1)^3 + (c+1)^3 - 3abc = 1.$$

Find all possible values of $a + b + c$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J688. Prove that positive real numbers x, y, z are the lengths of the altitudes in a triangle if and only if

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 2 \max \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\}$$

Proposed by Dan-Ştefan Marinescu, Romania

J689. Find the least value of the function

$$f(x) = \frac{2}{x^2 - 4x + 8} + \frac{1}{x^2 - 6x + 10} - \frac{2}{x^2 - 5x + 8},$$

where x is any real number.

Proposed by An Zhenping, Xianyang Normal University, China

J690. Let ABC be a triangle. The angle bisector of $\angle CAB$ intersects BC in D and the circumcircle in X . Let Y be the orthogonal projection of C on AX and Z the orthogonal projection of D on AB . The line YZ intersects BC in E . Prove that AE is perpendicular to BC .

Proposed by Mihaela Berindeanu, Bucharest, Romania

Senior Problems

S685. Let a, b be positive real numbers. Prove that

$$\frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} \geq \frac{a+b}{2(1+ab)}.$$

Proposed by An Zhenping, Xianyang Normal University, China

S686. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{\sqrt{b^2 + c^2}} + \frac{1}{\sqrt{c^2 + a^2}} + \frac{1}{\sqrt{a^2 + b^2}} \leq \frac{3\sqrt{2}}{ab + bc + ca} + \frac{1}{\sqrt{2}}.$$

Proposed by Marius Stănean, Zalău, Romania

S687. Solve in real numbers the system of equations:

$$\sqrt{\frac{x^2 + 1}{y^2 + 1}} = x, \quad 5\sqrt{\frac{y^2 + 1}{z^2 + 1}} = 6y, \quad 7\sqrt{\frac{z^2 + 1}{x^2 + 1}} = 20z.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S688. Prove that

$$\frac{ab^2 + a + b^2}{a^2 + ab + b}$$

is an integer for infinitely many positive integers a and infinitely many positive integers b .

Proposed by Mircea Becheanu, Canada

S689. Let ABC be an acute triangle with incenter I , A_1, B_1, C_1 the feet of the altitudes and A_2, B_2, C_2 the midpoints of the sides BC, AC, AB , respectively. The line A_2I intersects AA_1 in A_3 , the line B_2I intersects BB_1 in B_3 and the line C_2I intersects CC_1 in C_3 . Prove that if $\overrightarrow{AA_3} + \overrightarrow{BB_3} + \overrightarrow{CC_3} = \vec{0}$ then ABC is equilateral.

Proposed by Mihaela Berindeanu, Bucharest, Romania

S690. Let

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b},$$

where a, b, c are positive real numbers such that at most one of them is less than 1 and $abc = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 3 \geq 2(x + y + z).$$

Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

Undergraduate Problems

U685. Let $\{a_n\}$ be the sequence defined by $a_1 = 2$ and $a_n = 2a_{n-1}^2 - 1, n = 2, 3, \dots$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1)}{a_n}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U686. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a convex function such that $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x) dx \right)^2 \leq \frac{3}{4} \int_0^1 f(x)^2 dx.$$

Proposed by Robert Rogozsan, Baia Mare, Romania

U687. Prove that

$$\int_{\frac{4}{5}}^1 \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} dx < \int_0^1 \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt{x}} dx < \int_0^{\frac{1}{4}} \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} dx.$$

Proposed by Alessandro Ventullo, Milan, Italy

U688. Evaluate

$$\lim_{n \rightarrow \infty} \left(1 + \int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx \right)^n.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

U689. Evaluate the integral:

$$\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx.$$

Vasile Mircea Popa, Sibiu, Romania

U690. Find all the complex numbers z such that

$$(\Gamma(z+1))^3 + (z-1)^3 (\Gamma(z-1))^3 = ((z+1)^3 + 3) (\Gamma(z))^3,$$

where $\Gamma(z)$ is the Gamma function.

Proposed by Alessandro Ventullo, Milan, Italy

Olympiad Problems

O685. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + 3(ab + bc + ca) \geq 2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

O686. Let a, b, c, d be positive real numbers such that

$$2 \sum_{cyc} a + \sum_{cyc} ab + 3 = abcd.$$

Prove that

$$\sqrt[4]{bcd} + \sqrt[4]{cda} + \sqrt[4]{dab} + \sqrt[4]{abc} \leq \sqrt[4]{27(1+a)(1+b)(1+c)(1+d)}$$

Proposed by An Zhenping, Xianyang Normal University, China

O687. Let x, y, z be real numbers such that $xy + yz + zx = 1$ and $x + y + z > 0$. Prove that

$$\frac{xy\sqrt{z^2+1} + yz\sqrt{x^2+1} + zx\sqrt{y^2+1}}{x+y+z} \leq \frac{2}{3}.$$

Proposed by Marius Stănean, Zalău, Romania

O688. Inside the square $ABCD$ the points E, F are taken such that $\angle FDE = \angle EBF = 45^\circ$. It is given that $\text{area } FCD + \text{area } DEA + \text{area } EFB = a^2$. Find the length of the side AB .

Proposed by Mihaela Berindeanu, Bucharest, Romania

O689. Prove that $\frac{5}{2}$ is the minimum positive value of the constant k such that

$$\sqrt{\frac{b+c}{ka+b+c}} + \sqrt{\frac{c+a}{kb+c+a}} + \sqrt{\frac{a+b}{kc+a+b}} \geq 3\sqrt{\frac{2}{k+2}}$$

holds for any nonnegative real numbers a, b, c with $a + b + c > 0$.

Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

O690. Prove that there is an infinite sequence S of primes such that for any finite subset $A \subset S$ the sum of elements of A is not a perfect power.

Proposed by Mohammadreza Ghelichkhani, Tehran, Iran