

Junior Problems

J697. Jimmy plays with numbers and sees that a certain four-digit number $\overline{abcd}$ is equal to $(\overline{ab} + \overline{cd})^2$. Find all four-digit numbers with this property.

Proposed by Alessandro Ventullo, Milan, Italy

J698. Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

$$\frac{a}{2a+bc} + \frac{b}{2b+ca} + \frac{c}{2c+ab} \leq 1.$$

Proposed by Adrian Andreescu, Dallas, USA

J699. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{ab+1}{a+b} + \frac{bc+1}{b+c} + \frac{ca+1}{c+a} \geq \sqrt{6(1-ab-bc-ca)} + 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J700. Find all triples (x, y, z) of integers such that

$$(x^2 + y^2)(z^2 + 1) = 4(xyz + 16).$$

Proposed by Titu Andreescu, Dallas, USA

J701. In a circle of radius R , three chords of length R are given. Their ends are joined with segments to obtain a hexagon inscribed in the circle. Show that the midpoints of the new chords are the vertices of an equilateral triangle.

Proposed by Cristian Tudor Popescu, Bucharest, România

J702. Let ABC be a triangle with incircle Γ . The tangency points of Γ with sides BC, CA, AB are A_1, B_1, C_1 respectively. Line B_1C_1 intersects line BC at point A_2 . Similarly, points B_2 and C_2 are constructed. Prove that the perpendicular lines from A_2, B_2, C_2 to lines AA_1, BB_1, CC_1 respectively are concurrent.

Proposed by Mihaela Berindeanu, Bucharest, România

Senior Problems

S697. Let a, b, c be the side-lengths of a triangle. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 + 15 \geq 16 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

S698. Let $ABCD$ be a square and Γ the circle with center B and radius BC . Let X be a point on Γ inside the square. The tangent at Γ in X intersects DC at F and AD at E . On the same side as point B a square with side EF and center Y is constructed. The lines BE and AC intersect at H , the lines BF and AC intersect at G , FH and EG intersect at Z . Prove that $\vec{YZ} = \vec{YF} + \vec{YB} + \vec{YE}$.

Proposed by Mihaela Berindeanu, Bucharest, România

S699. Prove that a convex quadrilateral $ABCD$ is cyclic if and only if

$$\frac{AC^2 - AD^2}{AD \cdot CD} + \frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB}{BC} + \frac{CD}{AD}.$$

Proposed by Titu Andreescu, Dallas, USA

S700. Let a, b, c, x, y, z be positive real numbers. Prove that

$$\frac{x}{y+z}(b^3+c^3) + \frac{y}{z+x}(c^3+a^3) + \frac{z}{x+y}(a^3+b^3) \geq 3abc.$$

Proposed by An Zhenping, Xianyang Normal University, China

S701. Let x, y, z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{\sqrt{x^2+3}} + \frac{1}{\sqrt{y^2+3}} + \frac{1}{\sqrt{z^2+3}} \geq \frac{90}{11(x+y+z)+27}.$$

Proposed by Marius Stănean, Zalău, România

S702. Prove that for each positive integer c there are infinitely many positive integers n such that $n! + c$ is not the square of an integer.

Proposed by Mohammadreza Ghelichkhani, Shahed University, Tehran, Iran

Undergraduate Problems

U697. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Prove that

$$\left(\int_a^b f(x) \, dx \right)^4 \leq (b-a)^2 \int_a^b e^x f^2(x) \, dx \int_a^b \frac{f^2(x)}{e^x} \, dx.$$

Proposed by Mihaela Berindeanu, Bucharest, România

U698. Let $f(x + iy) = u(x, y) + iv(x, y)$ be a holomorphic function on $\mathbb{C}$ with real part

$$u(x, y) = 2x^2 - 3xy - 2y^2.$$

Find $\frac{|f'(z)|}{|z|}$ for $z \neq 0$.

Proposed by Alessandro Ventullo, Milan, Italy

- U699.** (a) Prove that if $r \in \{7, 11, 13, 17\}$ for each prime p such that $p \equiv r \pmod{24}$, there is a positive integer n such that $p \mid 2^n + 3^n$.
(b) Prove that for any prime $p \equiv 23 \pmod{24}$ there is not a positive integer n such that $p \mid 2^n + 3^n$.
(c) Prove that for infinitely many primes $p \equiv 1 \pmod{24}$ there is a positive integer n such that $p \mid 2^n + 3^n$.

Proposed by Mohammadreza Ghlichkhani, Shahed University, Tehran, Iran

U700. Let a, b, c, d be nonnegative real numbers such that $ab + ac + ad + bc + bd + cd = 1$. Prove that

$$\frac{1}{a+b+c} + \frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} \geq 3.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România

U701. Find all finite groups G of order $3n$ having at least two distinct subgroups H_1 and H_2 of order n such that $H_1 \cap H_2 = \{1\}$.

Proposed by Alessandro Ventullo, Milan, Italy

U702. Let a and b be real numbers. Let $(x_n)_{n \geq 0}$ be a sequence of real numbers such that $2x_n \leq x_{n-1} + x_{n+1}$, for all $n \geq 1$ and $\lim_{n \rightarrow \infty} (x_n - an - b) = 0$. Prove that $x_n \geq an + b$, for all $n \geq 0$.

Proposed by Dan-Ștefan Marinescu, Hunedoara, România

Olympiad Problems

O697. Let a, b, c, d be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$. Prove that

$$\sqrt{\frac{1}{a^2} + \sqrt{a} + 1} + \sqrt{\frac{1}{b^2} + \sqrt{b} + 1} + \sqrt{\frac{1}{c^2} + \sqrt{c} + 1} + \sqrt{\frac{1}{d^2} + \sqrt{d} + 1} \leq \frac{\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}}{2} + 3.$$

Proposed by Titu Andreescu, Dallas, USA

O698. Find the smallest positive constant k such that

$$\left(\frac{ka + b + c + d + e}{k + 4} \right)^2 \geq \frac{ab + bc + cd + de + ef + fa}{6}$$

whenever $a \geq b \geq c \geq d \geq e \geq f \geq 0$.

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, România

O699. Prove that the equation

$$y(x^2 - z^2 - y + 1) = x^2 + 3$$

has no integer solutions.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O700. Let $a_1, \dots, a_n$ be positive real numbers such that $a_1 \cdots a_n = 1$. Prove that

$$\sum_{i=1}^n \frac{a_i^5}{a_1 + \cdots + a_{i-1} + a_i^2 + a_{i+1} + \cdots + a_n} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

O701. Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

Prove that

$$8(a^3 + b^3 + c^3 + d^3) + 49abcd \geq 81.$$

Proposed by Marius Stănean, Zalău, România

O702. Let ABC be a triangle with $AB \neq AC$. Let H be its orthocenter, O be its circumcenter and I be its incenter. Let D be the reflection of A in O , let F be the midpoint of AH , and J be the intersection point of AI and BC . Let Q be the midpoint of the arc BAC . Let D_B, D_C are the feet of the perpendiculars dropped from the vertices B, C to the lines CD, BD respectively and E be the intersection point of $D_B D_C$ and BC . Knowing that $DE \parallel JO$, prove that line FQ is also parallel to them.

Proposed by Todor Zaharinov, Sofia, Bulgaria