

Junior problems

J679. Let n be a positive integer. Find a representation of $N = 6^{8n} + 6^{4n+1} + 1$ as the sum of the squares of three distinct positive integers.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

Note that

$$\begin{aligned} 6^{8n} + 6^{4n+1} + 1 &= 6^{8n} + 2 \cdot 6^{4n} + 1 + 4 \cdot 6^{4n} \\ &= (6^{4n} + 1)^2 + (2 \cdot 6^{2n})^2 \\ &= (6^{4n} + 1 - 2 \cdot 6^{2n})^2 + 2 \cdot (6^{4n} + 1) \cdot (2 \cdot 6^{2n}) \\ &= (6^{4n} - 2 \cdot 6^{2n} + 1)^2 + 4 \cdot 6^{6n} + 4 \cdot 6^{2n} \\ &= ((6^{2n} - 1)^2)^2 + (2 \cdot 6^{3n})^2 + (2 \cdot 6^n)^2. \end{aligned}$$

Note that $(6^{2n} - 1)^2$ is an odd positive integer and $2 \cdot 6^{3n} > 2 \cdot 6^n$ are even numbers, hence we have just written $6^{8n} + 6^{4n+1} + 1$ as sum of the squares of three distinct positive integers.

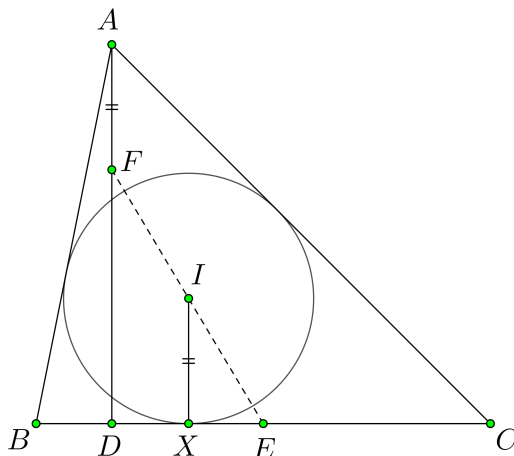
Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Israel Castillo Pilco, Universidad Nacional de Ingeniería, Peru; Ganghun Kim, The Lawrenceville School, NJ, USA; Kayla Song, Hopkins School, CT, USA; Polyhedra, Polk State College, FL, USA; Prasanna Pawar, AMMOC, Bengaluru, India; Suyeong Hahn, Choate Rosemary Hall, Connecticut, USA; William Han, Horace Mann School, Bronx, NY, USA; Yoonwoo Lee, Academic Affiliation Seoul International School, South Korea.

J680. Let ABC be a triangle with incenter I and inradius r . On altitude AD point F is taken such that $AF = r$. Let E be the midpoint of BC . Prove that E, I, F are collinear.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Kousik Sett, India

Without loss of generality (WLOG), we assume that AC is the largest side of triangle ABC . Let the incircle touch side BC at point X . We have $BE = CE = a/2$, $BX = (c + a - b)/2$, and $BD = c \cos B = (c^2 + a^2 - b^2)/(2a)$.



Also,

$$DE = BE - BD = \frac{a}{2} - \frac{c^2 + a^2 - b^2}{2a} = \frac{(b+c)(b-c)}{2a},$$

and

$$XE = BE - BX = \frac{a}{2} - \frac{c+a-b}{2} = \frac{b-c}{2}.$$

So,

$$\frac{DE}{XE} = \frac{b+c}{a}.$$

Let $AD = h$, and let Δ denotes the area of triangle ABC . Since $AF = IX = r$, we have

$$\frac{FD}{IX} = \frac{AD - AF}{IX} = \frac{h}{r} - 1 = \frac{2\Delta/a}{\Delta/s} - 1 = \frac{2s}{a} - 1 = \frac{b+c}{a}.$$

Hence,

$$\frac{DE}{XE} = \frac{FD}{IX}, \quad \text{i.e.,} \quad \frac{IX}{XE} = \frac{FD}{DE},$$

and since $\angle IXE = \angle FDE = 90^\circ$, we have triangles IXE and FDE are similar.

Therefore, $\angle XEI = \angle DEF$ and since E, X, D are collinear, we must have E, I, F are collinear.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Let a, b, c be the lengths of the sides of $\triangle ABC$ opposite A, B, C , semiperimeter s and area Δ .

If $b = c$, then F, I, E all lie on AD .

Suppose $b > c$. Let AI (extended) intersect BC at K . Then K divides BC in the ratio of the sides about A , i.e.,

$$BK = \frac{ac}{b+c}$$

and we have

$$\frac{AF}{FD} = \frac{AF}{AD - AF} = \frac{r}{\frac{2\Delta}{a} - r} = \frac{r}{\frac{2rs}{a} - r} = \frac{a}{2s - a} = \frac{a}{b+c},$$

$$\frac{DE}{EK} = \frac{BE - BD}{BE - BK} = \frac{\frac{a}{2} - c \cdot \cos B}{\frac{a}{2} - \frac{ac}{b+c}} = \frac{\frac{a}{2} - \frac{c^2+a^2-b^2}{2a}}{\frac{a(b-c)}{2(b+c)}} = \left(\frac{b+c}{a}\right)^2$$

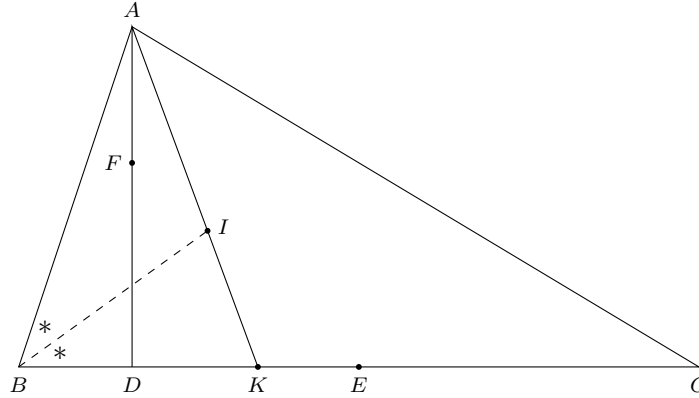
and

$$\frac{KI}{IA} = \left(\begin{array}{c} \text{by the angle bisector theorem} \\ \text{applied to } \triangle ABK \text{ at } B \end{array} \right) = \frac{BK}{AB} = \frac{\frac{ac}{b+c}}{c} = \frac{a}{b+c}.$$

Therefore

$$\frac{AF}{FD} \cdot \frac{DE}{EK} \cdot \frac{KI}{IA} = \frac{a}{b+c} \cdot \left(\frac{b+c}{a}\right)^2 \cdot \frac{a}{b+c} = 1,$$

whence F, I, E are collinear, by the converse of the celebrated Menelaus's theorem applied to the triad of points FIE on the sides of triangle DAK .



Also solved by Cristian Muth, UMFST Highschool in G. E. Palade University of Medicine, Pharmacy, Science, and Technology, Romania; Raja Oktovin Parhasian Damanik, Canberra ACT, Australia; Theo Koupelis, Clark College, Washington, USA; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Polyhedra, Polk State College, FL, USA.

J681. Solve in positive real numbers the system of equations

$$\begin{aligned}\frac{5xy + y + 5}{xy + 1} + \frac{xy + 1}{xy + y + 1} &= \frac{37}{6} \\ \frac{3yz + z + 3}{yz + 1} + \frac{yz + z + 1}{yz + 2z + 1} &= \frac{17}{4} \\ \frac{2zx + x + 2}{zx + 1} + \frac{zx + 3x + 1}{zx + 4x + 1} &= \frac{10}{3}.\end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

The given equations can be written as

$$\begin{aligned}5 + \frac{y}{xy + 1} + 1 - \frac{y}{xy + y + 1} &= 6 + \frac{1}{6} \\ 3 + \frac{z}{yz + 1} + 1 - \frac{z}{yz + 2z + 1} &= 4 + \frac{1}{4} \\ 2 + \frac{x}{zx + 1} + 1 - \frac{x}{zx + 4x + 1} &= 3 + \frac{1}{3},\end{aligned}$$

i.e.

$$\begin{aligned}\frac{1}{x + 1/y} - \frac{1}{x + 1/y + 1} &= \frac{1}{6} \\ \frac{1}{y + 1/z} - \frac{1}{y + 1/z + 2} &= \frac{1}{4} \\ \frac{1}{z + 1/x} - \frac{1}{z + 1/x + 4} &= \frac{1}{3}.\end{aligned}$$

Set $a = x + \frac{1}{y}$, $b = y + \frac{1}{z}$, $c = z + \frac{1}{x}$. We get

$$\begin{aligned}\frac{1}{a} - \frac{1}{a + 1} &= \frac{1}{6} \\ \frac{1}{b} - \frac{1}{b + 2} &= \frac{1}{4} \\ \frac{1}{c} - \frac{1}{c + 4} &= \frac{1}{3}.\end{aligned}$$

Since $a, b, c > 0$, solving each equation, we get $a = b = c = 2$. It follows that

$$\begin{aligned}x + \frac{1}{y} &= 2 \\ y + \frac{1}{z} &= 2 \\ z + \frac{1}{x} &= 2.\end{aligned}$$

Adding these three equations, we get

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 + \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^2 = 0,$$

so $x = y = z = 1$. It is easy to see that this is indeed a solution.

Solution 2 by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

By calling (1), (2) and (3) the equations of the system, note that

$$\begin{aligned}(1) &\Rightarrow 5 + \frac{y}{xy+1} + 1 = \frac{37}{6} + \frac{y}{xy+y+1} \Rightarrow \frac{1}{xy+1} = \frac{1}{6y} + \frac{1}{xy+y+1} \\(2) &\Rightarrow 3 + \frac{z}{yz+1} + 1 = \frac{17}{4} + \frac{z}{yz+2z+1} \Rightarrow \frac{1}{yz+1} = \frac{1}{4z} + \frac{1}{yz+2z+1} \\(3) &\Rightarrow 2 + \frac{x}{zx+1} + 1 = \frac{10}{3} + \frac{x}{zx+4x+1} \Rightarrow \frac{1}{zx+1} = \frac{1}{3x} + \frac{1}{zx+4x+1}.\end{aligned}$$

By the AM-HM inequality,

$$\begin{aligned}\frac{1}{xy+1} &= \frac{1}{6y} + \frac{1}{2(xy+y+1)} + \frac{1}{2(xy+y+1)} \geq \frac{9}{4xy+10y+4} \\ \frac{1}{yz+1} &= \frac{1}{4z} + \frac{1}{yz+2z+1} \geq \frac{4}{yz+6z+1} \\ \frac{1}{zx+1} &= \frac{1}{6x} + \frac{1}{6x} + \frac{1}{zx+4x+1} \geq \frac{9}{zx+16x+1}\end{aligned}$$

Rearranging,

$$\begin{aligned}4xy+10y+4 &\geq 9xy+9 \Leftrightarrow 2y \geq xy+1 \\ yz+6z+1 &\geq 4yz+4 \Leftrightarrow 2z \geq yz+1 \\ zx+16x+1 &\geq 9zx+9 \Leftrightarrow 2x \geq zx+1.\end{aligned}$$

But by multiplying the three inequalities and by using the AM-GM inequality, we have

$$8xyz = 2x \cdot 2y \cdot 2z \geq (zx+1) \cdot (xy+1) \cdot (yz+1) \geq 2\sqrt{zx} \cdot 2\sqrt{xy} \cdot 2\sqrt{yz} = 8xyz.$$

Hence, the equality must hold, and this happens if and only if $xy = yz = zx = 1$, i.e. $x = y = z = 1$. It is easy to check that $(x, y, z) = (1, 1, 1)$ is indeed a solution to our system.

Solution 3 by Diar Gashi, University of Primorska, Slovenia

We notice that the equations have the following form

$$\frac{nab+b+n}{ab+1} + \frac{ab+(t-1)b+1}{ab+tb+1} = \frac{(n+1)^2+1}{n+1}$$

for $t = \frac{4}{n-1}$, where $n = 5, 3$ and 2 respectively. And this can be simplified further as

$$\begin{aligned}0 &= \frac{nab+b+n}{ab+1} - n + \frac{ab+(t-1)b+1}{ab+tb+1} - 1 - \frac{(n+1)^2+1}{n+1} + (n+1) \\ &= \frac{b}{ab+1} - \frac{b}{ab+tb+1} - \frac{1}{n+1} \\ &= \frac{tb^2}{(ab+1)(ab+tb+1)} - \frac{1}{n+1} \\ &= -\frac{(ab+1)(ab+tb+1) - tb^2(n+1)}{(ab+1)(ab+tb+1)(n+1)} \\ &= -\frac{(ab+1)^2 + tb(ab+1) - t(n+1)b^2}{(ab+1)(ab+tb+1)(n+1)}\end{aligned}$$

The denominator is nonzero, so

$$0 = (ab+1)^2 + tb(ab+1) - t(n+1)b^2 \implies ab+1 = \frac{-tb \pm \sqrt{t^2b^2 + 4t(n+1)b^2}}{2}$$

$$\implies ab + 1 = \frac{-\frac{4}{n-1} \pm \sqrt{\frac{16}{(n-1)^2} + \frac{16(n+1)}{n-1}}}{2}b = 2\left(\frac{-1 \pm n}{n-1}\right)b.$$

Since $ab + 1 > 0 \implies ab + 1 = 2b$. So, by AM-GM and this we get that

$$b^2 + 1 \geq 2b = ab + 1 \implies b^2 \geq ab \implies b \geq a.$$

Hence, $x \geq y \geq z \geq x \implies \boxed{x = y = z = 1}$.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, VA, USA; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ganghun Kim, The Lawrenceville School, NJ, USA; Polyahedra, Polk State College, FL, USA; William Han, Horace Mann School, Bronx, NY, USA.

J682. Let a and b be positive real numbers. Prove that

$$\left(\frac{a}{b} + \frac{b^2}{a^2}\right) \left(\frac{a^2}{b^2} + \frac{b}{a}\right) + \frac{9ab}{a^2 - ab + b^2} \geq 13.$$

Proposed by An Zhenping, Xianyang National University, China

Solution 1 by Arkady Alt, San Jose, California, USA

Let $t := \frac{a}{b} + \frac{b}{a} \geq 2$. Then

$$\begin{aligned} \left(\frac{a}{b} + \frac{b^2}{a^2}\right) \left(\frac{a^2}{b^2} + \frac{b}{a}\right) + \frac{9ab}{a^2 - ab + b^2} - 13 &= \frac{a^3}{b^3} + \frac{b^3}{a^3} + \frac{9}{\frac{a}{b} + \frac{b}{a} - 1} - 11 \\ &= t^3 - 3t + \frac{9}{t-1} - 11 \\ &= \frac{(t^2 + 3t + 5)(t-2)^2}{t-1} \geq 0. \end{aligned}$$

Equality holds if and only if $a = b$.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Let $t = \frac{a}{b}$ and write the inequality to be proved as

$$\left(t + \frac{1}{t^2}\right) \left(t^2 + \frac{1}{t}\right) + \frac{9t}{t^2 - t + 1} \geq 13 \quad (t > 0). \quad (*)$$

Since

$$\left(t + \frac{1}{t^2}\right) \left(t^2 + \frac{1}{t}\right) = \frac{(t^3 + 1)^2}{t^3}, \quad 13 - \frac{9t}{t^2 - t + 1} = \frac{13t^2 - 22t + 13}{t^2 - t + 1},$$

and $t^2 - t + 1 > 0$ (because $t^2 + 1 \geq 2t > t$), $(*)$ can be written as

$$(t^3 + 1)^2(t^2 - t + 1) \geq t^3(13t^2 - 22t + 13) \quad (t > 0),$$

that is,

$$t^8 - t^7 + t^6 - 11t^5 + 20t^4 - 11t^3 + t^2 - t + 1 \geq 0 \quad (t > 0).$$

And this inequality holds because the polynomial

$$P(t) = t^8 - t^7 + t^6 - 11t^5 + 20t^4 - 11t^3 + t^2 - t + 1$$

factorizes as $P(t) = (t-1)^4(t^4 + 3t^3 + 7t^2 + 3t + 1)$. Equality holds if and only if $t = 1$, i.e. when $a = b$.

Solution 3 by Polyhedra, Polk State College, FL, USA

The inequality is equivalent to

$$\begin{aligned} 0 &\leq \frac{(a^3 + b^3)^2}{a^3b^3} - 4 + \frac{9ab}{a^2 - ab + b^2} - 9 \\ &= \frac{(a^3 - b^3)^2}{a^3b^3} - \frac{9(a-b)^2}{a^2 - ab + b^2} \\ &= (a-b)^2 (a^2 + ab + b^2) \left(\frac{a^2 + ab + b^2}{a^3b^3} - \frac{9}{a^4 + a^2b^2 + b^4} \right), \end{aligned}$$

which is clearly true since $(a^2 + ab + b^2)(a^4 + a^2b^2 + b^4) \geq 3ab \cdot 3a^2b^2 = 9a^3b^3$.

Solution 4 by Ioan Viorel Codreanu, Satulung, Maramures, Romania

The inequality is equivalent to

$$\frac{(a^3 + b^3)^2}{a^3 b^3} + \frac{9ab(a + b)}{a^3 + b^3} \geq 13.$$

Using the AM-GM Inequality, we get

$$\begin{aligned} \frac{(a^3 + b^3)^2}{a^3 b^3} + \frac{9ab}{a^2 - ab + b^2} &= \underbrace{\frac{(a^3 + b^3)^2}{4a^3 b^3} + \dots + \frac{(a^3 + b^3)^2}{4a^3 b^3}}_{4 \text{ times}} + \underbrace{\frac{ab(a + b)}{a^3 + b^3} + \dots + \frac{ab(a + b)}{a^3 + b^3}}_{9 \text{ times}} \\ &\geq 13 \sqrt[13]{\frac{(a^3 + b^3)^8}{4^4 a^{12} b^{12}} \cdot \frac{a^9 b^9 (a + b)^9}{(a^3 + b^3)^9}} \\ &= 13 \sqrt[13]{\frac{(a + b)^9}{2^8 a^3 b^3 (a^3 + b^3)}}. \end{aligned}$$

It is enough to prove that

$$(a + b)^9 \geq 2^8 a^3 b^3 (a^3 + b^3).$$

Let $s = a + b$, $p = ab$. The last inequality becomes

$$s^9 \geq 256p^3(s^2 - 3p),$$

equivalent to

$$s^8 + 256 \cdot 3p^4 \geq 256p^3 s^2.$$

Using the AM-GM Inequality, we get

$$s^8 + 256 \cdot 3p^4 = s^8 + 256p^4 + 256p^4 + 256p^4 \geq 4 \sqrt[4]{256^3 s^8 p^{12}} = 256p^3 s^2.$$

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Arshia Rahimi, Malard, Iran; Daniel Văcaru, National Economic College “Maria Teiuleanu”, Pitești, Romania; Henry Ricardo, Westchester Area Math Circle, New York, USA; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Raja Oktovin Parhasian Damanik, Canberra ACT, Australia; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Diar Gashi, University of Primorska, Slovenia; Israel Castillo Pilco, Universidad Nacional de Ingeniería, Peru; Ganghun Kim, The Lawrenceville School, NJ, USA; Kayla Song, Hopkins School, CT, USA; Suyeong Hahn, Choate Rosemary Hall, Connecticut, USA; William Han, Horace Mann School, Bronx, NY, USA; Yoonwoo Lee, Academic Affiliation Seoul International School, South Korea.

J683. Let a_0 be a real number and let $a_{n+1} = a_n^2 + \frac{1}{6}$. Prove that $a_{n+2} > a_{n-1}^2$ for all positive integers n .

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution 1 by the author

We have

$$\begin{aligned} a_n &= a_{n-1}^2 + \frac{1}{6} \\ a_{n+1} &= a_n^2 + \frac{1}{6} \\ a_{n+2} &= a_{n+1}^2 + \frac{1}{6} \end{aligned}$$

for all positive integers n . Summing up and completing squares yields

$$a_{n+2} = a_{n-1}^2 + \left(a_n - \frac{1}{2}\right)^2 + \left(a_{n+1} - \frac{1}{2}\right)^2.$$

Thus $a_{n+2} \geq a_{n-1}^2$. We cannot have equality, as that would imply $a_n = a_{n+1} = \frac{1}{2}$, yielding $\frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{6}$, a contradiction. Hence the conclusion.

Solution 2 by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

Note that by definition of the sequence and AM-GM inequality, for every $n \geq 1$,

$$\begin{aligned} a_{n+2} &= a_{n+1}^2 + \frac{1}{6} = \left(a_n^2 + \frac{1}{6}\right)^2 + \frac{1}{6} = a_n^4 + \frac{1}{3}a_n^2 + \frac{7}{36} = \left(a_n^4 + \frac{1}{9}\right) + \frac{1}{3}a_n^2 + \frac{1}{12} \\ &\geq \frac{2}{3}a_n^2 + \frac{1}{3}a_n^2 + \frac{1}{12} = a_n^2 + \frac{1}{12} = \left(a_{n-1}^2 + \frac{1}{6}\right)^2 + \frac{1}{12} = \left(a_{n-1}^4 + \frac{1}{9}\right) + \frac{1}{3}a_{n-1}^2 \\ &\geq \frac{2}{3}a_{n-1}^2 + \frac{1}{3}a_{n-1}^2 \\ &= a_{n-1}^2 \end{aligned}$$

The equalities in both AM-GM inequality applications will occur simultaneously iff $a_n^2 = a_{n-1}^2 = \frac{1}{3}$, which is impossible since $a_n^2 = (a_{n-1}^2 + \frac{1}{6})^2 = (\frac{1}{3} + \frac{1}{6})^2 = \frac{1}{4} \neq \frac{1}{3}$. So the inequality $a_{n+2} > a_{n-1}^2$ must be strict.

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J684. Prove that there are no rational numbers x, y such that

$$2x^2 + 12y^2 = 1.$$

Proposed by Dorin Andrica and George Cătălin Țurcaș, Cluj-Napoca, Romania

Solution 1 by Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA

It is easy to see that assuming $x = 0$ leads to $y = \pm 1/\sqrt{12}$, so y is irrational. Similarly, $y = 0$ gives $x = \pm 1/\sqrt{2}$, also irrational.

Suppose that there exist nonzero integers p, q, r, s with $\gcd(p, q) = 1$, $\gcd(r, s) = 1$, with $x = p/q$, $y = r/s$ and $2x^2 + 12y^2 = 1$.

Then, the initial equation can be written as

$$12q^2r^2 = s^2(q^2 - 2p^2).$$

Denoting by $v_p(n)$ the p -adic valuation of an integer n , we see that $v_3(12q^2r^2)$ is necessarily odd, since $12 = 2^2 \cdot 3^1$ and q^2r^2 is a perfect square. Since $v_3(s^2)$ is even (possibly 0), it follows that $v_3(q^2 - 2p^2)$ is necessarily odd.

In particular, $q^2 - 2p^2$ must be divisible by 3. But both p^2 and q^2 are congruent to 0 or 1 modulo 3. The only way for which $q^2 - 2p^2 \equiv 0 \pmod{3}$ is when both p and q are divisible by 3. But this contradicts the fact that $\gcd(p, q) = 1$.

We reached a contradiction and so the equation $2x^2 + 12y^2 = 1$ has no rational solutions.

Solution 2 by Polyhedra, Polk State College, USA

If there are such rational numbers x, y , then there is a least common denominator d such that $x = m/d$ and $y = n/d$. Therefore, $2m^2 + 12n^2 = d^2$, so d must be even, say, $d = 2a$. Then $m^2 + 6n^2 = 2a^2$, so m is even, say, $m = 2b$. Then $2b^2 + 3n^2 = a^2$. If n is even, then both m/d and n/d can be reduced, contradicting the fact that d is the least common denominator. Therefore, n and a must both be odd. Then $a^2 - 3n^2 \equiv -2 \pmod{8}$ while $2b^2 \equiv 0, 2 \pmod{8}$, a contradiction again.

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Senior problems

S679. Find all positive integers n for which there is a positive integer k such that $(n-1)! + 1 = (k-16)^2$ and $(n+1)! + 9 = (k+16)^2$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution 1 by the author

Clearly, k is odd. We see that $(n-1)! + 1$ is not a perfect square for $n = 1, 2, 3, 4$, while for $n = 5$ (and $k = 11$), $4! + 1 = (11-16)^2$ and $6! + 9 = (11+16)^2$. Suppose now that $n \geq 6$. We have

$$(n-1)! + (n+1)! + 10 = (k-16)^2 + (k+16)^2,$$

implying

$$(n-1)! + (n+1)! = 2k^2 + 502.$$

Because k is odd, $k^2 = 8s + 1$ for some positive integer s . The left-hand side is divisible by 16, while the right-hand side, which is equal to $8(2s + 63)$, is only divisible by 8, a contradiction. In conclusion, $n = 5$ is the only solution.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Subtracting the two given equations we get

$$(n-1)! [n(n+1) - 1] + 8 = 64k. \quad (*)$$

If $n \geq 7$, then $n-1 \geq 6$, and thus $(n-1)!$ is a multiple of 16. From $(*)$ we get that this is not possible because $16 \nmid 8$. Also, k is a positive integer, and therefore, from the second of the given equations we get $(n+1)! \geq (1+16)^2 - 9 = 280$, and thus $n \geq 5$, because $5! = 120$. Therefore, we only need examine the cases $n = 5$ and $n = 6$. For $n = 5$ we get $(k-16)^2 = 25 = 5^2$ and $(k+16)^2 = 729 = 27^2$, from which we get $k = 11$. For $n = 6$ we get $(k-16)^2 = 121 = 11^2$ and $(k+16)^2 = 5049$, which is not the square of an integer. Thus, $(n, k) = (5, 11)$ is the only acceptable solution.

Also solved by G. C. Greubel, Newport News, VA, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Sundares H. R., Shivamogga, Karnataka, India; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Ganghun Kim, The Lawrenceville School, NJ, USA; William Han, Horace Mann School, Bronx, NY, USA.

S680. Let x, y, z be positive real numbers. Prove that

$$\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \leq \sqrt{\frac{16(x+y+z)}{3} + \frac{2(xy+yz+zx)}{x+y+z}}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

There is an acute triangle ABC with sides $a = \sqrt{y+z}$, $b = \sqrt{z+x}$, $c = \sqrt{x+y}$. The inequality becomes

$$a+b+c \leq \sqrt{\frac{8(a^2+b^2+c^2)}{3} + \frac{2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)}{a^2+b^2+c^2}}.$$

We can rewrite this as

$$a+b+c \leq \sqrt{\frac{5(a^4+b^4+c^4) + 22(a^2b^2+b^2c^2+c^2a^2)}{3(a^2+b^2+c^2)}},$$

so that after squaring we get

$$3(a+b+c)^2(a^2+b^2+c^2) \leq 5(a^4+b^4+c^4) + 22(a^2b^2+b^2c^2+c^2a^2).$$

We prove that this inequality is valid in any triangle, not only in an acute triangle. Using the Ravi's substitutions i.e. $a = v+w$, $b = w+u$, $c = u+v$, $u, v, w > 0$, the inequality can be rewritten as follows

$$24(u+v+w)^2(u^2+v^2+w^2+uv+vw+wu) \leq 5 \sum_{cyc} (u+v)^4 + 22 \sum_{cyc} (v+w)^2(w+u)^2,$$

or after expanding and reducing terms, it becomes

$$u^4+v^4+w^4+uvw(u+v+w) \geq u^3(v+w)+v^3(w+u)+w^3(u+v),$$

or

$$u^2(u-v)(u-w)+v^2(v-w)(v-u)+w^2(w-u)(w-v) \geq 0,$$

which is true by Schur's Inequality. The equality holds when $x = y = z$.

Also solved by Arkady Alt, San Jose, California, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

S681. Show that there are no integers x and y satisfying

$$x^2 + y^{1013} = 2024^{2025}.$$

Proposed by Vlad Crişan, Switzerland

Solution by the author

Given that we are asked to show there are no integer solutions to the given equation, it is natural to wonder if we can find a positive integer n such that the given equation has no solutions modulo n .

We will now motivate how to choose a suitable value for n . For the given equation to not have solutions modulo n , we need $x^2 + y^{1013}$ to not cover all possible residues modulo n . Recall that if n is a positive integer and a is an integer coprime to n , then by Euler's theorem one has $a^{\phi(n)} \equiv 1 \pmod{n}$, where $\phi(n)$ stands for Euler's totient function.

Looking at the exponent 1013, a natural choice for n would be values for which 1013 is a divisor of $\phi(n)$, as then y^{1013} would cover a restricted set of values modulo n .

Notice that 1013 is prime, so we can't have $\phi(n) = 1013$. However, $2 \cdot 1013 = 2026$ and 2027 is prime, thus $\phi(2027) = 2026$. All this motivates trying $n = 2027$.

We will now show that indeed the equation has no solutions modulo 2027, and thus no integer solutions. From Euler's theorem, we know that y^{2026} is either 0 or 1 modulo 2027, and thus y^{1013} is one of $\{-1, 0, 1\}$ modulo 2027. Also, since $2024^{2026} \equiv 1 \pmod{2027}$ (again by Euler), it follows that 2024^{2025} is the inverse of 2024 modulo 2027. Since $2024 \equiv -3 \pmod{2027}$ and $2027 \equiv 2 \pmod{3}$, it follows that $2 \cdot 2027 \equiv 1 \pmod{3}$, which gives us $-3 \cdot 1351 \equiv 1 \pmod{2027}$, and thus $2024^{2025} \equiv 1351 \pmod{2027}$.

It follows that $2024^{2025} - y^{1013}$ can only be one of $\{1350, 1351, 1352\}$ modulo 2027. We need to show that none of the values in the set $\{1350, 1351, 1352\}$ is a quadratic residue modulo 2027. Notice that $3 \cdot 1351 \equiv -1 \pmod{2027}$. From the quadratic reciprocity law, we have that 3 is a quadratic residue modulo 2027, while -1 is not a quadratic residue modulo 2027, since $2027 \equiv 3 \pmod{4}$. It follows that 1351 is not a quadratic residue modulo 2027. Since $3 \cdot 1350 \equiv -4 \pmod{2027}$, and -4 is not a quadratic residue modulo 2027 (since -1 isn't), it follows that 1350 is also not a quadratic residue modulo 2027. Finally, since $3 \cdot 1352 \equiv 2 \pmod{2027}$ and we saw that 3 is a quadratic residue modulo 2027, it follows that 1352 is a quadratic residue if and only if 2 is. But 2 is not a quadratic residue, since 2027 is not ± 1 modulo 8. This concludes our proof.

S682. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{2bc} + \frac{b}{a+b} + \frac{c}{a+c} \geq \frac{a}{b+c} + 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, California, USA

By Cauchy-Schwarz inequality

$$\begin{aligned} \frac{a^2}{2bc} + \frac{b}{a+b} + \frac{c}{a+c} &= \frac{a^2}{2bc} + \frac{b^2}{ab+b^2} + \frac{c^2}{ca+c^2} \\ &\geq \frac{(a+b+c)^2}{2bc+ab+b^2+ca+c^2} \\ &= \frac{(a+b+c)^2}{(b+c)(a+b+c)} \\ &= \frac{a+b+c}{b+c} \\ &= \frac{a}{b+c} + 1. \end{aligned}$$

Equality occurs if and only if $a = b = c$.

Also solved by Daniel Văcaru, National Economic College “Maria Teiuleanu”, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Israel Castillo Pilco, Universidad Nacional de Ingeniería, Peru.

S683. Let OA, OB, OC be rays in the plane such that $OB^2 = 2OA \cdot OC$ and

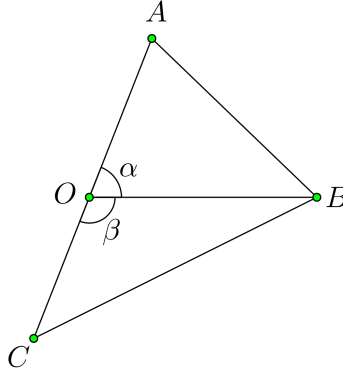
$$\frac{AB^2}{OA} + \frac{BC^2}{OC} = 3(OA + OC).$$

Find $\angle AOC$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Kousik Sett, India

Join A, B and B, C . Let $\angle AOB = \alpha$ and $\angle BOC = \beta$. Then $\angle AOC = \alpha + \beta$.



Applying the cosine law on triangles AOB and BOC , we obtain

$$\cos \alpha = \frac{OA^2 + OB^2 - AB^2}{2 \cdot OA \cdot OB} \quad \text{and} \quad \cos \beta = \frac{OC^2 + OB^2 - BC^2}{2 \cdot OC \cdot OB}.$$

Substituting $OB^2 = 2OA \cdot OC$ and using $\frac{AB^2}{OA} + \frac{BC^2}{OC} = 3(OA + OC)$, we have

$$\begin{aligned} \cos \alpha + \cos \beta &= \frac{OA}{2 \cdot OB} + \frac{OC}{OB} + \frac{OC}{2 \cdot OB} + \frac{OA}{OB} - \frac{1}{2 \cdot OB} \left(\frac{AB^2}{OA} + \frac{BC^2}{OC} \right) \\ &= \frac{3(OA + OC)}{2OB} - \frac{3(OA + OC)}{2OB} = 0. \end{aligned}$$

Hence,

$$\cos \alpha = -\cos \beta = \cos(\pi - \beta) \Rightarrow \alpha + \beta = \pi = 180^\circ.$$

Therefore,

$$\angle AOC = 180^\circ.$$

Remark. Observe that the two given conditions, when combined, can be written as

$$AB^2 \cdot OC + BC^2 \cdot OA = (OC + OA)(OB^2 + OC \cdot OA),$$

which is the Stewart's Theorem for triangle ABC .

Also solved by Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Theo Koupelis, Clark College, Washington, USA.

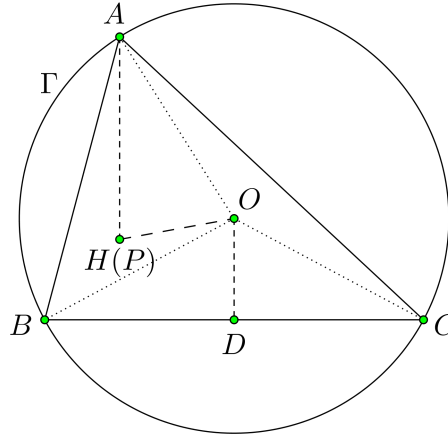
S684. Let O and P be points in the plane. Prove that there is a circle Γ of center O , whose radius is an integer and there are points A, B, C on Γ such that

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

Proposed by Dorin Andrica and George Cătălin Țurcaș, Cluj-Napoca, România

Solution 1 by Kousik Sett, India

If points A, B , and C on Γ then O is the circumcenter of triangle ABC . Let H be the orthocenter of triangle ABC and D be the midpoint of BC . It is well-known that $AH = 2OD$.



In triangle OBC , we have $\overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OD}$. Therefore,

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH}.$$

From the given condition, we have $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$. So, $P = H$.

Let $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = R$. It is known that $\angle BOC = 2A$, $\angle COA = 2B$, and $\angle AOB = 2C$. Therefore,

$$\begin{aligned} |\overrightarrow{OP}|^2 &= (\overrightarrow{OP})^2 = \overrightarrow{OP} \cdot \overrightarrow{OP} = (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) \cdot (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) \\ &= 3R^2 + 2(\overrightarrow{OB} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{OA} + \overrightarrow{OA} \cdot \overrightarrow{OB}) \\ &= 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C) \\ &= 3R^2 + 2R^2(1 - 2\sin^2 A + 1 - 2\sin^2 B + 1 - 2\sin^2 C) \\ &= 9R^2 - (a^2 + b^2 + c^2) \end{aligned}$$

where $BC = a = 2R \sin A$, $CA = b = 2R \sin B$, and $AB = c = 2R \sin C$.

Therefore,

$$a^2 + b^2 + c^2 + x^2 = 9R^2,$$

where $|OP| = x$.

Now, we demonstrate that for some integer values of R , the above equation admits acceptable solutions. We know that $R = abc/(4\Delta)$, where Δ denotes the area of triangle ABC . To find such an R , we need to find suitable triplets (a, b, c) for which the area of the triangle is an integer.

Let us choose, $a = 13k$, $b = 14k$, $c = 15k$, where k is a positive integer. Then using Heron's formula, we obtain $\Delta = 84k^2$. Hence, $R = 65k/8$. So, we need to choose $k = 8m$ where $m \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of Natural numbers. Therefore, for $m = 1$, i.e., for $k = 8$, we obtain a solution $(a, b, c, R) = (104, 112, 120, 65)$. Clearly, there exist infinitely many solutions since we can choose any integer value for m .

Other solutions can be found by choosing different triplets such as $(a, b, c) = (4k, 13k, 15k)$, $(9k, 10k, 17k)$, and so on.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $d = OP > 0$ in some units of distance. Consider the circle Γ of center O and radius r , such that r is any positive integer $\geq \lceil d \rceil$. Let C be the point of intersection of the line OP with Γ , such that P is on the segment OC . Let A, B be points on Γ , symmetrically positioned about the line OP , such that $\angle COA = \angle COB = \theta$, where $\frac{\pi}{2} \leq \theta < \frac{2\pi}{3}$. Then, in such a configuration, we get

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \iff d = 2r \cos \theta + r \iff \cos \theta = \frac{d - r}{2r}.$$

Such an angle always exists under the chosen conditions because $-\frac{1}{2} < \frac{d - r}{2r} \leq 0$.

Undergraduate problems

U679. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1 - \ln \cos x}{x^2 + (1 - \ln \cos x)^2} dx.$$

Proposed by Paolo Perfetti, Università degli studi "Tor Vergata", Roma, Italy

Solution by Theo Koupelis, Clark College, Washington, USA

For $|x| < \frac{\pi}{2}$ we have

$$1 - \ln \cos x - ix = -\ln(e^{-1+ix} \cos x) = -\ln\left(e^{-1+ix} \cdot \frac{e^{ix} + e^{-ix}}{2}\right) = -\ln \frac{1 + e^{i2x}}{2e}$$

Thus,

$$\begin{aligned} I &:= \int_0^{\frac{\pi}{2}} \frac{1 - \ln \cos x}{x^2 + (1 - \ln \cos x)^2} dx = - \int_0^{\frac{\pi}{2}} \Re \frac{1}{\ln \frac{1+e^{i2x}}{2e}} dx = -\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Re \frac{1}{\ln \frac{1+e^{i2x}}{2e}} dx \\ &= -\frac{1}{4} \text{PV} \int_{-\pi}^{\pi} \Re \frac{1}{\ln \frac{1+e^{ix}}{2e}} dx = -\frac{1}{4} \Re \text{PV} \int_{-\pi}^{\pi} \frac{1}{\ln \frac{1+e^{ix}}{2e}} dx. \end{aligned}$$

As a contour of integration we choose the counterclockwise contour that consists of a circle C_ϵ of radius 1, centered at the origin, and a semicircular indent γ_ϵ around -1 . We choose the branch cut to be the negative real axis. Let $z = e^{ix}$, and

$$f(z) = \frac{1}{z \ln \frac{1+z}{2e}}.$$

Then

$$\begin{aligned} I &= -\frac{1}{4} \Re \lim_{\epsilon \rightarrow 0^+} \int_{-\pi+\epsilon}^{\pi-\epsilon} \frac{1}{\ln \frac{1+e^{ix}}{2e}} dx = -\frac{1}{4} \Im \lim_{\epsilon \rightarrow 0^+} \left(\oint_{C_\epsilon} - \int_{\gamma_\epsilon} \right) \frac{dz}{z \ln \frac{1+z}{2e}} \\ &= -\frac{1}{4} \Im (2\pi i \cdot \text{Res}_{z=0} f(z) - 0) = -\frac{1}{4} \Im \left(2\pi i \cdot \frac{-1}{\ln(2e)} \right) = \frac{\pi}{2 \ln(2e)}, \end{aligned}$$

where we used the result that for $z = -1 + \epsilon \cdot e^{i\theta}$ we have

$$\lim_{\epsilon \rightarrow 0^+} \int \frac{dz}{z \ln \frac{1+z}{2e}} = \lim_{\epsilon \rightarrow 0^+} \int \frac{\epsilon i e^{i\theta} d\theta}{(-1 + \epsilon e^{i\theta}) \ln \frac{\epsilon e^{i\theta}}{2e}} = 0.$$

Also solved by Prasanna Pawar, AMMOC, Bengaluru, India.

U680. Prove that 18 is the largest positive constant k such that

$$\frac{1}{ab^2 + k} + \frac{1}{bc^2 + k} + \frac{1}{ca^2 + k} \geq \frac{3}{1 + k}$$

for all $a \geq b \geq c \geq 0$ such that $a + b + c = 3$.

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România

Solution by the author

Setting $a = b = \frac{3}{2}$ and $c = 0$, the inequality leads to $k \leq 18$. We will further show that the inequality is true for $k = 18$. Let us denote $p = a + b + c$, $q = ab + bc + ca$, $r = abc$ and

$$A = a^2b + b^2c + c^2a, \quad B = ab^2 + bc^2 + ca^2.$$

Since $p = 3$, we have $q \leq p^2/3 = 3$ and $r \leq p^3/27 = 1$. By expanding, the inequality can be restated as follows:

$$3k^2 \geq 3r^3 + (2k - 1)rA + k(k - 2)B,$$

$$6k^2 \geq 6r^3 + [(2k - 1)r + k(k - 2)](A + B) + [(2k - 1)r - k(k - 2)](A - B).$$

Since $(2k - 1)r - k(k - 2) \leq 2k - 1 - k(k - 2) = -k^2 + 4k - 1 < 0$ and $A - B = (a - b)(b - c)(a - c) \geq 0$, it suffices to show that

$$6k^2 \geq 6r^3 + [(2k - 1)r + k(k - 2)](A + B),$$

i.e.

$$6k^2 \geq 6r^3 + [(2k - 1)r + k(k - 2)](pq - 3r),$$

$$648 \geq 2r^3 + (35r + 288)(q - r).$$

Case 1: $0 \leq q \leq 9/4$. It suffices to show that

$$648 \geq 2r^3 + (35r + 288) \left(\frac{9}{4} - r \right),$$

which is equivalent to

$$r(837 + 140r - 8r^2) \geq 0.$$

Case 2: $9/4 \leq q \leq 3$. Let $z = q/3 \in [3/4, 1]$. For fixed z , we need to show that $648 \geq f(r)$, where $f(r) = 2r^3 + (35r + 288)(3z - r)$. By the fourth degree Schur's inequality, we have

$$r \leq \frac{(p^2 - q)(4q - p^2)}{6p} = \frac{(3 - z)(4z - 3)}{2} := r_0.$$

Since $f'(r) = 6r^2 - 70r + 105z - 288 \leq 6 - 70r + 105 - 288 < 0$, $f(r)$ is decreasing. So, we only need to show that $648 \geq f(r_0)$, i.e.

$$648 - 864z \geq 2r_0^3 - 35r_0^2 - (288 - 105z)r_0.$$

Since

$$648 - 864z = -216(4z - 3) = \frac{-432r_0}{3 - z},$$

we need to show that

$$\frac{-432}{3 - z} \geq 2r_0^2 - 35r_0 - (288 - 105z),$$

that is

$$\frac{-864}{3 - z} \geq 16z^4 - 120z^3 + 437z^2 - 585z - 180,$$

$$16z^5 - 168z^4 + 797z^3 - 1896z^2 + 1575z - 324 \geq 0,$$

$$(z - 1)g(z) \geq 0,$$

where

$$g(z) = 16z^4 - 152z^3 + 645z^2 - 1251z + 324.$$

The inequality holds if $g(z) \leq 0$. Indeed,

$$g(z) < 645z^2 - 1251z + 324 = -645z(1 - z) - 151(4z - 3) - 2z - 129 < 0.$$

For $k = 18$, the equality occurs when $a = b = c = 1$, and also when $a = b = \frac{3}{2}$ and $c = 0$.

Also solved by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA.

U681. Let a and b be integers. prove that

$$\prod_{k=1}^{n-1} \left(a^2 + b^2 + 1 - 2a \cos \frac{2k\pi}{n} - 2b \sin \frac{2k\pi}{n} \right)$$

is an integer which is a sum of two perfect squares.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let $z = a + bi$ and $\omega_k = \exp(i \cdot 2k\pi/n)$. Then

$$\begin{aligned} (z - \omega_k) \overline{(z - \omega_k)} &= \left(a - \cos \frac{2k\pi}{n} \right)^2 + \left(b - \sin \frac{2k\pi}{n} \right)^2 \\ &= a^2 + b^2 + 1 - 2a \cos \frac{2k\pi}{n} - 2b \sin \frac{2k\pi}{n}. \end{aligned}$$

It follows that

$$\begin{aligned} \prod_{k=1}^{n-1} \left(a^2 + b^2 + 1 - 2a \cos \frac{2k\pi}{n} - 2b \sin \frac{2k\pi}{n} \right) &= \prod_{k=1}^{n-1} (z - \omega_k) \overline{(z - \omega_k)} \\ &= \frac{\prod_{k=1}^n (z - \omega_k) \overline{(z - \omega_k)}}{(z - 1) \overline{(z - 1)}} \\ &= \frac{(z^n - 1) \overline{(z^n - 1)}}{(z - 1) \overline{(z - 1)}} \\ &= p_n(z) \overline{p_n(z)} \\ &= (\operatorname{Re}(p_n(z)))^2 + (\operatorname{Im}(p_n(z)))^2, \end{aligned}$$

where $p_n(z) = 1 + z + z^2 + \cdots + z^{n-1}$. Because a and b are integers, both the real part and the imaginary part of $p_n(z)$ are also integers. Thus,

$$\prod_{k=1}^{n-1} \left(a^2 + b^2 + 1 - 2a \cos \frac{2k\pi}{n} - 2b \sin \frac{2k\pi}{n} \right)$$

is an integer which is a sum of two perfect squares.

Also solved by Theo Koupelis, Clark College, Washington, USA; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Daniel Pascuas, Barcelona, Spain.

U682. Evaluate

$$\int \frac{\sin x \cos x}{\sqrt{5 + \sin 4x}} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Mingcan Fan, Huizhou University, Huizhou, China

Using substitution $x = \frac{\pi}{8} + \frac{t}{2}$, i.e., $t = 2x - \frac{\pi}{4}$, we have

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{5 + \sin 4x}} dx &= \int \frac{\sin 2x}{2\sqrt{5 + \sin 4x}} dx = \frac{1}{4\sqrt{2}} \int \frac{\sin t + \cos t}{\sqrt{5 + \cos 2t}} dt \\ &= \frac{1}{4\sqrt{2}} \left(\int \frac{d(\sin t)}{\sqrt{6 - 2\sin^2 t}} dt - \int \frac{d(\cos t)}{\sqrt{4 + 2\cos^2 t}} dt \right) \\ &= \frac{1}{8} \arcsin \frac{\sin t}{\sqrt{3}} - \frac{1}{8} \ln \frac{\cos t + \sqrt{2 + \cos^2 t}}{\sqrt{2}} + C \\ &= \frac{1}{8} \arcsin \frac{\sin(2x - \frac{\pi}{4})}{\sqrt{3}} - \frac{1}{8} \ln \left[\cos \left(2x - \frac{\pi}{4} \right) + \sqrt{2 + \cos^2 \left(2x - \frac{\pi}{4} \right)} \right] + C. \end{aligned}$$

where the following formula

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \sec u du = \ln |\tan u + \sec u| + C = \ln |x + \sqrt{1+x^2}| + C$$

is used by using substitution $x = \tan u$.

Solution 2 by Diar Gashi, University of Primorska, Slovenia

By Trigonometry identities,

$$\begin{aligned} I &:= \int \frac{\sin(x) \cos(x)}{\sqrt{5 + \sin(4x)}} dx = \int \frac{\sin(2x)}{2\sqrt{4 + \sin(2x)^2 + \cos(2x)^2 + 2\sin(2x) \cos(2x)}} dx \\ &= \frac{1}{2} \int \frac{\sin(2x)}{\sqrt{4 + (\sin(2x) + \cos(2x))^2}} dx \\ &= \frac{1}{2} \int \frac{\sin(2x)}{\sqrt{4 + (\sqrt{2} \sin(2x + \frac{\pi}{4}))^2}} dx \\ &= \frac{1}{2\sqrt{2}} \int \frac{\sin(2x)}{\sqrt{2 + (\sin(2x + \frac{\pi}{4}))^2}} dx. \end{aligned}$$

Substituting $u = 2x + \frac{\pi}{4} \implies du = 2dx$ and therefore

$$I = \frac{1}{2\sqrt{2}} \int \frac{\sin(u - \frac{\pi}{4})}{\sqrt{2 + \sin^2(u)}} \frac{du}{2} = \frac{1}{4\sqrt{2}} \int \frac{\frac{1}{\sqrt{2}}(\sin(u) - \cos(u))}{\sqrt{2 + \sin^2(u)}} du = \frac{1}{8} \int \frac{\sin(u) - \cos(u)}{\sqrt{2 + \sin^2(u)}} du.$$

Below, substituting $-\cos(u) = \sqrt{3}t \implies \sin(u)du = \sqrt{3}dt$, we see that

$$\begin{aligned} I_1 &:= \int \frac{\sin(u)}{\sqrt{2 + \sin^2(u)}} du = \int \frac{\sin(u)}{\sqrt{3 - \cos^2(u)}} du = \int \frac{\sqrt{3}dt}{\sqrt{3}\sqrt{1-t^2}} = \int \frac{dt}{\sqrt{1-t^2}} \\ &= \arcsin(t) + C_1 \\ &= \arcsin\left(\frac{-\cos(u)}{\sqrt{3}}\right) + C_1 \end{aligned}$$

Also, substituting below $\sin(u) = \sqrt{2}t \implies \cos(u) = \sqrt{2}dt$, we get

$$I_2 := \int \frac{\cos(u)}{\sqrt{2 + \sin^2(u)}} du = \int \frac{\sqrt{2}dt}{\sqrt{2 + 2t^2}} = \int \frac{dt}{\sqrt{1 + t^2}} = \sinh^{-1}(t) + C_1 = \sinh^{-1}\left(\frac{\sin(u)}{\sqrt{2}}\right) + C_2$$

Thus, in total we get

$$I = \frac{1}{8} \left(\arcsin\left(\frac{-\cos(u)}{\sqrt{3}}\right) - \sinh^{-1}\left(\frac{\sin(u)}{\sqrt{2}}\right) \right) + C.$$

$$\implies \boxed{I = -\frac{1}{8} \left(\arcsin\left(\frac{\cos(2x + \frac{\pi}{4})}{\sqrt{3}}\right) + \sinh^{-1}\left(\frac{\sin(2x + \frac{\pi}{4})}{\sqrt{2}}\right) \right) + C.}$$

Also solved by Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, VA, USA; Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil.

U683. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that has a primitive and let $F : \mathbb{R} \rightarrow \mathbb{R}$ be such a primitive. Prove that the following are equivalent:

(i) $(x - y)(f(x) - f(y)) \leq (x - y)^2$, for all $x, y \in \mathbb{R}$.

(ii) $F\left(x + \frac{1}{n}\right) - 2F(x) + F\left(x - \frac{1}{n}\right) \leq \frac{1}{n^2}$, for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Proposed by Dan-Ştefan Marinescu, Romania

Solution by Daniel Pascuas, Barcelona, Spain

First, note that (i) can be written as

$$f(x) - f(y) \leq x - y, \text{ for all } x, y \in \mathbb{R} \text{ such that } x \geq y,$$

or equivalently

(i') $y - f(y) \leq x - f(x)$, for all $x, y \in \mathbb{R}$ such that $x \geq y$.

Since F is a primitive of f , (i') says that the derivative of the function $G(x) = \frac{x^2}{2} - F(x)$ is nondecreasing. It follows that (i) means that G is a convex function. On the other hand, it is clear that (ii) can be written as

(ii') $0 \leq G(x + \frac{1}{n}) - 2G(x) + G(x - \frac{1}{n})$, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Now let us prove the equivalence of (i) and (ii).

(i) $\Rightarrow$ (ii) By (i), G is a convex function, so Jensen's inequality gives that

$$G(x) \leq \frac{G(x + h) + G(x - h)}{2} \quad (x \in \mathbb{R}, h > 0),$$

which can be written as

$$F(x + h) - 2F(x) + F(x - h) \leq h^2 \quad (x \in \mathbb{R}, h > 0).$$

Putting $h = \frac{1}{n}$, $n \in \mathbb{N}$, in this inequality we obtain (ii).

(ii) $\Rightarrow$ (i) We want to prove that (ii') implies that G is convex. To do that we need the following

THEOREM. *A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is convex if and only if it is continuous and $\overline{D}^2 g(x) \geq 0$, for all $x \in \mathbb{R}$, where $\overline{D}^2 g$ denotes the upper second symmetric derivative of g defined by*

$$\overline{D}^2 g(x) := \limsup_{h \rightarrow 0^+} \frac{g(x + h) - 2g(x) + g(x - h)}{h^2} \quad (x \in \mathbb{R}).$$

(See Theorem 1.4.7 in C. P. Niculescu and L.-P. Persson, *Convex Functions and Their Applications - A contemporary approach*, Second Edition, CMS Books in Mathematics, Springer, 2018.)

Since we already know that G is continuous (because it is differentiable), we only have to show that (ii') implies that $\overline{D}^2 G(x) \geq 0$, for all $x \in \mathbb{R}$, and that is easy:

$$\overline{D}^2 G(x) \geq \limsup_{n \rightarrow \infty} \frac{G(x + \frac{1}{n}) - 2G(x) + G(x - \frac{1}{n})}{\frac{1}{n^2}} \geq 0 \quad (x \in \mathbb{R}).$$

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Theo Koupelis, Clark College, Washington, USA.

U684. Find all pairs (a, b) of relatively prime positive integers a and b with $a > b$, and such that n^2 divides $a^n - b^n$ for only finitely many positive integers n .

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution by the authors

We show that n^2 divides $a^n - b^n$ (whenever a and b are relatively prime positive integers for which $a > b$) for only finitely many n if and only $a - b = 1$, so the required pairs are those of the form $(x + 1, x)$ for positive integer x .

For the "if" part, assume that $a - b = 1$, and let $n > 1$ be a positive integer such that n^2 divides $a^n - b^n$ (clearly, the divisibility $n^2 \mid a^n - b^n$ holds for $n = 1$), and let p be the smallest prime divisor of n . Note that $a - b = 1$ implies that a and b have different parities, so n is odd — and, consequently, p is odd, too. We have $a^n \equiv b^n \pmod{p}$ and, also, $a^{p-1} \equiv b^{p-1} (\equiv 1) \pmod{p}$, according to Fermat's little theorem. (Note that neither a , nor b is divisible by p , since if any of them was so, the other would be so, too, via $p \mid a^n - b^n$ — which is not possible as long as a and b are relatively prime.) Consequently (with $\gcd(x, y)$ denoting the greatest common divisor of the integers x and y), we get

$$a^{\gcd(n, p-1)} \equiv b^{\gcd(n, p-1)} \pmod{p},$$

that is (because, n being odd and p being its smallest prime divisor, we have $\gcd(n, p-1) = 1$), $a \equiv b \pmod{p}$. This is evidently absurd, so (in the case of $a - b = 1$) the divisibility of $a^n - b^n$ by n^2 only holds for $n = 1$. For the "only if" part we further show that, whenever a and b are relatively prime positive integers such that $a - b \geq 2$, we can find infinitely many positive integers n such that n^2 divides $a^n - b^n$. In order to do that we use one case of *lifting the exponent lemma*, whose proof we will skip (as it can already be considered well-known), and one more helping result. Here and further $v_p(N)$ denotes the exponent of the prime p in the prime factorization of the nonzero integer N .

Lemma 1. (*Lifting the exponent lemma*) Let a and b be distinct integers, and let p be an odd prime that divides $a - b$, but it does not divide any of a and b . Then we have

$$v_p(a^n - b^n) = v_p(a - b) + v_p(n).$$

Note that if a and b are relatively prime, we do not need to assume that p does not divide a , or b . And here is the second result, followed by its proof.

Lemma 2. Let $a > b$ be relatively prime positive integers, and let p be an odd prime that divides $a - b$. Then we can find an odd prime q such that q divides $a^p - b^p$, but it does not divide $a - b$. In particular, we have $q \neq p$.

Proof. By lifting the exponent lemma, we have $v_p(a^p - b^p) = v_p(a - b) + 1$, and $v_\pi(a^p - b^p) = v_\pi(a - b)$ for any prime $\pi \neq p$ that divides $a - b$. (Note that neither p , nor π divides a , or b , since if it did, a and b would not be relatively prime anymore. Also, observe that, for $\pi = 2$, the equality $v_2(a^p - b^p) = v_2(a - b)$ follows from the fact that $a^{p-1} + a^{p-2}b + \dots + ab^{p-2} + b^{p-1}$ is odd as long as a and b do not share the same parity; and they don't — see the final of the proof.) We can also write this equalities as

$$v_p\left(\frac{a^p - b^p}{a - b}\right) = 1 \quad \text{and} \quad v_\pi\left(\frac{a^p - b^p}{a - b}\right) = 0.$$

These observations imply that

$$\frac{a^p - b^p}{a - b} = pt,$$

where t is a positive integer not divisible by any prime divisor of $a - b$ (including p).

Since $p \geq 3$, $b > 0$, and $p \mid a - b$, we also have

$$pt = \frac{a^p - b^p}{a - b} = a^{p-1} + a^{p-2}b + \dots + ab^{p-2} + b^{p-1} > a^{p-1} \geq a > a - b \geq p$$

that is, $t > 1$. But then t surely has a prime divisor q , which is clearly different from all divisors of $a - b$. Also, since

$$\frac{a^p - b^p}{a - b} = a^{p-1} + a^{p-2}b + \dots + ab^{p-2} + b^{p-1}$$

is odd whenever a and b do not have the same parity (which is the case here, since they are relatively prime), q is also odd, and the lemma is proved.

Now let a and b be relatively prime positive integers with $a - b \geq 2$, and consider first that $a - b$ has an odd prime divisor p_1 . Then, by lifting the exponent lemma, $a^{p_1} - b^{p_1}$ is divisible by p_1^2 , while, by lemma 2, $a^{p_1} - b^{p_1}$ has an odd prime divisor $p_2 \neq p_1$ (actually distinct from any prime divisor of $a - b$). Again, lifting the exponent lemma tells us that $a^{p_1 p_2} - b^{p_1 p_2}$ is divisible by p_2^2 ; but $a^{p_1 p_2} - b^{p_1 p_2}$ is divisible by $a^{p_1} - b^{p_1}$, so it is divisible by p_1^2 too, consequently $p_1^2 p_2^2$ divides $a^{p_1 p_2} - b^{p_1 p_2}$. So far we found two distinct solutions for $n^2 \mid a^n - b^n$, namely $n = p_1$ and $n = p_1 p_2$. Applying again lemma 2 for a^{p_1} , b^{p_1} , and p_2 instead of a , b , and p respectively we get the existence of a prime p_3 that does not divide $a^{p_1} - b^{p_1}$ (so it differs from p_1), it is different from p_2 , and divides $a^{p_1 p_2} - b^{p_1 p_2}$. By lifting the exponent lemma once again we have that p_3^2 divides $a^{p_1 p_2 p_3} - b^{p_1 p_2 p_3}$; but $p_1^2 p_2^2$ also divides $a^{p_1 p_2 p_3} - b^{p_1 p_2 p_3}$ (because it divides $a^{p_1 p_2} - b^{p_1 p_2}$, which, on its turn, is a divisor of $a^{p_1 p_2 p_3} - b^{p_1 p_2 p_3}$), so for $n = p_1 p_2 p_3$ we have again $n^2 \mid a^n - b^n$. Clearly, $p_1 < p_1 p_2 < p_1 p_2 p_3$, thus we have by now three solutions for $n^2 \mid a^n - b^n$. Moreover, we can continue this procedure (in a similar manner) in order to get an infinite sequence $p_1, p_2, \dots$ of distinct odd primes such that each $n = p_1 \cdots p_j$ ($j = 1, 2, \dots$) verifies $n^2 \mid a^n - b^n$ — meaning that we achieved our goal in this case.

It remains to prove the same (that $n^2 \mid a^n - b^n$ for infinitely many n) when $a - b \geq 2$ has no odd prime divisor. This means that a and b are now relatively prime positive integers such that $a - b = 2^k$ for some positive integer k . Notice that a and b are necessarily odd, so we have $a^2 \equiv b^2 \equiv 1 \pmod{4}$, yielding that $a^2 - b^2$ (and, also, any $a^{2m} - b^{2m}$ for positive integer m) is divisible by 4. If $a + b$ has an odd prime divisor p_1 , then $a^2 - b^2$ is divisible by p_1 , thus we can apply the same procedure as above to a^2 and b^2 in the place of a and b respectively in order to get infinitely many distinct odd primes $p_1, p_2, \dots$ such that $(p_1 \cdots p_j)^2$ divides $(a^2)^{p_1 \cdots p_j} - (b^2)^{p_1 \cdots p_j}$ for all $j = 1, 2, \dots$. But this shows that for infinitely many n (namely $n = 2p_1 \cdots p_j$ with $j = 1, 2, \dots$) we have $n^2 \mid a^n - b^n$, as desired.

Finally, assume that $a - b = 2^k$ for some $k \geq 1$ and $a + b$ has no odd prime divisor, that is $a + b = 2^l$ for some positive integer l . Clearly $l > k$, so $k \geq 1$ implies $l \geq 2$. Then $a = 2^{l-1} + 2^{k-1}$, $b = 2^{l-1} - 2^{k-1}$, and hence

$$a^4 - b^4 = (a - b)(a + b)(a^2 + b^2) = 2^k \cdot 2^l \cdot (2^{2l-1} + 2^{2k-1}) = 2^{3k+l-1}(2^{2l-2k} + 1),$$

implying

$$v_2(a^4 - b^4) = 3k + l - 1 \geq 3 + 2 - 1 = 4.$$

This shows that $a^4 - b^4$ (and any $a^{4m} - b^{4m}$ with positive integer m) is divisible by 16 and, also, that $a^4 - b^4$ has an odd prime divisor p_1 (because its factor $2^{2l-2k} + 1$ is odd and greater than 1). Consequently, proceeding analogously as above for a^4 and b^4 instead of a and b , we obtain a sequence of distinct primes $p_1, p_2, \dots$ such that every $(p_1 \cdots p_j)^2$ divides $(a^4)^{p_1 \cdots p_j} - (b^4)^{p_1 \cdots p_j}$ ($j = 1, 2, \dots$). But this means that n^2 divides $a^n - b^n$ for every $n = 4p_1 \cdots p_j$ ($j = 1, 2, \dots$), and our proof ends here.

Olympiad problems

O679. Let P be a point inside a circle Γ of center O . Line OP intersects circle Γ in points A and Q . A variable chord through P intersects Γ in B and C . Prove that if $OP \leq PQ$ then $\angle BAC \geq \pi/3$.

Proposed by Mircea Becheanu, Canada

Solution 1 by the author

The solution is based on the following well known fact: in a given circle, the longest chord cuts the longest circular arc. When we speak about an arc cut by a chord we mean the smallest arc. We also mention the following result:

Lemma. Let AQ be a diameter in a circle and P a point on AQ . The shortest chord through P is that one which is perpendicular to the diameter AQ .

Proof of the Lemma. Assume that P is between O and Q . Let KL be the perpendicular in P to AQ and let MN be a chord through P such that M lies on the shortest arc cut by KL , between K and Q . The triangles KMP and NLP have equal angles, so they are similar. Therefore we have

$$\frac{KM}{LN} = \frac{MP}{PL} = \frac{KP}{PN}.$$

Because $PL < PN$ and $KP = PL$ we have

$$\frac{KM}{LN} = \frac{KP}{PN} < 1,$$

hence $KM < LN$. It follows that $\text{arc}(KM) < \text{arc}(LN)$ giving that $\text{arc}(KQL) < \text{arc}(MQN)$ and $KL < MN$. Back to the problem, assume for beginning that $OP = PQ$ and let BC the chord through P , perpendicular to AQ . Then the triangle ABC is equilateral and $\angle BAC = \pi/3$. If $B'C'$ is another chord through P we have $B'C' > BC$ and then $\angle B'AC' > \angle BAC = \pi/3$.

Assume that $OP < PQ$. Using the same notations like in the previous argument we have $\angle BAC > \pi/3$ and similarly $\angle B'AC' > \pi/3$.

Solution 2 by Kousik Sett, India

Join $A, B; A, C; O, B; O, C; Q, B;$ and Q, C . Let $OP = x$, $PQ = y$, $BC = a$, $CA = b$, and $AB = c$. Then $OQ = x + y = OA = OB = OC$, $AP = 2x + y$ and $AQ = 2(x + y)$.

Since AQ is the diameter of $\odot(ABC)$, we have $\angle ABQ = \angle ACQ = 90^\circ$. Therefore,

$$AB^2 + BQ^2 = AC^2 + CQ^2 = AQ^2 \quad \text{i.e.,} \quad c^2 + BQ^2 = b^2 + CQ^2 = 4(x + y)^2.$$

Since $BQ > 0$ and $CQ > 0$, we have $4(x + y)^2 > c^2$ and $4(x + y)^2 > b^2$. So,

$$4(x + y)^2 - bc > 0. \tag{1}$$

Moreover,

$$BQ^2 = 4(x + y)^2 - c^2 \quad \text{and} \quad CQ^2 = 4(x + y)^2 - b^2, \tag{2}$$

which yields

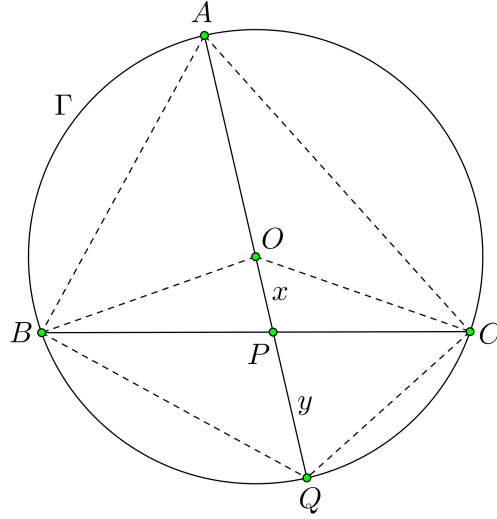
$$BQ^2 \cdot CQ^2 = 16(x + y)^4 - (b^2 + c^2) \cdot 4(x + y)^2 + b^2 c^2.$$

Since $b^2 + c^2 \geq 2bc$, we have

$$BQ^2 \cdot CQ^2 \leq (4(x + y)^2 - bc)^2.$$

By (1), it follows that

$$BQ \cdot CQ \leq 4(x + y)^2 - bc. \tag{3}$$



The similarity of triangles ABP and CQP yields

$$\frac{AB}{CQ} = \frac{AP}{CP} \Rightarrow \frac{c}{CQ} = \frac{2x+y}{CP}.$$

The similarity of triangles ACP and BQP yields

$$\frac{AC}{BQ} = \frac{CP}{PQ} \Rightarrow \frac{b}{BQ} = \frac{CP}{y}.$$

The above two relations give

$$\frac{bc}{BQ \cdot CQ} = \frac{2x+y}{y} \Rightarrow BQ \cdot CQ = bc \cdot \left(\frac{y}{2x+y} \right). \quad (4)$$

From (3) and (4), we have

$$bc \leq \left(\frac{2x+y}{y} \right) (4(x+y)^2 - bc),$$

which implies

$$bc \leq 2(x+y)(2x+y). \quad (5)$$

Applying Ptolemy's Theorem on cyclic quadrilateral $ABQC$, we get

$$c \cdot CQ + b \cdot BQ = a \cdot AQ.$$

By squaring and using (2) and (4), and simplifying, we obtain

$$\begin{aligned} & c^2 \cdot CQ^2 + b^2 \cdot BQ^2 + 2bc \cdot BQ \cdot CQ = a^2 \cdot AQ^2 \\ \Rightarrow & c^2(AQ^2 - b^2) + b^2(AQ^2 - c^2) + 2bc \cdot bc \left(\frac{y}{2x+y} \right) = a^2 \cdot AQ^2 \\ \Rightarrow & AQ^2(b^2 + c^2 - a^2) = 2b^2c^2 - 2b^2c^2 \left(\frac{y}{2x+y} \right) = 2b^2c^2 \left(\frac{2x}{2x+y} \right) \\ \Rightarrow & \frac{b^2 + c^2 - a^2}{bc} = 2bc \left(\frac{2x}{2x+y} \right) \left(\frac{1}{4(x+y)^2} \right). \end{aligned}$$

Using (5), the above equation can be written as

$$\frac{b^2 + c^2 - a^2}{bc} \leq 4(x+y)(2x+y) \left(\frac{2x}{2x+y} \right) \left(\frac{1}{4(x+y)^2} \right) = \frac{2x}{x+y}.$$

Since $OP \leq PQ$, i.e., $x \leq y$, we have $\frac{2x}{x+y} \leq 1$, which yields

$$\frac{b^2 + c^2 - a^2}{bc} \leq 1.$$

Therefore, we have

$$\frac{b^2 + c^2 - a^2}{2bc} \leq \frac{1}{2}, \quad \text{i.e.,} \quad \cos \angle BAC \leq \cos \pi/3,$$

which gives $\angle BAC \geq \pi/3$ and we are done!

Observe that, the equality holds when $b = c$ and $x = y$. These two conditions together imply that triangle ABC is equilateral. Therefore, the equality holds only when triangle ABC is equilateral.

Also solved by Theo Koupelis, Clark College, Washington, USA.

O680. Let $a \geq b \geq 1 \geq c \geq 0$ such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{3a^2 + 5} + \frac{1}{3b^2 + 5} + \frac{1}{3c^2 + 5} \leq \frac{3}{8}.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, Romania

Solution by the author

Denote $x = \frac{(a+b)^2}{4}$ and $p = ab$. From $(a-1)(b-1) \geq 0$, we obtain $a+b \leq ab+1$, hence $(a+b)^2 \leq (ab+1)^2$. Therefore,

$$1 \leq p \leq x \leq \frac{(p+1)^2}{4}.$$

Since

$$\frac{1}{3a^2 + 5} + \frac{1}{3b^2 + 5} = \frac{3(a^2 + b^2) + 10}{9a^2b^2 + 15(a^2 + b^2) + 25} = \frac{2(6x - 3p + 5)}{60x + 9p^2 - 30p + 25}$$

and

$$c^2 = \frac{(3-ab)^2}{(a+b)^2} = \frac{(3-p)^2}{4x},$$

the desired inequality can be written as $F \leq 0$, where

$$F = \frac{6x - 3p + 5}{60x + 9p^2 - 30p + 25} + \frac{2x}{20x + 3p^2 - 18p + 27} - \frac{3}{16}.$$

Moreover, for fixed p , the inequality $F \leq 0$ is equivalent to $f(x) \leq 0$, where

$$f(x) = -240x^2 + B(p)x + C(p).$$

Since $f(x)$ is concave and $x \in [p, (p+1)^2/4]$, it suffices to prove that $F \geq 0$ for $x = p$ and $x = (p+1)^2/4$. For $x = p$, we have

$$F = \frac{1}{3p+5} + \frac{2p}{3p^2+2p+27} - \frac{3}{16} = \frac{27(1-p)^3}{16(3p+5)(3p^2+2p+27)} \leq 0,$$

and for $x = (p+1)^2/4$, we have

$$16F = \frac{3p^2+13}{3p^2+5} + \frac{p^2+2p+1}{p^2-p+4} - 3 = \frac{-3(p-1)^4}{(3p^2+5)(p^2-p+4)} \leq 0.$$

The proof is completed. The equality occurs for $a = b = c = 1$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

O681. Let x, y, z be positive real numbers. Prove that

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} \leq \sqrt{\frac{9(x+y+z)}{2(xy+yz+zx)}}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by Alessandro Ventullo, Milan, Italy

Since $3(a^2+b^2+c^2) \geq (a+b+c)^2$ for any real numbers a, b, c , setting $a = \frac{1}{\sqrt{x+y}}$, $b = \frac{1}{\sqrt{y+z}}$, $c = \frac{1}{\sqrt{z+x}}$, we get

$$3 \left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) \geq \left(\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} \right)^2.$$

Therefore it is enough to prove that

$$\sqrt{\frac{9(x+y+z)}{2(xy+yz+zx)}} \geq \sqrt{3 \left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right)}.$$

By simplifying, this is equivalent to

$$\frac{3(x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3 - 2xyz(x+y+z))}{2(xy+yz+zx)(x+y)(y+z)(z+x)} \geq 0,$$

which is true because by the AM-GM Inequality we have

$$\frac{1}{2} \sum_{cyc} (x^3y + x^3z + xy^3 + xz^3 - 4x^2yz) \geq 0.$$

Equality holds if and only if $x = y = z$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

The desired inequality is equivalent to

$$3\sqrt{x+y+z} \prod_c \sqrt{x+y} \geq \sqrt{2(xy+yz+zx)} \sum_c \sqrt{(x+y)(y+z)}.$$

The function $f(t) = \sqrt{t}$ is clearly concave for $t > 0$, and thus, using Jensen's inequality we get

$$\sum_c \sqrt{(x+y)(y+z)} \leq 3\sqrt{\frac{\sum_c (x+y)(y+z)}{3}} = 3\sqrt{xy+yz+zx + \frac{x^2+y^2+z^2}{3}}.$$

Therefore, it suffices to show that

$$\sqrt{x+y+z} \prod_c \sqrt{x+y} \geq \sqrt{2(xy+yz+zx)} \sqrt{xy+yz+zx + \frac{x^2+y^2+z^2}{3}}.$$

Squaring and simplifying we get

$$x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3 \geq 2xyz(x+y+z),$$

or equivalently

$$\frac{1}{2} \sum_c (x^3y + x^3z + xy^3 + xz^3 - 4x^2yz) \geq 0,$$

which is obvious by AM-GM. Equality occurs when $x = y = z$.

Also solved by Arkady Alt, San Jose, California, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Titu Zvonaru, Comănești, Romania.

O682. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{a + b + c}{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

By GM-HM and Cauchy-Schwarz inequality we have

$$\sum_{\text{cyc}} \sqrt{ab} \geq \sum_{\text{cyc}} \frac{2ab}{a+b} = \sum_{\text{cyc}} \frac{2(ab)^2}{ab(a+b)} \geq \frac{2(ab+bc+ca)^2}{ab(a+b) + bc(b+c) + ca(c+a)}.$$

Therefore

$$\frac{a+b+c}{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}} \leq \frac{(a+b+c)[ab(a+b) + bc(b+c) + ca(c+a)]}{2(ab+bc+ca)^2}.$$

Now we need to show that

$$\frac{(a+b+c)[ab(a+b) + bc(b+c) + ca(c+a)]}{2(ab+bc+ca)^2} \leq \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

This is equivalent to

$$(a+b+c)[ab(a+b) + bc(b+c) + ca(c+a)] \leq 2(ab+bc+ca)(a^2 + b^2 + c^2),$$

$$\sum_{\text{cyc}} ab(a+b)^2 \leq \sum_{\text{cyc}} 2ab(a^2 + b^2),$$

$$\sum_{\text{cyc}} ab(a-b)^2 \geq 0.$$

The last inequality is clearly true and the proof is completed.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $a = x^2, b = y^2, c = z^2$, where x, y, z are positive real numbers. The desired inequality is equivalent to

$$(x^4 + y^4 + z^4)(xy + yz + zx) \geq (x^2y^2 + y^2z^2 + z^2x^2)(x^2 + y^2 + z^2),$$

or after expanding and simplifying,

$$xyz(x^3 + y^3 + z^3 - 3xyz) + \sum_c (x^5y + xy^5 - x^4y^2 - x^2y^4) \geq 0,$$

or

$$xyz(x^3 + y^3 + z^3 - 3xyz) + \sum_c xy(x-y)^2(x^2 + xy + y^2) \geq 0,$$

which is obvious by AM-GM. Equality occurs when $a = b = c$. If we were to allow at most one of the variables a, b, c to be zero, equality would also occur at $(a, b, c) = (a, a, 0)$ and cyclic permutations.

Solution 3 by Daniel Pascuas, Barcelona, Spain

Let $x = \sqrt{a}, y = \sqrt{b}, z = \sqrt{c}$ and write the inequality as

$$\frac{x^4 + y^4 + z^4}{x^2y^2 + y^2z^2 + z^2x^2} \geq \frac{x^2 + y^2 + z^2}{xy + yz + zx},$$

or equivalently

$$(x^2y^2 + y^2z^2 + z^2x^2)(x^2 + y^2 + z^2) \leq (x^4 + y^4 + z^4)(xy + yz + zx). \quad (*)$$

Now $(x^2y^2 + y^2z^2 + z^2x^2)(x^2 + y^2 + z^2) = A + 3x^2y^2z^2$ and $(x^4 + y^4 + z^4)(xy + yz + zx) = B + C$, where $A = x^4(y^2 + z^2) + y^4(x^2 + z^2) + z^4(x^2 + y^2)$, $B = x^5(y + z) + y^5(x + z) + z^5(x + y)$ and $C = x^4yz + xy^4z + xyz^4$. But, since

$$A = (x^5y)^{\frac{3}{4}}(xy^5)^{\frac{1}{4}} + (x^5z)^{\frac{3}{4}}(xz^5)^{\frac{1}{4}} + (y^5x)^{\frac{3}{4}}(yx^5)^{\frac{1}{4}} + (y^5z)^{\frac{3}{4}}(yz^5)^{\frac{1}{4}} + (z^5x)^{\frac{3}{4}}(zx^5)^{\frac{1}{4}} + (z^5y)^{\frac{3}{4}}(zy^5)^{\frac{1}{4}},$$

Hölder's inequality with exponents $p = \frac{4}{3}$ and $q = 4$ gives that $A \leq B$, while the AM-GM inequality shows that $3x^2y^2z^2 \leq C$, and hence $(*)$ holds.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, California, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Israel Castillo Pilco, Universidad Nacional de Ingeniería, Peru.

O683. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{b+c}{\sqrt{bc+2a^2+2a}} + \frac{c+a}{\sqrt{ca+2b^2+2b}} + \frac{a+b}{\sqrt{ab+2c^2+2c}} \geq 2.$$

Proposed by An Zhenping, Xiangyang National University, China

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By the AM-GM inequality $bc \leq \left(\frac{b+c}{2}\right)^2$, and so

$$\frac{b+c}{\sqrt{bc+2a^2+2a}} \geq \frac{b+c}{\sqrt{\left(\frac{b+c}{2}\right)^2+2a^2+2a}} = \frac{2(b+c)}{\sqrt{(b+c)^2+8a^2+8a}}$$

and because $b+c=1-a$, it follows that

$$\frac{b+c}{\sqrt{bc+2a^2+2a}} \geq \frac{2-2a}{\sqrt{9a^2+6a+1}} = \frac{2-2a}{3a+1}.$$

Then, applying the same argument to the left-hand side terms of the proposed inequality, it is enough to prove that

$$\frac{1-a}{3a+1} + \frac{1-b}{3b+1} + \frac{1-c}{3c+1} \geq 1.$$

Function $f(x) = \frac{1-x}{3x+1}$ is convex because $f''(x) = \frac{18}{(3x+1)^2} > 0$. Therefore, by Jensen's inequality

$$\frac{1-a}{3a+1} + \frac{1-b}{3b+1} + \frac{1-c}{3c+1} \geq 3 \cdot \frac{1-1/3}{3(1/3)+1} = 1.$$

Equality occurs if and only if $a = b = c = \frac{1}{3}$.

Solution 2 by Theo Koupeles, Clark College, Washington, USA

Using the condition $a + b + c = 1$ and AM-GM, we get

$$\begin{aligned} \sum_c \frac{b+c}{\sqrt{bc+2a^2+2a}} &\geq \sum_c \frac{b+c}{\sqrt{\frac{(b+c)^2}{4}+2a(a+1)}} = \sum_c \frac{2(1-a)}{\sqrt{(1-a)^2+8a(a+1)}} \\ &= \sum_c \frac{2(1-a)}{3a+1} \geq \sum_c 2 \cdot \frac{2-3a}{3} = 4 - 2 \sum_c a = 2. \end{aligned}$$

In the above we used the tangent-line trick to get

$$\frac{1-x}{3x+1} \geq \frac{2-3x}{3} \iff (3x-1)^2 \geq 0,$$

which is obvious for $x > 0$. Equality occurs when $a = b = c = 1/3$.

Solution 3 by Arkady Alt, San Jose, California, USA

Noting that

$$bc + 2a^2 + 2a = bc + 2a^2 + 2a(a+b+c) = (2a+c)(2a+b)$$

and

$$(2a+c)(2a+b) \leq \frac{((2a+c)+(2a+b))^2}{4} = \frac{(4a+b+c)^2}{4}$$

we obtain

$$\sum_{cyc} \frac{b+c}{\sqrt{bc+2a^2+2a}} \geq 2 \sum_{cyc} \frac{b+c}{4a+b+c}.$$

Thus it remains to prove that

$$\sum_{cyc} \frac{b+c}{4a+b+c} \geq 1.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{cyc} \frac{b+c}{4a+b+c} &= \sum_{cyc} \frac{(b+c)^2}{(b+c)(4a+(b+c))} \\ &\geq \frac{\left(\sum_{cyc} (b+c)\right)^2}{\sum_{cyc} (b+c)(4a+(b+c))} \\ &= \frac{4\left(\sum_{cyc} a\right)^2}{2\left(\sum_{cyc} a\right)^2 + 6\sum_{cyc} ab} \\ &= \frac{2}{1+3\sum_{cyc} ab} \end{aligned}$$

and

$$3\sum_{cyc} ab \leq \left(\sum_{cyc} a\right)^2 = 1.$$

Then

$$\sum_{cyc} \frac{b+c}{4a+b+c} \geq \frac{2}{1+3\sum_{cyc} ab} \geq \frac{2}{1+1} = 1.$$

Also solved by Cristian Muth, UMFST Highschool in G. E. Palade University of Medicine, Pharmacy, Science, and Technology, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Titu Zvonaru, Comănești, Romania; Daniel Pascuas, Barcelona, Spain; Israel Castillo Pilco, Universidad Nacional de Ingeniería, Peru.

O684. A sequence $(a_n)_{n \geq 1}$ of positive integers is called admissible if for all $n > 2$ the divisibility

$$1 + a_1 + \cdots + a_{n-1} \mid (1 + a_1)(1 + a_2) \cdots (1 + a_{n-1}) - 1$$

holds precisely when n is a prime number. Let f be a nonconstant polynomial with integer coefficients.

(a) Can the sequence $(a_n)_{n \geq 1}$ defined by $a_n = f(n)$ be admissible?

(b) Can the sequence $(a_n)_{n \geq 1}$ defined by $a_n = f(2^n)$ be admissible?

Proposed by Gabriel Dospinescu, Lyon, France

Solution by the author

We refer the reader to the previous proposed problem for part b), which shows that taking $f(X) = X$ gives an admissible sequence $a_n = 2^n$. Let us focus on part a) and let us suppose that $a_n = f(n)$ is admissible. By a standard application of the binomial formula, we see that there is a polynomial $g \in \mathbf{Q}[X]$ such that $g(n) = 1 + f(1) + \cdots + f(n-1)$ for all $n \geq 1$. Note that since f is not constant, g has degree at least 2. Let $N > 1$ be a positive integer such that $h := Ng$ has integer coefficients and let $A = 1 + f$. Then A, h are nonconstant, have integer coefficients and $h(p) \mid NA(1)A(2) \cdots A(p-1) - N$ for all odd primes p .

First, we claim that $h(0) \neq 0$. Indeed, otherwise p divides $h(p) = 1 + f(1) + \cdots + f(p-1)$ for all primes p and so $p \mid g(p)$ for all primes $p > N$. By a classical result p also divides $f(0) + f(1) + \cdots + f(p-1)$ for all large enough primes (for instance all $p > \deg f$), so that necessarily $f(0) = 1$ and $A(0) = 2$. On the other hand, by Schur's theorem we can find a prime $p > N$ for which the congruence $A(x) \equiv 0 \pmod{p}$ is solvable. Since $p \mid N \prod_{k=1}^{p-1} A(k) - N$, p cannot divide any of $A(1), \dots, A(p-1)$ and so necessarily $p \mid A(0) = 2$, a contradiction. This shows that $h(0) \neq 0$.

Since A and h are nonconstant, a classical but nontrivial result ensures that there are infinitely many primes p for which both congruences $h(x) \equiv 0 \pmod{p}$ and $A(y) \equiv 0 \pmod{p}$ are solvable. Pick a large prime $p > N$ which does not divide $h(0)$ and for which $p \mid h(x)$ for some x . Necessarily p is prime to x , so by Dirichlet's theorem there is a prime $q > p$ of the form $q = x + pk$. We have $p \mid h(q) \mid N \prod_{k=1}^{q-1} A(k) - N$. But this is impossible since by assumption there is $k \in \{1, 2, \dots, q-1\}$ with $p \mid A(k)$, and thus $p \mid N$, contradicting the inequality $p > N$.