

Junior Problems

- J691.** Let a, b, c be real numbers such that $a^2(b+c) + b^2(c+a) + c^2(a+b) = 3(a+b+c-1)$ and $a+b+c \neq 0$. Prove that $ab+bc+ca = 3$ if and only if $abc = 1$.

Proposed by Adrian Andreescu, Dallas, USA

- J692.** Let a, b, c be positive real numbers such that $a+b+c+2 = abc$ and denote $t = ab+bc+ca$. Prove that

$$\sqrt{t+a^2} + \sqrt{t+b^2} + \sqrt{t+c^2} \leq t.$$

Proposed by An Zhenping, Xianyang Normal University, China

- J693.** Let a, b, c be positive real numbers such that $a+b+c = 3abc$. Prove that

$$a^2 + b^2 + c^2 + 3 \geq 2(ab+bc+ca).$$

Proposed by Titu Andreescu, Dallas, USA

- J694.** Let x, y, z be positive real numbers such that $xy+yz+zx = 12$. Prove that

$$\sqrt{(4x+y+z)(x+y)(y+z)(z+x)} \geq 2\sqrt{3}(2x+y+z).$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

- J695.** In a chess tournament with $n \geq 5$ players, each player played all other players. One gets a point for a win, half a point for a draw, and zero points for a loss. At the end of the tournament, each player had a different number of points. Prove that the second and third ranked players had together more points than the winner of the tournament.

Proposed by Titu Andreescu, Dallas, USA

- J696.** Let a, b, c be pairwise distinct real numbers such that

$$\left(\frac{a-b}{b-c}\right)^2 + \left(\frac{b-c}{c-a}\right)^2 + \left(\frac{c-a}{a-b}\right)^2 = 5.$$

Evaluate

$$\frac{a-b}{b-c} + \frac{b-c}{c-a} + \frac{c-a}{a-b}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Senior Problems

S691. Let S be the set of positive real numbers and $f : S \times S \longrightarrow S$ be a function such that

$$f(f(x, y), z) = \frac{xy + yz + zx}{xyz}$$

for all $x, y, z \in S$. The sequence $(a_n)_{n \geq 0}$ is defined as follows: $a_0 = 1$ and $a_{n+1} = f(f(a_n, 1), 1)$, for all $n \geq 0$. Evaluate a_n .

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

S692. Let a, b, c be positive real numbers such that $ab + bc + ca = 3\sqrt{2}$. Prove that

$$\frac{ab}{\sqrt{a^2 + b^2}} + \frac{bc}{\sqrt{b^2 + c^2}} + \frac{ca}{\sqrt{c^2 + a^2}} \leq \frac{abc}{2} + \frac{6}{a + b + c}.$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

S693. Let a and b be distinct positive integers.

- (a) Prove that there are finitely many integers n such that $\sqrt{n+a} + \sqrt{n+b}$ is an integer.
- (b) Determine all rational numbers r such that $\sqrt{r+a} + \sqrt{r+b}$ is rational.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

S694. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$a^3b + b^3c + c^3a + 9abc \leq 12.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S695. Prove that in any triangle ABC

$$\sum \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) \sqrt{\left(\tan \frac{B}{2} + \tan \frac{C}{2} \right) \left(\tan \frac{C}{2} + \tan \frac{A}{2} \right)} \geq 4.$$

Proposed by Adrian Andreescu, Dallas, USA

S696. Let $ABCD$ be a tetrahedron and M be a point inside it. Let N, P, Q be the intersections of BM, CM, DM with planes $(ACD), (ADB), (ABC)$, respectively. Find the point M inside the tetrahedron $ABCD$ such that the volume of the tetrahedron $MNPQ$ is a maximum.

Proposed by Nguyen Minh Ha and Vu Phuong Thuy, Hanoi, Vietnam

Undergraduate Problems

U691. Let $I_n(k)$ be the number of permutations in S_n with exactly k inversions. For $n \geq 4$, evaluate

$$\sum_{j \geq 1} (-1)^{j-1} I_n(2j).$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

U692. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and a, b, c real numbers such that $a + b + c = 0$. Prove that

$$f(a - b) + f(b - c) + f(c - a) \geq f(a) + f(b) + f(c).$$

Proposed by Dan-Ştefan Marinescu, Hunedoara, Romania

U693. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + ab + bc + ca = 6$. Prove that

$$\frac{1}{a^2 + 5} + \frac{1}{b^2 + 5} + \frac{1}{c^2 + 5} \leq \frac{1}{2}.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploieşti, Romania

U694. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{\arctan a}{1 + a + ab} + \frac{\arctan b}{1 + b + bc} + \frac{\arctan c}{1 + c + ca} \leq \frac{\pi}{4}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

U695. Consider the positive integer $n \geq 2$, the complex numbers k, p, r such that $k \neq 0, r \neq \pm 1, \pm i$, and the equations with complex coefficients

$$x^n + 2kx^{n-1} + px^{n-2} + a_{n-3}x^{n-3} + a_{n-4}x^{n-4} + \cdots + a_1x + a_0 = 0,$$

$$y^n + 2kr^2y^{n-1} + py^{n-2} + b_{n-3}y^{n-3} + b_{n-4}y^{n-4} + \cdots + b_1y + b_0 = 0.$$

Define the following transformation of a finite set of complex numbers (or a finite multiset if we have multiplicities of the elements): Replace every element with the sum of the remaining numbers. Let $x_1, x_2, \dots, x_n$ be the roots of the first equation. It is possible to obtain the roots of the second equation applying successively this transformation, after a finite number of steps and starting with $x_1, x_2, \dots, x_n$?

Proposed by Sorin Monel Budişan, Beclean, Romania

U696. Let $x_1, \dots, x_n$ be nonnegative real numbers such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ and let $S = x_1 + \cdots + x_n$. Prove that

$$\prod_{i=1}^n \prod_{j=1}^n (1 + x_i x_j) \geq \left(\frac{n+1}{n-1} \right)^{nS^2} \prod_{i=1}^n \prod_{j=1}^n (1 - x_i x_j).$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

Olympiad Problems

O691. Let $x \geq y \geq z \geq 0$ be real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$x + y + z + (\sqrt{x} - \sqrt{z})^2 \geq 3.$$

Proposed by Titu Andreescu, Dallas, USA and Marius Stănean, Zalău, Romania

O692. Alice and Barbara play a game on a 45×45 table. The table is initially empty. They take turns filling one cell at a time with either $+1$ or -1 . Alice moves first. At the end of the game (when all cells are filled), the score of the table is computed as the sum of the products of all rows:

$$S = \sum_{i=1}^{45} \prod_{j=1}^{45} a_{ij},$$

where a_{ij} is the value in cell (i, j) . Alice wins if $S > 0$, Barbara wins if $S < 0$. Who has a winning strategy?

Proposed by Alessandro Ventullo, Milan, Italy

O693. Find the largest positive value of k such that

$$\frac{a_1 a_2 + a_2 a_3 + \cdots + a_n a_1}{n} \geq \left(\frac{ka_2 + a_3 + \cdots + a_n}{k + n - 2} \right)^2,$$

for any sequence of real numbers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, where $n \geq 3$.

Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

O694. Prove that the equation

$$x^2 + y^2 - z^2 + 2 = xyz$$

has no integer solutions.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O695. Find all functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$f(xy + f(y)) = f(x)y + f(y)$$

for all $x, y > 0$.

Proposed by Prodromos Fotiadis, University of Crete, Greece.

O696. Prove that for every odd prime p there are infinitely many positive integers n such that the exponents of 2 and p in the prime factorization of $n!$ are even.

Proposed by Mohammadreza Ghelichkhani, Tehran, Iran