

Junior problems

J685. Let a be a nonzero real number. Prove that

$$a^3 - \frac{1}{a^3} + a^2 + \frac{1}{a^2} - 3\left(a - \frac{1}{a}\right) = 3$$

if and only if

$$a^5 - \frac{1}{a^5} - 5\left(a^3 - \frac{1}{a^3}\right) + 11\left(a - \frac{1}{a}\right) = 1.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Daniel Pascuas, Barcelona, Spain

Let $x = a - \frac{1}{a}$. Then

$$x^2 = a^2 + \frac{1}{a^2} - 2, \quad x^3 = a^3 - \frac{1}{a^3} - 3\left(a - \frac{1}{a}\right) \quad \text{and} \quad x^5 = a^5 - \frac{1}{a^5} - 5\left(a^3 - \frac{1}{a^3}\right) + 10\left(a - \frac{1}{a}\right),$$

so

$$a^3 - \frac{1}{a^3} + a^2 + \frac{1}{a^2} - 3\left(a - \frac{1}{a}\right) = x^3 + x^2 + 2,$$

and

$$a^5 - \frac{1}{a^5} - 5\left(a^3 - \frac{1}{a^3}\right) + 11\left(a - \frac{1}{a}\right) = x^5 + x.$$

Therefore

$$a^3 - \frac{1}{a^3} + a^2 + \frac{1}{a^2} - 3\left(a - \frac{1}{a}\right) = 3 \iff x^3 + x^2 - 1 = 0$$

and

$$a^5 - \frac{1}{a^5} - 5\left(a^3 - \frac{1}{a^3}\right) + 11\left(a - \frac{1}{a}\right) = 1 \iff x^5 + x - 1 = 0.$$

Since $x^5 + x - 1 = (x^2 - x + 1)(x^3 + x^2 - 1) = ((x - \frac{1}{2})^2 + \frac{3}{4})(x^3 + x^2 - 1)$ and $x \in \mathbb{R}$, it is clear that $x^5 + x - 1 = 0$ is equivalent to $x^3 + x^2 - 1 = 0$. Hence the two identities in the statement are equivalent.

Also solved by Arkady Alt, San Jose, California, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Ivan Hadinata, Department of Mathematics, Gadjah Mada University, Indonesia; Kousik Sett, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Raja Oktovin Parhasian Damanik, Canberra ACT, Australia; Sundares H. R., Shivamogga, Karnataka, India; Telemachus Baltasvias, Kerameies Junior High School, Kefallonia, Greece; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Anderson Torres, Sao Paulo, Brazil; Nikolai N. Mnev, AMMOC, Bengaluru, India; Polyhedra, Polk State College, FL, USA; Samhith Kadiyala, North Carolina Cyber Academy, NC, USA; Srijan Sundar, Oxford, UK; Sumanyu Nandecha, AMMOC, India.

J686. Let a and b be positive real numbers. Prove that

$$\sqrt{x^2 - ax + a^2} + \sqrt{x^2 - bx + b^2} \geq \sqrt{a^2 + ab + b^2}$$

for all real numbers x . When does equality occur?

Proposed by Mircea Becheanu, Canada

Solution 1 by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

If $x \leq 0$, let $x = -y$ with $y \geq 0$. The inequality becomes $\sqrt{y^2 + ay + a^2} + \sqrt{y^2 + by + b^2} \geq \sqrt{a^2 + ab + b^2}$. Clearly, $\sqrt{y^2 + ay + a^2} + \sqrt{y^2 + by + b^2} \geq \sqrt{y^2 - ay + a^2} + \sqrt{y^2 - by + b^2}$. So, it suffices to prove the original inequality for positive real number x .

Let PA, PB, PC be rays with $PA = a, PB = x$, and $PC = b$ such that $\angle APB = \angle BPC = 60^\circ$ and $\angle APC = 120^\circ$.

By cosine law, we have

$$AB^2 = PA^2 + PB^2 - 2 \cdot PA \cdot PB \cdot \cos 60^\circ = a^2 + x^2 - ax$$

$$BC^2 = PB^2 + PC^2 - 2 \cdot PB \cdot PC \cdot \cos 60^\circ = x^2 + b^2 - bx$$

$$CA^2 = PA^2 + PC^2 - 2 \cdot PA \cdot PC \cdot \cos 120^\circ = a^2 + b^2 + ab.$$

By triangle inequality, we have $AB + BC \geq CA$, and hence $\sqrt{x^2 - ax + a^2} + \sqrt{x^2 - bx + b^2} \geq \sqrt{a^2 + ab + b^2}$. The equality occurs when B lies on segment AC , and since PB bisects $\angle ABC$, we calculate PB using $[APB] + [BPC] = [APC]$, i.e.

$$\frac{1}{4}ax\sqrt{3} + \frac{1}{4}bx\sqrt{3} = \frac{1}{4}ab\sqrt{3} \iff x = \frac{ab}{a+b}.$$

Solution 2 by Daniel Pascuas, Barcelona, Spain

Let d be the Euclidean distance on $\mathbb{R}^2$, i.e. $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Then our inequality directly follows from the triangle inequality for d :

$$\begin{aligned} \sqrt{x^2 - ax + a^2} + \sqrt{x^2 - bx + b^2} &= d\left(\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right), (x, 0)\right) + d\left((x, 0), \left(\frac{b}{2}, -\frac{b\sqrt{3}}{2}\right)\right) \\ &\geq d\left(\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right), \left(\frac{b}{2}, -\frac{b\sqrt{3}}{2}\right)\right) = \sqrt{\frac{(a-b)^2}{4} + \frac{3(a+b)^2}{4}} \\ &= \sqrt{a^2 + ab + b^2}. \end{aligned}$$

The equality holds if and only if the point $(x, 0)$ belongs to the segment determined by $(\frac{a}{2}, \frac{a\sqrt{3}}{2})$ and $(\frac{b}{2}, -\frac{b\sqrt{3}}{2})$, which means that $x = \frac{ab}{a+b}$. Thus the equality only occurs when $x = \frac{ab}{a+b}$.

Solution 3 by Polyhedra, Polk State College, FL, USA

By squaring both sides we see that the inequality is equivalent to

$$2\sqrt{(x^2 - ax + a^2)(x^2 - bx + b^2)} \geq ab + (a+b)x - 2x^2.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \sqrt{(2x^2 - 2ax + 2a^2)(2x^2 - 2bx + 2b^2)} &= \sqrt{[a^2 + (a-x)^2 + x^2][b^2 + x^2 + (b-x)^2]} \\ &\geq ab + (a-x)x + x(b-x) = ab + (a+b)x - 2x^2. \end{aligned}$$

Equality holds if and only if $\frac{b}{a} = \frac{x}{a-x} = \frac{b-x}{x}$, which is equivalent to $x = \frac{ab}{a+b}$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ivan Hadinata, Department of Mathematics, Gadjah Mada University, Indonesia; Kousik Sett, India; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Anderson Torres, Sao Paulo, Brazil; Nihad Hashimov, AMMOC, Azerbaijan; Srijan Sundar, Oxford, UK; Sumanyu Nandecha, AMMOC, India.

J687. Let a, b, c be real numbers such that $ab + bc + ca = 169$ and

$$(a+1)^3 + (b+1)^3 + (c+1)^3 - 3abc = 1.$$

Find all possible values of $a + b + c$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ivan Hadinata, Department of Mathematics, Gadjah Mada University, Indonesia

Let $x = a + b + c$ and $y = ab + bc + ca = 169$. Then

$$\begin{aligned} 0 &= (a+1)^3 + (b+1)^3 + (c+1)^3 - 3abc - 1 \\ &= (a^3 + b^3 + c^3 - 3abc) + 3(a^2 + b^2 + c^2) + 3(a + b + c) + 2 \\ &= x(x^2 - 3y) + 3(x^2 - 2y) + 3x + 2 \\ &= x^3 + 3x^2 - 504x - 1012 \\ &= (x+2)(x+23)(x-22), \end{aligned}$$

hence $x = -2$ or $x = -23$ or $x = 22$. Since $a, b, c \in \mathbb{R}$, then $x^2 \geq 3y = 507$, and the only possibility is $x = -23$. One example of (a, b, c) when $x = -23$ is

$$a = -8 + \sqrt{7}, \quad b = -8 - \sqrt{7}, \quad c = -7.$$

In conclusion, the only possible value of $a + b + c$ is -23 .

Editor's Comment. Many readers failed to check the condition $(a + b + c)^2 \geq 3(ab + bc + ca)$.

Raja Oktovin Parhasian Damanik found another triple (a, b, c) satisfying the conditions, arguing as follows. Take $a = -x, b = -y, c = -z$, then $x + y + z = 23$ and $xy + yz + zx = 169$. Take $z = 5$, then $x + y = 18$ and $xy = 79$. This has solution, for example, $x, y = 9 \pm \sqrt{2}$. Hence we can take $a = -9 - \sqrt{2}, b = -9 + \sqrt{2}$, and $c = -5$.

Theo Koupelis noticed that there are infinitely many solutions that satisfy the given equations, arguing as follows.

We have the system $b + c = -23 - a$ and $bc = 169 - a(b + c) = a^2 + 23a + 169$; thus, b, c are the solutions of the quadratic $t^2 + (23 + a)t + (a^2 + 23a + 169) = 0$, whose discriminant is $\Delta = -(3a^2 + 46a + 147)$. For real solutions we must have $\Delta \geq 0$; thus, for any $\frac{-23 - 2\sqrt{22}}{3} \leq a \leq \frac{-23 + 2\sqrt{22}}{3}$, the values of b and c are given by $\frac{-23 - a \pm \sqrt{-(3a^2 + 46a + 147)}}{2}$.

Also solved by Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, VA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Kalkin Gandhi, AMMOC Bengaluru, India; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Raja Oktovin Parhasian Damanik, Canberra ACT, Australia; Sundares H. R., Shivamogga, Karnataka, India; Telemachus Baltasvias, Kerameies Junior High School, Kefallonia, Greece; Theo Koupelis, Clark College, Washington, USA; Vishwesh Ravi Shrimali, Jaipur, Rajasthan, India; Aaditya Bhaumik Sahu, AMMOC, India; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Madhav R Pillai, Delhi Public School, Ranchi, Jharkhand, India; Medha Shrivastava, AMMOC, India; Nihad Hashimov, AMMOC, Azerbaijan; Nikolai N. Mnev, AMMOC, Bengaluru, India; Polyahedra, Polk State College, FL, USA; Samhith Kadiyala, North Carolina Cyber Academy, NC, USA; Srijan Sundar, Oxford, UK.

J688. Prove that positive real numbers x, y, z are the lengths of the altitudes in a triangle if and only if

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 2 \max \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\}.$$

Proposed by Dan-Ştefan Marinescu, Romania

Solution 1 by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

Suppose that a triangle has altitudes of lengths x, y, z which are respectively perpendicular to the sides of length a, b, c . The area $[ABC]$ satisfies $2[ABC] = ax = by = cz$. By triangle inequality, we have that $a + b + c > 2\max\{a, b, c\}$. Hence,

$$\begin{aligned} \frac{[ABC]}{x} + \frac{[ABC]}{y} + \frac{[ABC]}{z} &> 2 \max \left\{ \frac{[ABC]}{x}, \frac{[ABC]}{y}, \frac{[ABC]}{z} \right\} \\ &= 2[ABC] \max \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\} \end{aligned}$$

Hence,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 2 \max \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\}.$$

Now let x, y, z satisfy the inequality and hence $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are sides of a triangle; let A be the area of that triangle. There exists a triangle whose sides are of length $\frac{1}{2Ax}, \frac{1}{2Ay}, \frac{1}{2Az}$. The area of this triangle is

$$\frac{1}{4A^2} \cdot A = \frac{1}{4A}. \text{ Hence, the length of the altitude to side of the length } \frac{1}{2Ax} \text{ will be } \frac{2 \cdot \frac{1}{4A}}{\frac{1}{2Ax}} = x. \text{ Similarly,}$$

the other two altitudes will have length y and z , as desired.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Let ABC be a triangle and, as usual, let a, b, c be the lengths of the sides BC, AC, AB , respectively. Let x, y, z be the lengths of the altitudes of ABC corresponding to the sides BC, AC, AB , respectively. Then $xa = yb = zc = 2S$, where S is the area of the triangle ABC . Therefore $a = \frac{2S}{x}$, $b = \frac{2S}{y}$, and $c = \frac{2S}{z}$ must satisfy that $a + b + c > 2 \max\{a, b, c\}$, which means that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 2 \max \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\}.$$

Conversely, assume that x, y, z are positive real numbers satisfying this inequality. Then

$$\Lambda := \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \left(\frac{1}{y} + \frac{1}{z} - \frac{1}{x} \right) \left(\frac{1}{x} + \frac{1}{z} - \frac{1}{y} \right) \left(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} \right) > 0.$$

Thus $a = \frac{2}{x\sqrt{\Lambda}}$, $b = \frac{2}{y\sqrt{\Lambda}}$, $c = \frac{2}{z\sqrt{\Lambda}}$ are positive real numbers which satisfy $a + b + c > 2 \max\{a, b, c\}$, and so they are the lengths of the sides BC, AC, AB , respectively, of a triangle ABC . The semiperimeter of this triangle is $p = \frac{1}{2}(a + b + c) = \frac{1}{\sqrt{\Lambda}}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})$, so Heron's formula shows that its area is equal to

$$\sqrt{p(p-a)(p-b)(p-c)} = \frac{1}{\Lambda} \sqrt{\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \left(\frac{1}{y} + \frac{1}{z} - \frac{1}{x} \right) \left(\frac{1}{x} + \frac{1}{z} - \frac{1}{y} \right) \left(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} \right)} = \frac{1}{\sqrt{\Lambda}}.$$

Hence the lengths of the altitudes corresponding to the sides BC, AC, AB are $\frac{2}{a\sqrt{\Lambda}} = x$, $\frac{2}{b\sqrt{\Lambda}} = y$, $\frac{2}{c\sqrt{\Lambda}} = z$, respectively.

Also solved by Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Diar Gashi, University of Primorska, Slovenia; Medha Shrivastava, AMMOC, India; Polyhedra, Polk State College, FL, USA; Srijan Sundar, Oxford, UK.

J689. Find the least value of the function

$$f(x) = \frac{2}{x^2 - 4x + 8} + \frac{1}{x^2 - 6x + 10} - \frac{2}{x^2 - 5x + 8},$$

where x is any real number.

Proposed by An Zhenping, Xianyang Normal University, China

Solution 1 by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

We claim that $f(x) \geq 0$ for all real numbers x .

We want to prove that

$$\frac{2}{x^2 - 4x + 8} + \frac{1}{x^2 - 6x + 10} \geq \frac{2}{x^2 - 5x + 8}.$$

Note that if $x = 2$, then $LHS = \frac{2}{4} + \frac{1}{2} = 1$ and $RHS = \frac{2}{2} = 1$, so the equality can occur.

To prove the inequality, we use AM-HM inequality:

$$\frac{1}{\frac{1}{2}x^2 - 2x + 4} + \frac{1}{x^2 - 6x + 10} \geq \frac{4}{\frac{3}{2}x^2 - 8x + 14}.$$

It suffices to prove that $2x^2 - 10x + 16 \geq \frac{3}{2}x^2 - 8x + 14$. Rearranging, we have $\frac{1}{2}x^2 - 2x + 2 \geq 0$, i.e. $x^2 - 4x + 4 \geq 0$, i.e. $(x - 2)^2 \geq 0$ which is true.

Hence, the least value of $f(x)$ is 0, and it is attained by $x = 2$.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Just note that

$$\begin{aligned} f(x) &= \frac{1}{x^2 - 6x + 10} - \frac{2x}{(x^2 - 4x + 8)(x^2 - 5x + 8)} \\ &= \frac{(x^2 - 4x + 8)(x^2 - 5x + 8) - 2x(x^2 - 6x + 10)}{(x^2 - 4x + 8)(x^2 - 5x + 8)(x^2 - 6x + 10)} \\ &= \frac{x^4 - 11x^3 + 48x^2 - 92x + 64}{(x^2 - 4x + 8)(x^2 - 5x + 8)(x^2 - 6x + 10)} \\ &= \frac{(x - 2)^2(x^2 - 7x + 16)}{(x^2 - 4x + 8)(x^2 - 5x + 8)(x^2 - 6x + 10)} \\ &= \frac{(x - 2)^2\left((x - \frac{7}{2})^2 + \frac{15}{4}\right)}{\left((x - 2)^2 + 4\right)\left((x - \frac{5}{2})^2 + \frac{7}{4}\right)\left((x - 3)^2 + 1\right)} \geq 0 = f(2), \quad \text{for every } x \in \mathbb{R}, \end{aligned}$$

so the the least value of $f(x)$ is 0.

Also solved by Arkady Alt, San Jose, California, USA; Henry Ricardo, Westchester Area Math Circle, New York, USA; Kalkin Gandhi, AMMOC Bengaluru, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Karma Pillai, Polyhedra, Polk State College, FL, USA; Sumanyu Nandecha, AMMOC, India.

J690. Let ABC be a triangle. The angle bisector of $\angle CAB$ intersects BC in D and the circumcircle in X . Let Y be the orthogonal projection of C on AX and Z the orthogonal projection of D on AB . The line YZ intersects BC in E . Prove that AE is perpendicular to BC .

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Raja Oktovin Parhasian Damanik, Canberra ACT, Australia

Let $\angle BAC, \angle ABC, \angle BCA$ be denoted by α, β, γ , respectively.

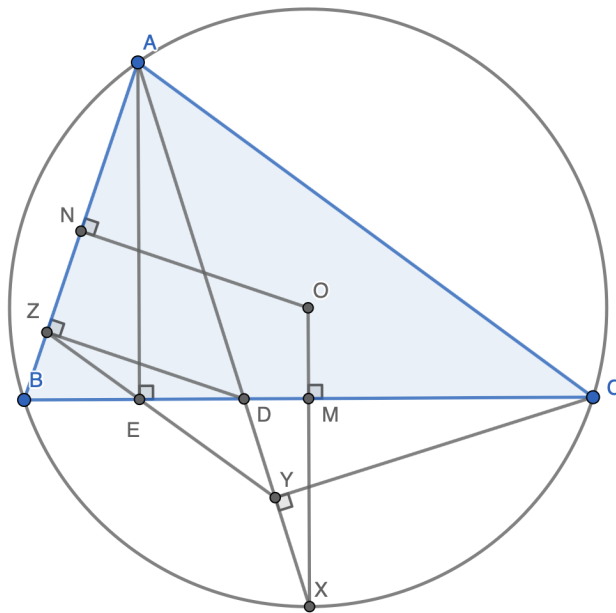
We want to prove that $\angle AEC = \angle AYC = 90^\circ$, that is, we want to prove that $AEYC$ is a cyclic quadrilateral.

Note that $\angle DAE = \angle DAB - \angle BAE = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\alpha}{2} + \beta - 90^\circ$.

Also, note that $\angle ECY = \angle DCY = \angle ACY - \angle ACB = \left(90^\circ - \frac{\alpha}{2}\right) - \gamma$.

But it is clear now that $\frac{\alpha}{2} + \beta - 90^\circ = 90^\circ - \frac{\alpha}{2} - \gamma$, i.e. $\alpha + \beta + \gamma = 180^\circ$, which is true for angles of a triangle α, β, γ .

So, $\angle YAE = \angle DAE = \angle ECY$, which implies that $AEYC$ is a cyclic quadrilateral.



Solution 2 by Diar Gashi, University of Primorska, Slovenia

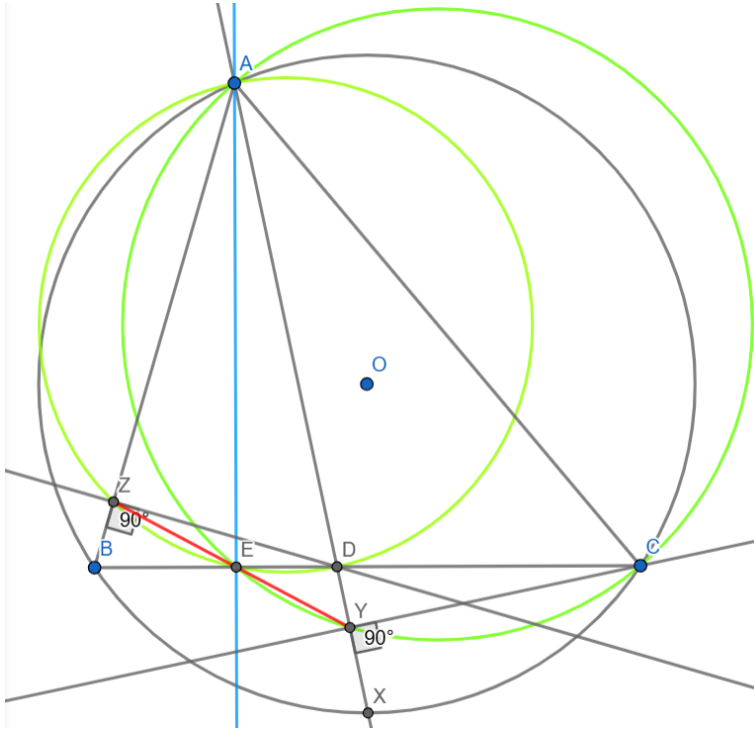
Let E' be the orthogonal projection of A onto BC . Since $\angle AE'C = 90 = \angle AYC$, we have that $AE'YC$ is cyclic. So,

$$\angle AE'Y = 90 + \angle CE'Y = 90 + \angle CAY = 90 + \angle BAD.$$

Since $\angle AZD = 90 = \angle AE'D$, we have that $AZE'D$ is cyclic. Thus,

$$\angle AE'Z = \angle ADZ = 90 - \angle BAD.$$

Therefore $\angle AE'Y + \angle AE'Z = 180$ and this means that Z, E' and Y are collinear. So, $E = E'$ and therefore AE is indeed perpendicular to BC .

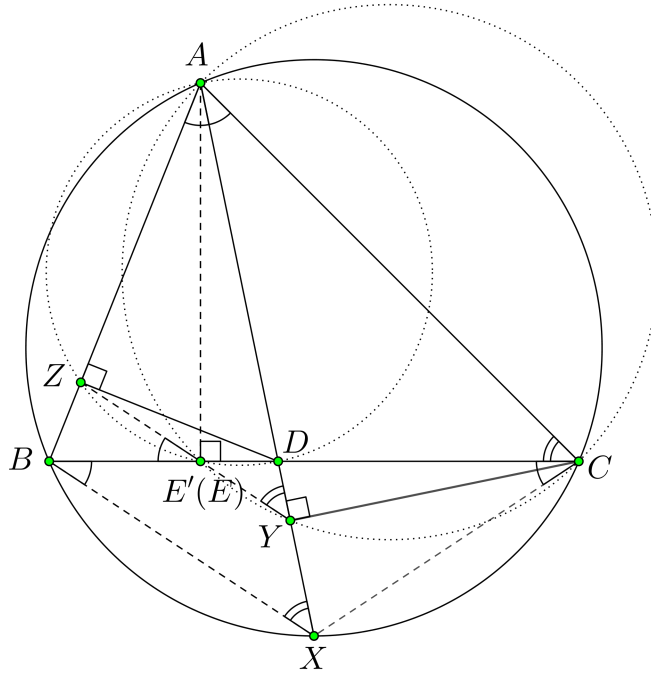


Solution 3 by Kousik Sett, India

Let the circumcircle of triangle ACY intersect BC at E' . Then $\angle AE'C = \angle AYC = 90^\circ$.

Also, $\angle E'YA = \angle E'CA = \angle BCA = \angle BXA$, which implies $E'Y \parallel BX$.

Since $\angle DE'A = \angle DZA = 90^\circ$, quadrilateral $AZE'D$ is cyclic. So, $\angle BE'Z = \angle ZAD = \angle BAX$.



Since the angle bisector of $\angle CAB$ intersects the circumcircle of triangle ABC at X , we infer that X is the midpoint of arc BC not containing A . So, $\angle XBC = \angle BCX = \angle BAX$.

Therefore, we obtain $\angle BE'Z = \angle XBC$, which implies $ZE' \parallel BX$.

Since $E'Y \parallel BX$ and $ZE' \parallel BX$, it follows that Z , E' , and Y are collinear, and $E' = BC \cap ZY$. However, it is given that $E = BC \cap ZY$. Therefore, we conclude that $E' = E$ and consequently, we have AE is perpendicular to BC .

Also solved by Theo Koupelis, Clark College, Washington, USA; Karma Pillai, Polyhedra, Polk State College, FL, USA; Prasanna Pawar, AMMOC, Bengaluru, India.

Senior problems

S685. Let a, b be positive real numbers. Prove that

$$\frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} \geq \frac{a+b}{2(1+ab)}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution 1 by Arkady Alt, San Jose, California, USA

By Cauchy-Schwarz Inequality, we have

$$(1+b)^2 \leq \left(\frac{b}{a} + 1\right)(ab+1), \quad (1+a)^2 \leq \left(\frac{a}{b} + 1\right)(ab+1).$$

Then

$$\begin{aligned} \frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} &\geq \frac{a}{\left(\frac{b}{a} + 1\right)(ab+1)} + \frac{b}{\left(\frac{a}{b} + 1\right)(ab+1)} \\ &= \frac{1}{ab+1} \left(\frac{a^2}{a+b} + \frac{b^2}{a+b} \right) \\ &= \frac{1}{(ab+1)(a+b)} \cdot (a^2 + b^2) \\ &\geq \frac{1}{(ab+1)(a+b)} \cdot \frac{(a+b)^2}{2} \\ &= \frac{a+b}{2(1+ab)}. \end{aligned}$$

The equality holds if and only if $a = b = 1$.

Solution 2 by Ivan Hadinata, Department of Mathematics, Gadjah Mada University, Indonesia

By the AM-GM Inequality, $a + b + a^2b + ab^2 \geq 4ab$. Hence

$$2(a+b)(1+ab) \geq a+b+4ab+a^2b+ab^2 = a(1+b)^2 + b(1+a)^2.$$

Then, by Cauchy-Schwarz Inequality we have

$$\frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} = \frac{a^2}{a(1+b)^2} + \frac{b^2}{b(1+a)^2} \geq \frac{(a+b)^2}{a(1+b)^2 + b(1+a)^2} \geq \frac{(a+b)^2}{2(a+b)(1+ab)} = \frac{a+b}{2(1+ab)},$$

as wanted. The equality holds if and only if $a = b = 1$.

Also solved by Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, New York, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Raja Oktovin Parhasian Damanik, Canberra ACT, Australia; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Nihad Hashimov, AMMOC, Azerbaijan; Prasanna Pawar, AMMOC, Bengaluru, India; Srijan Sundar, Oxford, UK; Sumanyu Nandecha, AMMOC, India.

S686. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{\sqrt{b^2 + c^2}} + \frac{1}{\sqrt{c^2 + a^2}} + \frac{1}{\sqrt{a^2 + b^2}} \leq \frac{3\sqrt{2}}{ab + bc + ca} + \frac{1}{\sqrt{2}}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

By homogenizing the inequality becomes

$$\frac{1}{\sqrt{2(b^2 + c^2)}} + \frac{1}{\sqrt{2(c^2 + a^2)}} + \frac{1}{\sqrt{2(a^2 + b^2)}} \leq \frac{a + b + c}{ab + bc + ca} + \frac{3}{2(a + b + c)}.$$

We have

$$\sum_{cyc} \frac{1}{\sqrt{2(b^2 + c^2)}} \leq \sum_{cyc} \frac{1}{b + c} = \frac{(a + b + c)^2 + ab + bc + ca}{(a + b + c)(ab + bc + ca) - abc}.$$

Hence it suffices to prove that

$$\frac{(a + b + c)^2 + ab + bc + ca}{(a + b + c)(ab + bc + ca) - abc} \leq \frac{a + b + c}{ab + bc + ca} + \frac{3}{2(a + b + c)}.$$

Clearing denominators and expanding, the inequality becomes

$$(a + b + c)(ab + bc + ca)^2 \geq 2abc(a + b + c)^2 + 3abc(ab + bc + ca).$$

But this follows from the AM-GM Inequality and $(x + y + z)^2 \geq 3(xy + yz + zx)$. Indeed, we have

$$\begin{aligned} \frac{2(a + b + c)(ab + bc + ca)^2}{3} &= \frac{2(a + b + c)}{3} \cdot (ab + bc + ca)^2 \\ &\geq \frac{2(a + b + c)}{3} \cdot 3abc(a + b + c) = 2abc(a + b + c)^2 \end{aligned}$$

and

$$\begin{aligned} \frac{(a + b + c)(ab + bc + ca)^2}{3} &= \frac{(a + b + c)(ab + bc + ca)}{3} \cdot (ab + bc + ca) \\ &\geq 3abc(ab + bc + ca). \end{aligned}$$

Summing up these inequalities, we get the desired result.

Observation. This is stronger than problem 1 from 2024 Junior Balkan MO:

Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 = \frac{1}{4}.$$

Prove that

$$\frac{1}{\sqrt{b^2 + c^2}} + \frac{1}{\sqrt{c^2 + a^2}} + \frac{1}{\sqrt{a^2 + b^2}} \leq \frac{\sqrt{2}}{(a + b)(b + c)(c + a)}.$$

It can be shown, not hard, like

$$\frac{a + b + c}{ab + bc + ca} + \frac{3}{2(a + b + c)} \leq \frac{3(a + b + c)}{2(ab + bc + ca)} \leq \frac{4(a^2 + b^2 + c^2)}{(a + b)(b + c)(c + a)}.$$

Also solved by Arkady Alt, San Jose, California, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Nguyen Thanh Nam, Tu Son Secondary School, Vietnam.

S687. Solve in real numbers the system of equations:

$$\sqrt{\frac{x^2+1}{y^2+1}} = x, \quad 5\sqrt{\frac{y^2+1}{z^2+1}} = 6y, \quad 7\sqrt{\frac{z^2+1}{x^2+1}} = 20z.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution 1 by the author

Because x, y, z are positive, let $x = \tan u$, $y = \tan v$, $z = \tan w$, where $u, v, w \in \left(0, \frac{\pi}{2}\right)$. Then

$$\frac{\cos v}{\cos u} = \frac{\sin u}{\cos u}, \quad \frac{5 \cos w}{\cos v} = \frac{6 \sin v}{\cos v}, \quad \frac{7 \cos u}{\cos w} = \frac{20 \sin w}{\cos w},$$

implying

$$\cos v = \sin u, \quad 5 \cos w = 6 \sin v, \quad 7 \cos u = 20 \sin w.$$

It follows that $\sin v = \cos u$ and so $4(5 \cos w) = 4(6 \cos u)$ and, since $20 \sin w = 7 \cos u$, we get

$$20^2(\cos^2 w + \sin^2 w) = (24^2 + 7^2) \cos^2 u,$$

yielding $\cos u = \frac{4}{5}$. Then $\sin u = \frac{3}{5}$, $\sin w = \frac{7}{25}$, $\cos w = \frac{24}{25}$, $\sin v = \frac{4}{5}$, $\cos v = \frac{3}{5}$, hence

$$x = \tan u = \frac{3}{4}, \quad y = \tan v = \frac{4}{3}, \quad z = \tan w = \frac{7}{24}.$$

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Clearly we have $x, y, z > 0$. Multiplying the three given equations we get $35 = 120xyz$ or

$$24xyz = 7. \tag{1}$$

Squaring and simplifying the first of the given equations we get $x^2 + 1 = x^2(y^2 + 1)$ or

$$xy = 1. \tag{2}$$

From (1) and (2) we get $z = 7/24$. Substituting into the second of the given equations, squaring and simplifying we get

$$25 \cdot \frac{y^2 + 1}{(7/24)^2 + 1} = 36y^2 \implies y^2 = \frac{16}{9} \implies y = \frac{4}{3}.$$

Therefore, $(x, y, z) = \left(\frac{3}{4}, \frac{4}{3}, \frac{7}{24}\right)$ is the only solution.

Also solved by Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, VA, USA; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Raja Oktovin Parhasian Damanik, Canberra ACT, Australia; Sundares H. R., Shivamogga, Karnataka, India; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Madhav R Pillai, Delhi Public School, Ranchi, Jharkhand, India; Nihad Hashimov, AMMOC, Azerbaijan; Nikolai N. Mnev, AMMOC, Bengaluru, India; Srijan Sundar, Oxford, UK.

S688. Prove that

$$\frac{ab^2 + a + b^2}{a^2 + ab + b}$$

is an integer for infinitely many positive integers a and infinitely many positive integers b .

Proposed by Mircea Becheanu, Canada

Solution 1 by Theo Koupelis, Clark College, Washington, USA

For any positive integer a we set $b = a[a(a-1)+1] = a^3 - a^2 + a$, which is clearly a positive integer. Then

$$\frac{ab^2 + a + b^2}{a^2 + ab + b} = \frac{(a+1)(a^3 - a^2 + a)^2 + a}{(a+1)(a^3 - a^2 + a) + a^2} = \frac{a(a^3 + a + 1)(a^3 - a^2 + 1)}{a(a^3 + a + 1)} = a^3 - a^2 + 1,$$

which is clearly a positive integer.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Just note that $ab^2 + a + b^2 = (a^2 + ab + b)(b - a + 1) + a^3 - a^2 + a - b$ so

$$\frac{ab^2 + a + b^2}{a^2 + ab + b} = b - a + 1 \text{ is an integer if } b = p(a), \text{ where } p(x) = x^3 - x^2 + x.$$

Since $p(a)$ is a positive integer, for any positive integer a , and $p(x)$ is increasing on $\mathbb{R}$ (because $p'(x) = 3x^2 - 2x + 1 = 3(x^2 - \frac{2}{3}x + \frac{1}{3}) = 3(x - \frac{1}{3})^2 + \frac{2}{3} > 0$, for any $x \in \mathbb{R}$), the values of $p(x)$ on the positive integers form an infinite set of positive integers. Therefore we conclude that

$$\frac{ab^2 + a + b^2}{a^2 + ab + b}$$

is an integer for infinitely many positive integers a and infinitely many positive integers b .

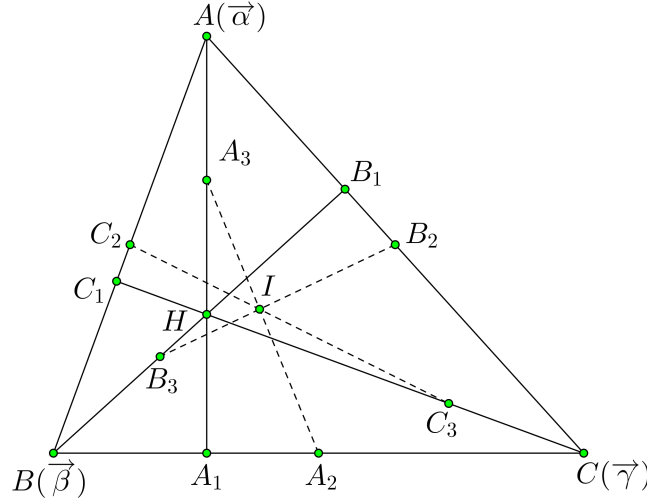
Also solved by Anderson Torres, Sao Paulo, Brazil.

S689. Let ABC be an acute triangle with incenter I , A_1, B_1, C_1 the feet of the altitudes and A_2, B_2, C_2 the midpoints of the sides BC, AC, AB , respectively. The line A_2I intersects AA_1 in A_3 , the line B_2I intersects BB_1 in B_3 and the line C_2I intersects CC_1 in C_3 . Prove that if $\overrightarrow{AA_3} + \overrightarrow{BB_3} + \overrightarrow{CC_3} = \vec{0}$ then ABC is equilateral.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Kousik Sett, India

Let the altitudes AA_1 , BB_1 , and CC_1 meet at H (the orthocenter of triangle ABC). Since $A_3 \in AA_1$, $B_3 \in BB_1$, $C_3 \in CC_1$, and $\overrightarrow{AA_3} + \overrightarrow{BB_3} + \overrightarrow{CC_3} = \vec{0}$, we infer that $\overrightarrow{AH} + \overrightarrow{BH} + \overrightarrow{CH} = \vec{0}$.



Let the position vectors of the vertices A, B, C be $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$ respectively. From right triangles AA_1B and AA_1C , we have

$$\frac{\tan C}{\tan B} = \frac{AA_1}{A_1C} \cdot \frac{BA_1}{AA_1} = \frac{BA_1}{A_1C}.$$

Consequently, if the position vector of A_1 is $\vec{a_1}$, then

$$\vec{a_1} = \frac{(\tan B)\vec{\beta} + (\tan C)\vec{\gamma}}{\tan B + \tan C}. \quad (1)$$

Also, we have

$$\frac{\tan C}{\tan B + \tan C} = \frac{BA_1}{BA_1 + A_1C} = \frac{BA_1}{BC} = \frac{BA_1}{a} \Rightarrow BA_1 = \frac{a \tan C}{\tan B + \tan C}.$$

Again, since $\angle CBB_1 = 90^\circ - C$ and $\angle B_1BA = 90^\circ - A$, we have

$$\frac{A_1H}{HA} = \frac{[A_1BH]}{[HBA]} = \frac{\frac{1}{2} \cdot BA_1 \cdot BH \cdot \sin CBB_1}{\frac{1}{2} \cdot BH \cdot BA \cdot \sin B_1BA} = \frac{BA_1 \cdot \cos C}{c \cdot \cos A}.$$

Substituting the expressions for BA_1 , and using the Sine law, we obtain

$$\frac{A_1H}{HA} = \frac{a \tan C}{\tan B + \tan C} \cdot \frac{\cos C}{c \cdot \cos A} = \frac{\sin A \cdot \sin C}{\tan B + \tan C} \cdot \frac{1}{\sin C \cdot \cos A} = \frac{\tan A}{\tan B + \tan C}.$$

Consequently, if the position vector of H is $\vec{h}$, then using (3), we obtain

$$\vec{h} = \frac{(\tan A)\vec{\alpha} + (\tan B + \tan C)\vec{a_1}}{\tan A + \tan B + \tan C} = \frac{(\tan A)\vec{\alpha} + (\tan B)\vec{\beta} + (\tan C)\vec{\gamma}}{\tan A + \tan B + \tan C}.$$

Since $\overrightarrow{AH} + \overrightarrow{BH} + \overrightarrow{CH} = \vec{0}$, we have $(\vec{h} - \vec{\alpha}) + (\vec{h} - \vec{\beta}) + (\vec{h} - \vec{\gamma}) = \vec{0}$, which implies

$$\begin{aligned} (\tan B)(\vec{\beta} - \vec{\alpha}) + (\tan C)(\vec{\gamma} - \vec{\alpha}) + (\tan A)(\vec{\alpha} - \vec{\beta}) + (\tan C)(\vec{\gamma} - \vec{\beta}) \\ + (\tan A)(\vec{\alpha} - \vec{\gamma}) + (\tan B)(\vec{\beta} - \vec{\gamma}) = \vec{0}, \end{aligned}$$

which yields

$$(2 \tan A - \tan B - \tan C)\vec{\alpha} + (2 \tan B - \tan C - \tan A)\vec{\beta} + (2 \tan C - \tan A - \tan B)\vec{\gamma} = \vec{0}.$$

Since $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$ are non-collinear vector, we must have

$$2 \tan A - \tan B - \tan C = 0, \quad 2 \tan B - \tan C - \tan A = 0, \quad \text{and} \quad 2 \tan C - \tan A - \tan B = 0.$$

Therefore,

$$3 \tan A = \tan A + \tan B + \tan C = 3 \tan B = 3 \tan C,$$

and since A , B , and C are acute angles, we conclude that $A = B = C$. Hence the triangle must be equilateral.

Also solved by Theo Koupelis, Clark College, Washington, USA.

S690. Let

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b},$$

where a, b, c are positive real numbers such that at most one of them is less than 1 and $abc = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 3 \geq 2(x + y + z).$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

Solution by the author

Assume that $a \geq b \geq 1 \geq c$ and denote $p = a + b + c$ and $q = ab + bc + ca$. By the AM-GM inequality, we have

$$p \geq 3(abc)^{1/3} = 3.$$

In addition, from $(a-1)(b-1)(c-1) \leq 0$, we get

$$q \geq p.$$

Write the inequality as follows:

$$\begin{aligned} \sum_{cyc} \frac{b+c}{a} + 6 &\geq \sum_{cyc} \frac{8a}{b+c}, \\ \sum_{cyc} bc(b+c) + 6 &\geq \frac{8}{(b+c)(c+a)(a+b)} \sum_{cyc} a(a+b)(a+c), \\ pq + 3 &\geq \frac{8(p^3 - 2pq + 3)}{pq - 1}. \end{aligned}$$

So, for fixed p , we need to show that $f(q) \geq 0$, where

$$f(q) = p^2 q^2 + 18pq - 8p^3 - 27.$$

Since $f(q)$ is increasing, we have

$$f(q) \geq f(p) = p^4 - 8p^3 + 18p^2 - 27 = (p+1)(p^3 - 9p^2 + 27p - 27) = (p+1)(p-3)^3 \geq 0.$$

The equality occurs for $a = b = c = 1$.

Also solved by Arkady Alt, San Jose, California, USA; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

Undergraduate problems

U685. Let $\{a_n\}$ be the sequence defined by $a_1 = 2$ and $a_n = 2a_{n-1}^2 - 1, n = 2, 3, \dots$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1)}{a_n}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by 1 the author

From $a_n = 2a_{n-1}^2 - 1$ we obtain

$$a_n - 1 = 2(a_{n-1}^2 - 1) = 2(a_{n-1} - 1)(a_{n-1} + 1).$$

This gives us

$$\frac{a_n - 1}{a_{n-1} - 1} = 2(a_{n-1} + 1).$$

Similarly

$$\begin{aligned} \frac{a_{n-1} - 1}{a_{n-2} - 1} &= 2(a_{n-2} + 1), \\ &\dots \quad \dots \quad \dots, \\ \frac{a_2 - 1}{a_1 - 1} &= 2(a_1 + 1). \end{aligned}$$

Multiplying these relations we get

$$2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1) = \frac{a_n - 1}{a_1 - 1} = a_n - 1.$$

Therefore

$$\frac{2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1)}{a_n} = 1 - \frac{1}{a_n}.$$

It's easy to check that $a_n > 1$ for all n . Furthermore

$$a_n - a_{n-1} = 2a_{n-1}^2 - a_{n-1} - 1 = (a_{n-1} - 1)(2a_{n-1} + 1) > 0 \quad \forall n.$$

Hence $\{a_n\}$ is a increasing sequence. Now if $\{a_n\}$ has a finite upper bound, then there exists $\ell = \lim_{n \rightarrow \infty} a_n$ where ℓ is finite. Since $a_n > a_1 = 2$ for all n then $\ell \geq 2$. From $a_n = 2a_{n-1}^2 - 1$ we take $n \rightarrow \infty$ to get $\ell = 2\ell^2 - 1$. This yields $(\ell - 1)(2\ell + 1) = 0$ or $\ell = 1$ which is a contradiction. So $\lim_{n \rightarrow \infty} a_n = \infty$. From here we obtain

$$\lim_{n \rightarrow \infty} \frac{2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1)}{a_n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{a_n}\right) = 1.$$

Solution by 2 Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

Denote the limit to be found by L . We show that $L = 1$.

We first show that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. To do this, an explicit expression for a_n will first be found, before taking the appropriate limit. Let

$$a_1 = \frac{1}{2} \left(x + \frac{1}{x} \right), \quad x > 1.$$

Since $a_1 = 2$, on solving for x , for $x > 1$ we find $x = 2 + \sqrt{3}$. Next

$$a_2 = 2a_1^2 - 1 = 2 \left[\frac{1}{2} \left(x + \frac{1}{x} \right) \right]^2 - 1 = \frac{1}{2} \left(x^2 + \frac{1}{x^2} \right).$$

By induction on n , it can be readily shown that

$$a_n = \frac{1}{2} \left(x^{2^{n-1}} + \frac{1}{x^{2^{n-1}}} \right),$$

where n is a positive integer and $x = 2 + \sqrt{3}$. From this explicit expression we readily see that

$$a_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

Now consider the limit. Using the given recurrence relation for the sequence $\{a_n\}_{n \geq 1}$, namely $a_1 = 2$ and

$$a_n = 2a_{n-1}^2 - 1, \quad n = 2, 3, \dots$$

allows the limit to be rewritten as

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1)}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{2^{n-1} \cdot 2a_{n-2}^2 \cdot 2a_{n-3}^2 \cdots 2a_1^2 \cdot 3}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{2^{n-1} \frac{1}{2} (2a_{n-2})^2 \cdot \frac{1}{2} (2a_{n-3})^2 \cdots \frac{1}{2} (2a_1)^2 \cdot 3}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{2^{n-1} (2a_{n-2})^2 (2a_{n-3})^2 \cdots (2a_1)^2 \cdot 3}{2^{n-2} a_n} \\ &= 6 \lim_{n \rightarrow \infty} \frac{1}{a_n} \prod_{k=1}^{n-2} (2a_k)^2. \end{aligned}$$

Now

$$(2a_k)^2 = 2(2a_k^2) = 2(a_{k+1} + 1) = \frac{2(a_{k+1}^2 - 1)}{a_{k+1} - 1} = \frac{2(a_{k+1}^2 - 1)}{(2a_k^2 - 1) - 1} = \frac{a_{k+1}^2 - 1}{a_k^2 - 1}.$$

Returning to the limit and making use of the above result we have

$$\begin{aligned} L &= 6 \lim_{n \rightarrow \infty} \frac{1}{a_n} \prod_{k=1}^{n-2} \frac{a_{k+1}^2 - 1}{a_k^2 - 1} \\ &= 6 \lim_{n \rightarrow \infty} \frac{1}{a_n} \cdot \frac{a_{n-1}^2 - 1}{a_1^2 - 1} \\ &= 2 \lim_{n \rightarrow \infty} \frac{a_{n-1}^2 - 1}{a_n} \\ &= 2 \lim_{n \rightarrow \infty} \frac{a_{n-1}^2 - 1}{2a_{n-1}^2 - 1} \\ &= 2 \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{a_{n-1}^2}}{2 - \frac{1}{a_{n-1}^2}} \\ &= 2 \cdot \frac{1}{2} = 1, \end{aligned}$$

since $a_n \rightarrow \infty$ as $n \rightarrow \infty$, as announced.

Solution by 3 Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let $x = \ln(2 + \sqrt{3})$. Then

$$\cosh x = \frac{2 + \sqrt{3} + \frac{1}{2 + \sqrt{3}}}{2} = \frac{2 + \sqrt{3} + 2 - \sqrt{3}}{2} = 2,$$

so $a_1 = \cosh x$. Next, $a_2 = 2 \cosh^2 x - 1 = \cosh(2x)$ and by a straightforward induction argument, $a_n = \cosh(2^{n-1}x)$ for all $n \geq 1$. From here, we find

$$2(a_j + 1) = e^{2^{j-1}x} + 2 + e^{-2^{j-1}x} = \left(e^{2^{j-2}x} + e^{-2^{j-2}x}\right)^2$$

and

$$\begin{aligned} 2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1) &= \prod_{j=1}^{n-1} \left(e^{2^{j-2}x} + e^{-2^{j-2}x}\right)^2 \\ &= \left(\sum_{j=1}^{2^{n-1}} e^{(2^{n-1}-2j+1)x/2}\right)^2 \\ &= \left(\frac{e^{(2^{n-1}-1)x/2} - e^{(-2^{n-1}-1)x/2}}{1 - e^{-x}}\right)^2 \\ &= \left(\frac{e^{2^{n-2}x} - e^{-2^{n-2}x}}{e^{x/2} - e^{-x/2}}\right)^2 \\ &= \frac{\sinh^2(2^{n-2}x)}{\sinh^2(x/2)}. \end{aligned}$$

Note that

$$\sinh^2(x/2) = \frac{2 + \sqrt{3} - 2 + \frac{1}{2+\sqrt{3}}}{4} = \frac{\sqrt{3} + 2 - \sqrt{3}}{4} = \frac{1}{2},$$

so

$$\begin{aligned} \frac{\sinh^2(2^{n-2}x)}{\sinh^2(x/2)} &= 2 \sinh^2(2^{n-2}x) = 2(\cosh^2(2^{n-2}x) - 1) \\ &= (2 \cosh^2(2^{n-2}x) - 1) - 1 = \cosh(2^{n-1}x) - 1. \end{aligned}$$

Finally,

$$\lim_{n \rightarrow \infty} \frac{2^{n-1}(a_{n-1} + 1)(a_{n-2} + 1) \cdots (a_1 + 1)}{a_n} = \lim_{n \rightarrow \infty} \frac{\cosh(2^{n-1}x) - 1}{\cosh(2^{n-1}x)} = 1.$$

Also solved by Arkady Alt, San Jose, California, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Joshua Pit  , Cambridge Rindge and Latin School, MA, USA; Juan Jos   Granier, Cencosud, Chile; Nicu  or Zlota, "Traian Vuia" Technical College, Foc  ani, Romania; Paolo Perfetti, Universit   degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Diar Gashi, University of Primorska, Slovenia.

U686. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a convex function such that $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x) dx \right)^2 \leq \frac{3}{4} \int_0^1 f(x)^2 dx.$$

Proposed by Robert Rogozsan, Baia Mare, Romania

Solution by Daniel Pascuas, Barcelona, Spain

The inequality at the statement is equivalent to

$$\left| \int_0^1 f(x) dx \right| \leq \frac{\sqrt{3}}{2} \left(\int_0^1 f(x)^2 dx \right)^{\frac{1}{2}}. \quad (1)$$

We are going to prove that

$$\int_0^1 f(x) dx \leq \frac{\sqrt{3}}{2} \left(\int_0^1 f(x)^2 dx \right)^{\frac{1}{2}}, \quad (2)$$

which clearly implies (1) whenever $\int_0^1 f(x) dx \geq 0$. However we will show that (1) fails for many convex functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = 0$. By the way, note that the inequality (2) is sharp since it becomes an equality for $f(x) = x$.

Let us prove (2). Since f is convex and $f(0) = 0$, we have that

$$f(tx) = f(tx + (1-t)0) \leq tf(x) + (1-t)f(0) = tf(x) \quad (t, x \in [0, 1]),$$

so

$$\int_0^x f(t) dt = x \int_0^1 f(tx) dt \leq xf(x) \int_0^1 t dt = \frac{1}{2} xf(x) \quad (x \in [0, 1]).$$

By integrating this inequality and applying Fubini's theorem, we get that

$$\begin{aligned} \frac{1}{2} \int_0^1 xf(x) dx &\geq \int_0^1 \left(\int_0^x f(t) dt \right) dx = \int_0^1 \left(\int_t^1 f(t) dx \right) dt \\ &= \int_0^1 (1-t)f(t) dt = \int_0^1 f(x) dx - \int_0^1 xf(x) dx, \end{aligned}$$

and therefore

$$\int_0^1 f(x) dx \leq \frac{3}{2} \int_0^1 xf(x) dx.$$

Note that we may apply Fubini's theorem because the convexity of f on $[0, 1]$ implies that f is bounded and continuous on $(0, 1)$ (see, for example, Proposition 1.1.3 in C. P. Niculescu and L.-P. Persson, *Convex Functions and Their Applications - A contemporary approach*, Second Edition, CMS Books in Mathematics, Springer, 2018). Hence Cauchy-Schwarz inequality gives (2):

$$\int_0^1 f(x) dx \leq \frac{3}{2} \int_0^1 xf(x) dx \leq \frac{3}{2} \left(\int_0^1 x^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 f(x)^2 dx \right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2} \left(\int_0^1 f(x)^2 dx \right)^{\frac{1}{2}}.$$

Finally, for any $c > \frac{1}{2}$, consider $f_c(x) = (x - c)^2 - c^2 = x^2 - 2cx$, which is a convex function on $[0, 1]$ and satisfies $f_c(0) = 0$. Then

$$\int_0^1 f_c(x) dx = \frac{1}{3} - c \quad \text{and} \quad \int_0^1 f_c(x)^2 dx = \int_0^1 (x^4 - 4cx^3 + 4c^2x^2) dx = \frac{1}{5} - c + \frac{4c^2}{3},$$

so

$$\left| \int_0^1 f_c(x) dx \right| = c - \frac{1}{3} > \left(\frac{3}{20} - \frac{3c}{4} + c^2 \right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2} \left(\int_0^1 f_c(x)^2 dx \right)^{\frac{1}{2}},$$

because $(c - \frac{1}{3})^2 - (\frac{3}{20} - \frac{3c}{4} + c^2) = \frac{c}{12} - \frac{7}{180} = \frac{15c-7}{180} > 0$, since $c > \frac{1}{2}$. Hence (1) fails for f_c .

Editor's Comment. Paolo Perfetti provided the following counterexample for f . Let $f(x)$ be the convex function equal to $-xn$ for $0 \leq x \leq 1/n$, $f(x) = -1$ for $1/n \leq x \leq 1-1/n$, $f(x) = (x-1)n$ for $1-1/n \leq x \leq 1$.

$$\left(\int_0^1 f(x)dx\right)^2 = \left(1 - \frac{1}{n}\right)^2 \not\leq \frac{3}{4} \int_0^1 (f(x))^2 dx = \frac{3}{4} \left(1 - \frac{4}{3n}\right)^2$$

for any $n \geq 2$.

Also solved by Artem Borisov; G. C. Greubel, Newport News, VA, USA; Joshua Pité, Cambridge Rindge and Latin School, MA, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

U687. Prove that

$$\int_{\frac{4}{5}}^1 \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} dx < \int_0^1 \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt{x}} dx < \int_0^{\frac{1}{4}} \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} dx.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Since $\frac{1}{6} + \frac{8}{9} \log 2 < \frac{4}{5}$ and $\frac{5}{6} - \frac{8}{9} \log 2 < \frac{1}{4}$, it is sufficient to prove that

$$\int_{\frac{1}{6} + \frac{8}{9} \log 2}^1 \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} dx \leq \int_0^1 \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt{x}} dx \leq \int_0^{\frac{5}{6} - \frac{8}{9} \log 2} \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} dx.$$

First, we observe that the last two improper integrals converge because

$$\frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt{x}} \sim \frac{\frac{2}{3}x}{\sqrt{x}} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

and

$$\frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}} \sim \frac{2}{\sqrt{x}}$$

and the integral of the last function converges as $x \rightarrow 0$. In order to conclude, it is sufficient to use Steffensen's Inequality: if f and g are two integrable functions on $[a, b]$ such that f is decreasing and $0 \leq g(x) \leq 1$, then

$$\int_{b-\lambda}^b f(x) dx \leq \int_a^b f(x)g(x) dx \leq \int_a^{a+\lambda} f(x) dx,$$

where $\lambda = \int_a^b g(x) dx$.

By taking $f(x) = \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x}}{\sqrt{x}}$ and $g(x) = \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt[3]{1+x} + \sqrt[3]{1-x}}$, it is easy to check that these functions are integrable on $[0, 1]$, $f(x)$ is decreasing and $0 \leq g(x) \leq 1$ for all $x \in [0, 1]$. Moreover, from Problem U531 proposed by the author we know that

$$\lambda = \int_0^1 \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt[3]{1+x} + \sqrt[3]{1-x}} dx = \frac{5}{6} - \frac{8}{9} \log 2,$$

and the conclusion follows.

Solution 2 by Daniel Pascuas, Barcelona, Spain

By applying the substitution $t = \sqrt{x}$ we may write the inequality to be proved as

$$\int_{\frac{2}{\sqrt{5}}}^1 (\sqrt[3]{1+t^2} + \sqrt[3]{1-t^2}) dt < \int_0^1 (\sqrt[3]{1+t^2} - \sqrt[3]{1-t^2}) dt < \int_0^{\frac{1}{2}} (\sqrt[3]{1+t^2} + \sqrt[3]{1-t^2}) dt. \quad (*)$$

Taking into account the binomial expansion

$$\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}} = \sum_{n=0}^{\infty} c_n x^n \quad (-1 < x < 1),$$

where $c_n = \binom{\frac{1}{3}}{n}$, we get that

$$\sqrt[3]{1+x} - \sqrt[3]{1-x} = 2 \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \quad \text{and} \quad \sqrt[3]{1+x} + \sqrt[3]{1-x} = 2 \sum_{k=0}^{\infty} c_{2k} x^{2k},$$

for $0 < x < 1$, and, in particular,

$$\begin{aligned}\sqrt[3]{1+t^2} - \sqrt[3]{1-t^2} &= 2 \sum_{k=0}^{\infty} c_{2k+1} t^{4k+2} = \frac{2}{3} + 2 \sum_{k=1}^{\infty} c_{2k+1} t^{4k+2} \quad \text{and} \\ \sqrt[3]{1+t^2} + \sqrt[3]{1-t^2} &= 2 \sum_{k=0}^{\infty} c_{2k} t^{4k} = 2 + 2 \sum_{k=1}^{\infty} c_{2k} t^{4k}, \quad \text{for } 0 < t < 1.\end{aligned}$$

But $c_n = \frac{1}{3}(\frac{1}{3} - 1) \cdots (\frac{1}{3} - (n-1)) \frac{1}{n!}$, for $n \geq 1$, so $c_{2k} < 0 < c_{2k+1}$, for $k \geq 1$, and therefore

$$\sqrt[3]{1+t^2} + \sqrt[3]{1-t^2} < 2 \quad \text{and} \quad \frac{2}{3} < \sqrt[3]{1+t^2} - \sqrt[3]{1-t^2}, \quad \text{for } 0 < t < 1.$$

Hence we obtain the first inequality in (*):

$$\int_{\frac{2}{\sqrt{5}}}^1 (\sqrt[3]{1+t^2} + \sqrt[3]{1-t^2}) dt < 2 \left(1 - \frac{2}{\sqrt{5}}\right) < \frac{2}{3} < \int_0^1 (\sqrt[3]{1+t^2} - \sqrt[3]{1-t^2}) dt.$$

The second inequality in (*) can be written as

$$\int_{\frac{1}{2}}^1 (\sqrt[3]{1+t^2} - \sqrt[3]{1-t^2}) dt < 2 \int_0^{\frac{1}{2}} \sqrt[3]{1-t^2} dt,$$

which holds because

$$\int_{\frac{1}{2}}^1 (\sqrt[3]{1+t^2} - \sqrt[3]{1-t^2}) dt = \int_0^{\frac{1}{2}} (\sqrt[3]{1+(t+\frac{1}{2})^2} - \sqrt[3]{1-(t+\frac{1}{2})^2}) dt$$

and

$$\sqrt[3]{1+(t+\frac{1}{2})^2} - \sqrt[3]{1-(t+\frac{1}{2})^2} < 2\sqrt[3]{1-t^2} \quad (0 < t < \frac{1}{2}),$$

since the function $f(t) = 2\sqrt[3]{1-t^2} + \sqrt[3]{1-(t+\frac{1}{2})^2} - \sqrt[3]{1+(t+\frac{1}{2})^2}$ satisfies that:

- f is decreasing on $[0, \frac{1}{2}]$: Just note that

$$\begin{aligned}f'(t) &= \frac{4}{3}t(1-t^2)^{-\frac{2}{3}} - \frac{2}{3}(t+\frac{1}{2})(1-(t+\frac{1}{2})^2)^{-\frac{2}{3}} - \frac{2}{3}(t+\frac{1}{2})(1+(t+\frac{1}{2})^2)^{-\frac{2}{3}} \\ &< \frac{1}{3}(4t-2(t+\frac{1}{2}))(1-(t+\frac{1}{2})^2)^{-\frac{2}{3}} = \frac{1}{3}(2t-1)(1-(t+\frac{1}{2})^2)^{-\frac{2}{3}} \leq 0,\end{aligned}$$

for any $t \in [0, \frac{1}{2}]$.

- $f(\frac{1}{2}) = 2\sqrt[3]{\frac{3}{4}} - \sqrt[3]{2} = \sqrt[3]{6} - \sqrt[3]{2} > 0$.

Also solved by Joshua Pit , Cambridge Rindge and Latin School, MA, USA; Paolo Perfetti, Universit  degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

U688. Evaluate

$$\lim_{n \rightarrow \infty} \left(1 + \int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx \right)^n.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

Denote the limit to be evaluated by L . We show that

$$L = e^{\log^2(2)}.$$

Write the limit as

$$L = \lim_{n \rightarrow \infty} (1 + I_n)^n,$$

where

$$I_n = \int_0^1 \frac{x^n \log(1+x)}{1+x^n} dx.$$

Setting $x = t^{1/n}$ in the integral gives

$$I_n = \frac{1}{n} \int_0^1 \frac{t^{1/n} \log(1+t^{1/n})}{1+t} dt.$$

Note that as $n \rightarrow \infty$, for $t \in (0, 1)$ we see that $t^{1/n} \rightarrow 1$. So for $n \rightarrow \infty$, by the dominated convergence theorem we have

$$I_n \rightarrow \frac{1}{n} \int_0^1 \frac{1 \cdot \log(1+t)}{1+t} dt = \frac{\log(2)}{n} \int_0^1 \frac{dt}{1+t} = \frac{1}{n} \log^2(2).$$

So for the limit, by the dominated convergence theorem we have

$$L = \lim_{n \rightarrow \infty} \left(1 + \frac{\log^2(2)}{n} \right)^n = e^{\log^2(2)},$$

as announced. Here the well known result of

$$e^x = \lim_{x \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n,$$

with $x = \log^2(2)$ has been used.

Solution 2 by Daniel Pascuas, Barcelona, Spain

First, we integrate by parts to obtain that

$$\begin{aligned} \int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx &= \frac{1}{n} \int_0^1 x \ln(1+x) d \ln(1+x^n) \\ &= \frac{1}{n} \left\{ \left[x \ln(1+x) \ln(1+x^n) \right]_0^1 - \int_0^1 \ln(1+x^n) d(x \ln(1+x)) \right\} \\ &= \frac{1}{n} \left\{ (\ln 2)^2 - \int_0^1 \ln(1+x^n) \ln(1+x) dx - \int_0^1 \frac{x \ln(1+x^n)}{1+x} dx \right\}. \end{aligned}$$

Since $0 \leq \ln(1+t) \leq t$, for any $t \geq 0$, we have that

$$\begin{aligned} 0 &\leq \int_0^1 \frac{x \ln(1+x^n)}{1+x} dx \leq \int_0^1 x^{n+1} dx = \frac{1}{n+2} \quad \text{and} \\ 0 &\leq \int_0^1 \ln(1+x^n) \ln(1+x) dx \leq (\ln 2) \int_0^1 x^n dx = \frac{\ln 2}{n+1}. \end{aligned}$$

It follows that

$$\int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx = \frac{(\ln 2)^2}{n} + O\left(\frac{1}{n^2}\right),$$

so, taking into account that $\ln(1+t) = t + O(t^2)$, as $t \rightarrow 0$, we deduce that

$$\begin{aligned} \left(1 + \int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx\right)^n &= \exp\left\{n \ln\left(1 + \int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx\right)\right\} \\ &= \exp\left\{n \ln\left(1 + \frac{(\ln 2)^2}{n} + O\left(\frac{1}{n^2}\right)\right)\right\} \\ &= \exp\left\{(\ln 2)^2 + O\left(\frac{1}{n}\right)\right\}. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \left(1 + \int_0^1 \frac{x^n \ln(1+x)}{1+x^n} dx\right)^n = e^{(\ln 2)^2} = 2^{\ln 2}.$$

Also solved by G. C. Greubel, Newport News, VA, USA; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Srijan Sundar, Oxford, UK.

U689. Evaluate the integral:

$$\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx.$$

Proposed by Vasile Mircea Popa, Sibiu, Romania

Solution 1 by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

Denote the integral to be evaluated by I . We show that

$$I = \frac{5\pi^4}{81\sqrt{3}}.$$

We begin by enforcing a substitution of $x \mapsto \frac{1}{x}$ in the integral. This gives

$$I = \int_0^\infty \frac{\arctan(\frac{1}{x}) \log^2(x)}{1 - x + x^2} dx.$$

Since

$$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan(x), \quad x > 0,$$

allows the integral to be rewritten as

$$I = \frac{\pi}{2} \int_0^\infty \frac{\log^2(x)}{x^2 - x + 1} dx - \int_0^\infty \frac{\arctan(x) \log^2(x)}{x^2 - x + 1} dx = \frac{\pi}{2} \int_0^\infty \frac{\log^2(x)}{x^2 - x + 1} dx - I,$$

or

$$I = \frac{\pi}{4} \int_0^\infty \frac{\log^2(x)}{x^2 - x + 1} dx.$$

Since

$$1 + x^3 = (1 + x)(x^2 - x + 1),$$

the above integral for I can be rewritten as

$$I = \frac{\pi}{4} \int_0^\infty \frac{(1 + x) \log^2(x)}{1 + x^3} dx = \frac{\pi}{4} \left[\int_0^\infty \frac{\log^2(x)}{1 + x^3} dx + \int_0^\infty \frac{x \log^2(x)}{1 + x^3} dx \right]. \quad (2)$$

To find the values of the two integrals appearing in (2), we consider the parametric integral

$$J(a, b) = \int_0^\infty \frac{x^{a-1}}{1 + x^b} dx, \quad 0 < a < b.$$

Enforcing a substitution of $u = x^b$ produces

$$J(a, b) = \frac{1}{b} \int_0^\infty \frac{u^{\frac{a}{b}-1}}{1 + u} du.$$

The integral that has appeared can be expressed in terms of a beta function. Here

$$J(a, b) = \frac{1}{b} B\left(\frac{a}{b}, 1 - \frac{a}{b}\right) = \frac{1}{b} \Gamma\left(\frac{a}{b}\right) \Gamma\left(1 - \frac{a}{b}\right) = \frac{\pi}{b \sin\left(\frac{a\pi}{b}\right)}.$$

Here the connection between the beta and gamma function of

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

has been used together with Euler's reflexion formula for the gamma function of

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Thus

$$\frac{\partial^2 J}{\partial \alpha^2} = \int_0^\infty \frac{x^{a-1} \log^2(x)}{1+x^b} dx = \frac{\pi^3(\cos(\frac{2a\pi}{b}) + 3)}{2b^3 \sin^3(\frac{a\pi}{b})}.$$

Returning to the integral in (2) we see that

$$I = \frac{\pi}{4} \{J(1, 3) + J(2, 3)\} = \frac{\pi}{4} \left\{ \frac{\pi^3(\cos(\frac{2\pi}{3}) + 3)}{2 \cdot 3^3 \sin^3(\frac{\pi}{3})} + \frac{\pi^3(\cos(\frac{4\pi}{3}) + 3)}{2 \cdot 3^3 \sin^3(\frac{2\pi}{3})} \right\} = \frac{5\pi^4}{81\sqrt{3}},$$

as desired.

Solution 2 by Joshua Pit  , Cambridge Rindge and Latin School, MA, USA

The function $f(x) = \frac{\arctan x \ln^2 x}{x^2-x+1}$ is continuous over $(0, \infty)$.

Near 0^+ we have $f(x) \sim x(\ln x)^2$, and near ∞ , $f(x) = o(x^{-3/2})$. Hence the given integral converges. Denote

$$I = \int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2-x+1} dx.$$

Using the substitution $x = \frac{1}{u}$ we have

$$I = \int_\infty^0 \frac{-\arctan(\frac{1}{u}) (\ln u)^2}{1-u+u^2} du = \int_0^\infty \frac{\arctan(\frac{1}{u}) (\ln u)^2}{u^2-u+1} du.$$

For $u > 0$, we know the identity $\arctan(\frac{1}{u}) = \frac{\pi}{2} - \arctan u$.

Thus

$$\begin{aligned} I + I &= \int_0^\infty \frac{\arctan u (\ln u)^2}{u^2-u+1} du + \int_0^\infty \frac{\left(\frac{\pi}{2} - \arctan u\right) (\ln u)^2}{u^2-u+1} du \\ &= \int_0^\infty \frac{\left(\arctan u + \frac{\pi}{2} - \arctan u\right) (\ln u)^2}{u^2-u+1} du = \frac{\pi}{2} \int_0^\infty \frac{(\ln u)^2}{u^2-u+1} du. \end{aligned}$$

That is

$$I = \frac{\pi}{4} \int_0^\infty \frac{(\ln u)^2}{u^2-u+1} du =: \frac{\pi}{4} J.$$

Writing $J = \int_0^1 \frac{\ln^2 x}{x^2-x+1} dx + \int_1^\infty \frac{\ln^2 x}{x^2-x+1} dx$ and $u = 1/x$ again, we obtain that $J = 2 \int_0^1 \frac{\ln^2 x}{x^2-x+1} dx$.

We can write, for all $x \in \mathbb{R}$, with $|x| < 1$, $\frac{1}{x^2-x+1} = \sum_{n \geq 0} a_n x^n$, where the coefficients a_n are

$$a_n = \begin{cases} 1 & n \equiv 0, 1 \pmod{6} \\ -1 & n \equiv 3, 4 \pmod{6} \\ 0 & \text{otherwise} \end{cases}$$

so that

$$J = 2 \int_0^1 \frac{\ln^2 x}{x^2-x+1} dx = 2 \sum_{n \geq 0} a_n \int_0^1 x^n \ln^2 x dx.$$

By integration by parts, with $u = \ln^2 x$, $dv = x^n dx$, we get

$$\int_0^1 x^n \ln^2 x dx = -\frac{2}{n+1} \int_0^1 \frac{x^{n+1}}{x} \ln x dx = \frac{-2}{n+1} \int_0^1 x^n \ln x dx$$

With one more integration by parts,

$$\int_0^1 x^n \ln^2 x dx = \frac{2}{(n+1)^3}.$$

$$\begin{aligned}
J &= 4 \sum_{m \geq 0} \left[\frac{1}{(6m+1)^3} + \frac{1}{(6m+2)^3} - \frac{1}{(6m+4)^3} - \frac{1}{(6m+5)^3} \right] \\
&= \frac{4}{6^3} \sum_{m \geq 0} \left[\frac{1}{(m+1/6)^3} + \frac{1}{(m+1/3)^3} - \frac{1}{(m+2/3)^3} - \frac{1}{(m+5/6)^3} \right]
\end{aligned}$$

Recall that the Hurwitz Zeta function is $\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$ for $\Re(s) > 1$ and $a > 0$. We can then write

$$I = \frac{\pi}{6^3} \left(\zeta(3, 1/6) + \zeta(3, 1/3) - \zeta(3, 2/3) - \zeta(3, 5/6) \right).$$

The polygamma function is $\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x)$, where $\Gamma(x)$ is the Gamma function. We'll use $\psi^{(2)}(x)$, which is related to $\zeta(3, x)$ by $\psi^{(2)}(x) = -2\zeta(3, x)$.

We can use a reflection formula for the polygamma function:

$$\forall x \in (0, 1), \psi^{(2)}(1-x) - \psi^{(2)}(x) = \pi \frac{d^2}{dx^2} \cot(\pi x) = 2\pi^3 \cot(\pi x) \csc^2(\pi x).$$

Hence $\psi^{(2)}\left(\frac{5}{6}\right) - \psi^{(2)}\left(\frac{1}{6}\right) = 2\pi^3 \cot\left(\frac{\pi}{6}\right) \csc^2\left(\frac{\pi}{6}\right) = 8\sqrt{3}\pi^3$.

Similarly $\psi^{(2)}\left(\frac{4}{6}\right) - \psi^{(2)}\left(\frac{2}{6}\right) = \frac{8\sqrt{3}}{9}\pi^3$.

After simplification, we conclude that

$$\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx = \frac{5\sqrt{3}}{243} \pi^4 \approx 3.47155.$$

Solution 3 by Daniel Pascuas, Barcelona, Spain

The substitution $y = 1/x$ yields that

$$\begin{aligned}
\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx &= \int_0^\infty \frac{(\frac{\pi}{2} - \arctan y) \ln^2 y}{1 - y + y^2} dy \\
&= \frac{\pi}{2} \int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx - \int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx,
\end{aligned}$$

so

$$\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx = \frac{\pi}{4} \int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx = \frac{5\pi^4}{81\sqrt{3}},$$

because

$$\int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx = \frac{20\pi^3}{81\sqrt{3}}. \quad (*)$$

We give two different proofs of (*). The first one is just a standard evaluation of our integral by means of the residues theorem. In the second proof we apply the monotone convergence theorem to write our integral as the sum of a series which is evaluated by using the classical simple fraction expansion of the cotangent (that is also a consequence of the residues theorem).

Proof 1 of (*). Note that $\int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx = F''(0)$, where F is the holomorphic function on the strip $|\operatorname{Re} z| < 1$ defined by

$$F(z) = \int_0^\infty \frac{x^z dx}{x^2 - x + 1} \quad (|\operatorname{Re} z| < 1).$$

Now we are going to compute the integral $F(z)$ by applying the residue theorem to the meromorphic function $f_z(w) = \frac{w^z}{w^2 - w + 1} = \frac{(w+1)w^z}{w^3 + 1}$ and the keyhole contour $\gamma_{\delta, r, R}$, which we are going to define in a while. Here w^z denotes the branch of the z -power in the domain $\Omega = \mathbb{C} \setminus [0, \infty)$ (where $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$, as usual)

whose value at $w = -1$ equals to $e^{i\pi}$. Thus f_z is a meromorphic function on Ω whose only poles are $e^{\pm i\frac{\pi}{3}}$ (simple poles).

For $-\frac{\pi}{2} < \delta < \frac{\pi}{2}$ and $0 < r < R$, let $\alpha_{\delta,r,R}$ be the line segment which goes from $re^{i\delta}$ to $Re^{i\delta}$.

For $0 < \delta < \frac{\pi}{2}$ and $\varrho > 0$, let $\beta_{\delta,\varrho}$ be the arc of the circle centered at the origin with radius ϱ which goes from $\varrho e^{i\delta}$ to $\varrho e^{-i\delta}$ counterclockwise. Moreover, if α is a path on the complex plane, then α^{-1} denotes the path α traversed in the opposite direction. For $0 < \delta < \frac{\pi}{2}$ and $0 < r < 1 < R$, $\gamma_{\delta,r,R}$ is the concatenation of $\alpha_{\delta,r,R}$, $\beta_{\delta,R}$, $\alpha_{-\delta,r,R}^{-1}$, and $\beta_{\delta,r}^{-1}$. Note that $\gamma_{\delta,r,R}$ is a closed rectifiable curve in $\Omega \setminus \{e^{\pm i\frac{\pi}{3}}\}$. Then the residue theorem shows that

$$\int_{\gamma_{\delta,r,R}} f_z(w) dw = 2\pi i \left\{ \text{Res}(f_z(w); w = e^{i\frac{\pi}{3}}) + \text{Res}(f_z(w); w = e^{-i\frac{\pi}{3}}) \right\}.$$

Since

$$\begin{aligned} \text{Res}(f_z(w); w = e^{i\frac{\pi}{3}}) &= \lim_{w \rightarrow e^{i\frac{\pi}{3}}} (w - e^{i\frac{\pi}{3}}) f_z(w) = \lim_{w \rightarrow e^{i\frac{\pi}{3}}} \frac{w^z}{w - e^{-i\frac{\pi}{3}}} = \frac{1}{i\sqrt{3}} e^{\frac{\pi}{3}iz} \quad \text{and} \\ \text{Res}(f_z(w); w = e^{-i\frac{\pi}{3}}) &= \lim_{w \rightarrow e^{-i\frac{\pi}{3}}} (w - e^{-i\frac{\pi}{3}}) f_z(w) = \lim_{w \rightarrow e^{-i\frac{\pi}{3}}} \frac{w^z}{w - e^{i\frac{\pi}{3}}} = -\frac{1}{i\sqrt{3}} e^{\frac{5\pi}{3}iz}, \end{aligned}$$

we have that

$$\int_{\gamma_{\delta,r,R}} f_z(w) dw = \frac{2\pi}{\sqrt{3}} (e^{\frac{\pi}{3}iz} - e^{\frac{5\pi}{3}iz}) = \frac{2\pi}{\sqrt{3}} e^{\pi iz} (e^{-\frac{2\pi}{3}iz} - e^{\frac{2\pi}{3}iz}).$$

Therefore

$$\frac{2\pi}{\sqrt{3}} e^{\pi iz} (e^{-\frac{2\pi}{3}iz} - e^{\frac{2\pi}{3}iz}) = \int_{\gamma_{\delta,r,R}} f_z(w) dw = \left\{ \int_{\alpha_{\delta,r,R}} + \int_{\beta_{\delta,R}} - \int_{\alpha_{-\delta,r,R}} - \int_{\beta_{\delta,r}} \right\} f_z(w) dw.$$

Then let $\delta \searrow 0$ to get that

$$\frac{2\pi}{\sqrt{3}} e^{\pi iz} (e^{-\frac{2\pi}{3}iz} - e^{\frac{2\pi}{3}iz}) = (1 - e^{2\pi iz}) \int_r^R \frac{x^z dx}{x^2 - x + 1} + \int_0^{2\pi} \frac{R^z e^{iz\theta} i R e^{i\theta} d\theta}{R^2 e^{2i\theta} - R e^{i\theta} + 1} - \int_0^{2\pi} \frac{r^z e^{iz\theta} i r e^{i\theta} d\theta}{r^2 e^{2i\theta} - r e^{i\theta} + 1}.$$

But

$$\begin{aligned} \left| \int_0^{2\pi} \frac{R^z e^{iz\theta} i R e^{i\theta} d\theta}{R^2 e^{2i\theta} - R e^{i\theta} + 1} \right| &\leq 2\pi \frac{R^{1+\text{Re } z} e^{2\pi|\text{Re } z|}}{R^2 - R - 1} \rightarrow 0, \quad \text{as } R \nearrow \infty, \quad \text{and} \\ \left| \int_0^{2\pi} \frac{r^z e^{iz\theta} i r e^{i\theta} d\theta}{r^2 e^{2i\theta} - r e^{i\theta} + 1} \right| &\leq 2\pi \frac{r^{1+\text{Re } z} e^{2\pi|\text{Re } z|}}{1 - r - r^2} \rightarrow 0, \quad \text{as } r \searrow 0, \quad \text{for } |\text{Re } z| < 1. \end{aligned}$$

It follows that

$$F(z) = \int_0^\infty \frac{x^z dx}{x^2 - x + 1} = \frac{2\pi}{\sqrt{3}} \frac{e^{i\pi z} (e^{-\frac{2\pi}{3}iz} - e^{\frac{2\pi}{3}iz})}{(1 - e^{2\pi iz})} = \frac{2\pi}{\sqrt{3}} \frac{\sin(\frac{2\pi}{3}z)}{\sin(\pi z)} = \frac{4\pi}{3\sqrt{3}} \frac{\text{sinc}(\frac{2\pi}{3}z)}{\text{sinc}(\pi z)} \quad (|\text{Re } z| < 1),$$

where

$$\text{sinc } z = \frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k} \quad (z \in \mathbb{C}),$$

which is an entire function without zeros on the strip $|\text{Re } z| < 1$. As a consequence,

$$F''(z) = \frac{4\pi}{3\sqrt{3}} \frac{d^2}{dz^2} \left(\frac{\text{sinc}(\frac{2\pi}{3}z)}{\text{sinc}(\pi z)} \right) = \frac{4\pi}{3\sqrt{3}} \frac{d}{dz} \left(\frac{\frac{2\pi}{3} \text{sinc}'(\frac{2\pi}{3}z) \text{sinc}(\pi z) - \pi \text{sinc}(\frac{2\pi}{3}z) \text{sinc}'(\pi z)}{\text{sinc}^2(\pi z)} \right),$$

for $-1 < \text{Re } z < 1$, and therefore $F''(0) = -\frac{20\pi^3}{27\sqrt{3}} \text{sinc}''(0) = \frac{20\pi^3}{81\sqrt{3}}$, since $\text{sinc}(0) = 1$, $\text{sinc}'(0) = 0$, and $\text{sinc}''(0) = -\frac{1}{3}$. Hence we finally obtain (*).

Proof 2 of (*). The substitution $y = 1/x$ yields that

$$\int_1^\infty \frac{\ln^2 x}{x^2 - x + 1} dx = \int_0^1 \frac{\ln^2 y}{1 - y + y^2} dy = \int_0^1 \frac{\ln^2 x}{x^2 - x + 1} dx,$$

so

$$\int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx = \left\{ \int_0^1 + \int_1^\infty \right\} \frac{\ln^2 x}{x^2 - x + 1} dx = 2 \int_0^1 \frac{\ln^2 x}{x^2 - x + 1} dx = 2 \int_0^1 \frac{(1+x) \ln^2 x}{1+x^3} dx.$$

Since

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} ((x^3)^{2n} - (x^3)^{2n+1}) = \sum_{n=0}^{\infty} (1-x^3)x^{6n} \quad (0 < x < 1),$$

the monotone convergence theorem shows that

$$\begin{aligned} \int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx &= 2 \int_0^1 \frac{(1+x) \ln^2 x}{1+x^3} dx = 2 \sum_{n=0}^{\infty} \int_0^1 (1+x)(1-x^3)x^{6n} \ln^2 x dx \\ &= 2 \sum_{n=0}^{\infty} (I_{6n} + I_{6n+1} - I_{6n+3} - I_{6n+4}), \end{aligned}$$

where

$$I_m = \int_0^1 x^m \ln^2 x dx \quad (m = 0, 1, 2, \dots).$$

We evaluate I_m by integration by parts:

$$\begin{aligned} I_m &= \frac{1}{m+1} \int_0^1 \ln^2 x dx^{m+1} = \frac{1}{m+1} \left\{ \left[x^{m+1} \ln^2 x \right]_0^1 - 2 \int_0^1 x^m \ln x dx \right\} \\ &= -\frac{2}{(m+1)^2} \int_0^1 \ln x dx^{m+1} = -\frac{2}{(m+1)^2} \left\{ \left[x^{m+1} \ln x \right]_0^1 - \int_0^1 x^m dx \right\} = \frac{2}{(m+1)^3}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx &= 2 \sum_{n=0}^{\infty} (I_{6n} + I_{6n+1} - I_{6n+3} - I_{6n+4}) \\ &= 4 \sum_{n=0}^{\infty} \left(\frac{1}{(6n+1)^3} + \frac{1}{(6n+2)^3} - \frac{1}{(6n+4)^3} - \frac{1}{(6n+5)^3} \right) \\ &= 4 \sum_{n=0}^{\infty} \left(\frac{1}{(6n+1)^3} + \frac{1}{(6n+2)^3} + \frac{1}{(2-6(n+1))^3} + \frac{1}{(1-6(n+1))^3} \right) \\ &= 4 \sum_{n=-\infty}^{\infty} \left(\frac{1}{(6n+1)^3} + \frac{1}{(6n+2)^3} \right) = \frac{1}{54} \sum_{n=-\infty}^{\infty} \left(\frac{1}{(n+\frac{1}{6})^3} + \frac{1}{(n+\frac{1}{3})^3} \right). \end{aligned}$$

Now we evaluate this series by a nice complex variable argument. Recall the classical simple fraction expansion of the cotangent:

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} \quad (z \in \mathbb{C} \setminus \mathbb{Z}),$$

where the series converges uniformly on any compact subset of $\mathbb{C} \setminus \mathbb{Z}$. It follows that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^3} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{d^2}{dz^2} \left(\frac{1}{z+n} \right) = \frac{\pi}{2} \frac{d^2}{dz^2} \cot(\pi z) = \pi^3 \frac{\cos(\pi z)}{\sin^3(\pi z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}).$$

Hence we finally obtain (*):

$$\int_0^\infty \frac{\ln^2 x}{x^2 - x + 1} dx = \frac{\pi^3}{54} \left(\frac{\cos(\frac{\pi}{6})}{\sin^3(\frac{\pi}{6})} + \frac{\cos(\frac{\pi}{3})}{\sin^3(\frac{\pi}{3})} \right) = \frac{2\pi^3}{27} \left(\sqrt{3} + \frac{1}{3\sqrt{3}} \right) = \frac{20\pi^3}{81\sqrt{3}}.$$

Editor's Comment. In order to calculate the integral

$$I = \frac{\pi}{4} \int_0^\infty \frac{\ln^2 u}{u^2 - u + 1} du,$$

Henry Ricardo concluded immediately by using Formula 4.261.2 in *Table of Integrals, Series, and Products* (Sixth Edition) by I. S. Gradshteyn and I. M. Ryzhik, which states that

$$\int_0^\infty \frac{\ln^2 u}{u^2 - u + 1} du = \frac{20\pi^3}{81\sqrt{3}},$$

giving $I = 5\pi^4/81\sqrt{3}$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, New York, USA; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Prasanna Pawar, AMMOC, Bengaluru, India; Srijan Sundar, Oxford, UK.

U690. Find all the complex numbers z such that

$$(\Gamma(z+1))^3 + (z-1)^3 (\Gamma(z-1))^3 = ((z+1)^3 + 3) (\Gamma(z))^3,$$

where $\Gamma(z)$ is the Gamma function.

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

The given equality can be rewritten as

$$(\Gamma(z+1))^3 - (z+1)^3 (\Gamma(z))^3 + (z-1)^3 (\Gamma(z-1))^3 = 3 (\Gamma(z))^3.$$

If we set $a = \Gamma(z+1)$, $b = -(z+1)\Gamma(z)$ and $c = (z-1)\Gamma(z-1)$, the given equality becomes

$$a^3 + b^3 + c^3 = 3(\Gamma(z))^3.$$

Now, observe that for every complex number z we have $\Gamma(z+1) = z\Gamma(z)$, so

$$a + b + c = 0,$$

and from the identity $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$, we get

$$3abc = 3 (\Gamma(z))^3,$$

i.e.

$$-(z^2 - 1)\Gamma(z+1)\Gamma(z)\Gamma(z-1) = (\Gamma(z))^3.$$

By using once again $\Gamma(z+1) = z\Gamma(z)$, we get

$$-(z^2 - 1)z (\Gamma(z))^2 \Gamma(z-1) = (\Gamma(z))^3$$

and since $\Gamma(z) \neq 0$ for all complex numbers z , then

$$-(z^2 - 1)z\Gamma(z-1) = \Gamma(z),$$

i.e.

$$-(z+1)z = 1.$$

Therefore, $z^2 + z + 1 = 0$, which gives $z = \frac{-1 \pm i\sqrt{3}}{2}$.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Because

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad (z-1)\Gamma(z-1) = \Gamma(z),$$

the given equation is equivalent to

$$z^3(\Gamma(z))^3 + (\Gamma(z))^3 = ((z+1)^3 + 3)(\Gamma(z))^3,$$

i.e.

$$3(\Gamma(z))^3(z^2 + z + 1) = 0.$$

Since there is no $z \in \mathbb{C}$ for which $\Gamma(z) = 0$, it follows that $z^2 + z + 1 = 0$, i.e. $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

Also solved by G. C. Greubel, Newport News, VA, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Srijan Sundar, Oxford, UK.

Olympiad problems

O685. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + 3(ab + bc + ca) \geq 2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution 1 by Srijan Sundar, Oxford, UK

By the AM-GM Inequality we have

$$\begin{aligned}(a + b + c) \left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \right) &= ab + bc + ca + \sum_{cyc} a^2 \left(\frac{b}{c} + \frac{c}{b} \right) \\ &\geq ab + bc + ca + 2(a^2 + b^2 + c^2).\end{aligned}$$

Hence, since $a + b + c = 1$, we get

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + 3(ab + bc + ca) \geq 2(a + b + c)^2 = 2.$$

The equality holds if and only if $a = b = c = \frac{1}{3}$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

The desired inequality is equivalent to

$$a^2b^2 + b^2c^2 + c^2a^2 + 3abc(ab + bc + ca) \geq 2abc.$$

Homogenizing the desired inequality, it suffices to show that

$$(a + b + c)(a^2b^2 + b^2c^2 + c^2a^2) + 3abc(ab + bc + ca) \geq 2abc(a + b + c)^2.$$

Expanding and simplifying we get

$$a^3(b - c)^2 + b^3(c - a)^2 + c^3(a - b)^2 \geq 0,$$

which is obvious. The equality occurs when $a = b = c = \frac{1}{3}$.

Also solved by Hoang-Bao Le, Dublin, Ireland; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundaresh H. R., Shivamogga, Karnataka, India; Anderson Torres, Sao Paulo, Brazil; Daniel Pascuas, Barcelona, Spain; Nguyen Thanh Nam, Tu Son Secondary School, Vietnam.

O686. Let a, b, c, d be positive real numbers such that

$$2 \sum_{cyc} a + \sum_{sym} ab + 3 = abcd.$$

Prove that

$$\sqrt[4]{bcd} + \sqrt[4]{cda} + \sqrt[4]{dab} + \sqrt[4]{abc} \leq \sqrt[4]{27(1+a)(1+b)(1+c)(1+d)}$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Clark College, Washington, USA

The given condition is equivalent to

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 1.$$

Let $(x, y, z, w) = \left(\frac{1}{a+1}, \frac{1}{b+1}, \frac{1}{c+1}, \frac{1}{d+1} \right)$, where $x, y, z, w \in (0, 1)$ and $x + y + z + w = 1$. Thus, the desired inequality is equivalent to

$$\sum \sqrt[4]{\frac{(1-y)(1-z)(1-w)}{yzw}} \leq \sqrt[4]{\frac{27}{xyzw}},$$

or

$$\sum \sqrt[4]{3x(1-y)(1-z)(1-w)} \leq 3.$$

Using AM-GM we get

$$2 + 4x = 3x + (1-y) + (1-z) + (1-w) \geq 4\sqrt[4]{3x(1-y)(1-z)(1-w)}$$

with similar expressions for $2 + 4y, 2 + 4z, 2 + 4w$. Adding these expressions we get

$$\sum \sqrt[4]{3x(1-y)(1-z)(1-w)} \leq \frac{8 + 4(x+y+z+w)}{4} = 3,$$

which completes the proof.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania.

O687. Let x, y, z be real numbers such that $xy + yz + zx = 1$ and $x + y + z > 0$. Prove that

$$\frac{xy\sqrt{z^2+1} + yz\sqrt{x^2+1} + zx\sqrt{y^2+1}}{x+y+z} \leq \frac{2}{3}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

The inequality can be rewritten as

$$\frac{xy\sqrt{(y+z)(z+x)} + yz\sqrt{(x+y)(z+x)} + zx\sqrt{(y+z)(x+y)}}{(x+y+z)(xy+yz+zx)} \leq \frac{2}{3}.$$

We have two cases: $x+y, y+z, z+x \leq 0$ or $x+y, y+z, z+x \geq 0$. In first case with the substitutions $x \rightarrow -x, y \rightarrow -y, z \rightarrow -z$, we are in the second case. Hence, we will prove the inequality for $x+y, y+z, z+x \geq 0$. It is obvious that only one of the numbers x, y, z can be negative. Also, we have

$$\begin{aligned} \sqrt{x+z} + \sqrt{y+z} &= \sqrt{x+z+y+z+2\sqrt{(x+z)(y+z)}} \\ &= \sqrt{x+y+2z+2\sqrt{z^2+1}} \\ &> \sqrt{x+y}. \end{aligned}$$

Therefore, there is a triangle ABC with sides $a = \sqrt{y+z}, b = \sqrt{z+x}, c = \sqrt{x+y}$. The inequality becomes

$$3 \sum_{cyc} ab(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \leq (a^2 + b^2 + c^2) \sum_{cyc} (b^2 + c^2 - a^2)(c^2 + a^2 - b^2),$$

or

$$6 \sum_{cyc} a^3 b^3 + 6abc \sum_{cyc} a^3 - 3 \sum_{cyc} ab \sum_{cyc} a^4 \leq \sum_{cyc} a^2 \left(2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4 \right),$$

or

$$6 \sum_{cyc} a^3 b^3 + 6abc \sum_{cyc} a^3 - 6 \sum_{cyc} ab \sum_{cyc} a^2 b^2 \leq \left(\sum_{cyc} a^2 - 3 \sum_{cyc} ab \right) \left(2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4 \right),$$

or

$$6abc \sum_{cyc} a \left(2 \sum_{cyc} ab - \sum_{cyc} a^2 \right) + \left(\sum_{cyc} a^2 - 3 \sum_{cyc} ab \right) \left(2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4 \right) - 36a^2 b^2 c^2 \geq 0.$$

Using the known identities in a triangle

$$abc = 4Rrs, \quad ab + bc + ca = s^2 + r^2 + 4Rr, \quad a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr),$$

after performing the calculations, it becomes

$$48Rrs^2(4r^2 + 16Rr) - 16r^2s^2(s^2 + 5r^2 + 20Rr) - 36(16R^2r^2s^2) \geq 0,$$

or

$$s^2 \leq 12R^2 - 8Rr - 5r^2.$$

Since by Gerretsen's Inequality, we have

$$s^2 \leq 4R^2 + 4Rr + 3r^2,$$

it suffices to show that

$$4R^2 + 4Rr + 3r^2 \leq 12R^2 - 8Rr - 5r^2.$$

This factors as

$$(R - 2r)(2R + r) \geq 0,$$

which is true by Euler's Inequality. The equality holds when $x = y = z$.

Also solved by Arkady Alt, San Jose, California, USA; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA; Srijan Sundar, Oxford, UK.

O688. Inside the square $ABCD$ the points E, F are taken such that $\angle FDE = \angle EBF = 45^\circ$. It is given that $\text{area } FCD + \text{area } DEA + \text{area } EFB = a^2$. Find the length of the side AB .

Proposed by Mihaela Berindeanu, Bucharest, Romania

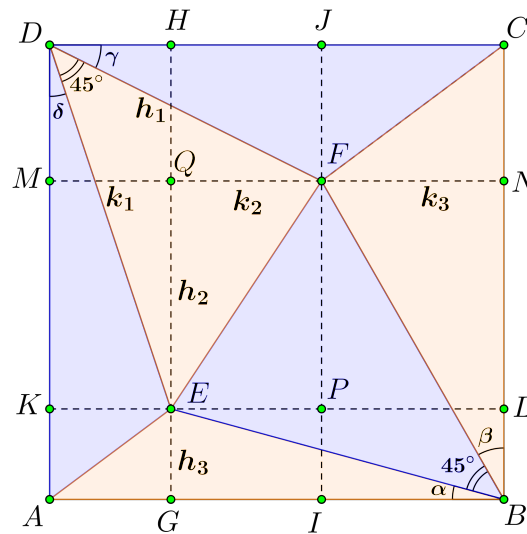
Solution by Kousik Sett, India

Let lines parallel to AD , passing through E, F meet AB, DC at G, I and H, J , respectively. Similarly, let lines parallel to AB , passing through E, F meet AD, BC at K, M and L, N , respectively.

Let $P = IJ \cap KL$, $Q = GH \cap MN$, and let x be the side length of the square $ABCD$ such that $x = h_1 + h_2 + h_3 = k_1 + k_2 + k_3$, where $h_1 = HQ$, $h_2 = QE$, $h_3 = EG$, and $k_1 = MQ$, $k_2 = QF$, $k_3 = FN$, respectively.

Assume that $\angle ABE = \alpha$, $\angle CBF = \beta$, $\angle CDF = \gamma$, and $\angle ADE = \delta$. Since $\angle FDE = \angle EBF = 45^\circ$, we infer that $\alpha + \beta = 45^\circ$ and $\gamma + \delta = 45^\circ$.

Lines GH, IJ, KL , and MN divide the square $ABCD$ into nine rectangles as shown at the figure below. Let $[XYZ \cdots W]$ denotes the area of the polygon $[XYZ \cdots W]$.



From right triangles BGE and BNF , we get

$$\tan \alpha = \frac{h_3}{x - k_1} \quad \text{and} \quad \tan \beta = \frac{k_3}{x - h_1}.$$

Since $\alpha + \beta = 45^\circ$, we have

$$\tan \alpha + \tan \beta + \tan \alpha \cdot \tan \beta = 1.$$

Substituting the expressions for $\tan \alpha$, $\tan \beta$, and simplifying, we obtain

$$x^2 - x(h_1 + k_1 + h_3 + k_3)x + h_1h_3 + k_1k_3 + h_1k_1 - h_3k_3 = 0. \quad (3)$$

From right triangles DJF and DKE , we get

$$\tan \gamma = \frac{h_1}{x - k_3} \quad \text{and} \quad \tan \delta = \frac{k_1}{x - h_3}.$$

Since $\gamma + \delta = 45^\circ$, we have

$$\tan \gamma + \tan \delta + \tan \gamma \cdot \tan \delta = 1.$$

Substituting the expressions for $\tan \gamma$, $\tan \delta$, and simplifying, we obtain

$$x^2 - x(h_1 + k_1 + h_3 + k_3)x + h_1h_3 + k_1k_3 - h_1k_1 + h_3k_3 = 0. \quad (4)$$

From (3) and (4), we obtain

$$h_1k_1 = h_3k_3, \quad \text{i.e.,} \quad [MQHD] = [IBLP]. \quad (5)$$

Observe that

$$\begin{aligned} & [FCD] + [DEA] + [EFB] \\ &= \frac{1}{2}([MNCD] + [AGHD]) + [GBNQ] - [GBE] - [BNF] - [EFQ] \\ &= \frac{1}{2}([MNCD] + [AGHD]) + [GBNQ] - \frac{1}{2}([GBLE] + [IBNF] + [EPFQ]) \\ &= \frac{1}{2}([MNCD] + [AGHD]) + [GBNQ] - \frac{1}{2}([GBNQ] + [IBLP]) \\ &= \frac{1}{2}([MQHD] + [QNCH] + [MQHD] + [AGQM]) + \frac{1}{2}([GBNQ] - [IBLP]) \\ &= \frac{1}{2}([MQHD] + [QNCH] + [AGQM] + [GBNQ]) + \frac{1}{2}([MQHD] - [IBLP]) \\ &= \frac{1}{2}([ABCD]). \quad [\text{by (5)}] \end{aligned}$$

Since $[FCD] + [DEA] + [EFB] = a^2$, we have $[ABCD] = 2a^2$, i.e., $x^2 = 2a^2$. So, $x = \sqrt{2}a$. Therefore, the length of the side $AB = \sqrt{2}a$.

Also solved by Theo Koupelis, Clark College, Washington, USA.

O689. Prove that $\frac{5}{2}$ is the minimum positive value of the constant k such that

$$\sqrt{\frac{b+c}{ka+b+c}} + \sqrt{\frac{c+a}{kb+c+a}} + \sqrt{\frac{a+b}{kc+a+b}} \geq 3\sqrt{\frac{2}{k+2}}$$

holds for any nonnegative real numbers a, b, c with $a+b+c > 0$.

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

Solution 1 by the author

For $b = c = 0$, the inequality becomes $2 \geq 3\sqrt{\frac{2}{k+2}}$, which is equivalent to $k \geq \frac{5}{2}$. To show that $\frac{5}{2}$ is the minimum value of the constant k , we need to prove the inequality

$$\sqrt{\frac{b+c}{5a+2b+2c}} + \sqrt{\frac{c+a}{5b+2c+2a}} + \sqrt{\frac{a+b}{5c+2a+2b}} \geq \sqrt{2}.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Applying the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned} \sum \sqrt{\frac{b+c}{5a+2b+2c}} &\geq \frac{[\sum(b+c)]^2}{\sum(b+c)^{3/2}\sqrt{5a+2b+2c}} \geq \frac{4(a+b+c)^2}{\sqrt{[\sum(b+c)][\sum(b+c)^2(5a+b+c)]}} \\ &= \frac{4p^2}{\sqrt{2p[4\sum a^3 + 11\sum ab(a+b) + 30abc]}} = \sqrt{\frac{8p^3}{4p^3 - pq + 9r}}. \end{aligned}$$

So, it suffices to show that

$$\frac{8p^3}{4p^3 - pq + 9r} \geq 2,$$

which is equivalent to the known inequality $pq \geq 9r$.

The proof is completed. The equality occurs for $a = b = c$, and also for $a = b = 0$ or $b = c = 0$ or $c = a = 0$.

Solution 2 by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let $b = c = 0$. The inequality becomes

$$2 \geq 3\sqrt{\frac{2}{k+2}} \iff k \geq 5/2$$

By taking $k = 5/2$ and using Hölder's inequality we get

$$\sum_{\text{cyc}} \sqrt{\frac{2(b+c)}{5a+2b+2c}} \sum_{\text{cyc}} \sqrt{\frac{2(b+c)}{5a+2b+2c}} \sum_{\text{cyc}} (2b+2c)^2(5a+2b+2c) \geq \left(\sum_{\text{cyc}} (2b+2c) \right)^3$$

The inequality is equivalent to

$$\left(\sum_{\text{cyc}} (2b+2c) \right)^3 \geq \sum_{\text{cyc}} (2b+2c)^2(5a+2b+2c) \cdot 9 \cdot \frac{2}{\frac{5}{2}+2}$$

Upon simplifying this last inequality is

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 6abc$$

which is evidently true by the AM-GM Inequality. The equality occurs when $a = b = c$ or when $a = b = 0$ or $b = c = 0$ or $c = a = 0$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA.

O690. Prove that there is an infinite sequence S of primes such that for any finite subset $A \subset S$ the sum of elements of A is not a perfect power.

Proposed by Mohammadreza Ghelichkhani, Tehran, Iran

Solution by the author

We construct S inductively. First, we add 2 to S . Now, assume that at some stage $S = \{p_1, p_2, \dots, p_t\}$, and let the set of all possible sums of elements from its subsets be $A = \{a_1, a_2, \dots, a_n\}$, where $n = 2^t$.

Let q_i be the i -th prime number that does not divide any element in A . We choose the next prime number, p_{t+1} , such that for each $1 \leq i \leq n$,

$$p_{t+1} + a_i \equiv q_i \pmod{q_i^2}.$$

It is clear that the solution to this congruence is coprime to q_i for each i . Therefore, by Dirichlet's theorem, there exists such a prime number.

Also solved by Juan José Granier, Cencosud, Chile.