

Junior problems

J697. Jimmy plays with numbers and sees that a certain four-digit number $\overline{abcd}$ is equal to $(\overline{ab} + \overline{cd})^2$. Find all four-digit numbers with this property.

Proposed Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Observe that $\overline{abcd} = 100 \cdot \overline{ab} + \overline{cd}$. Then

$$(\overline{ab} + \overline{cd})^2 = 100 \cdot \overline{ab} + \overline{cd}.$$

Now, let $x = \overline{ab}$, $y = \overline{cd}$ and let $s = x + y$. We have

$$(x + y)^2 = 100x + y \implies s^2 = s + 99x,$$

so

$$x = \frac{s(s-1)}{99}.$$

Since $10 \leq x \leq 99$, then

$$32 \leq s \leq 99. \tag{1}$$

Moreover, since $99 \mid s(s-1)$ and $\gcd(s-1, s) = 1$, then either $11 \mid (s-1)$ or $11 \mid s$. Similarly, either $9 \mid (s-1)$ or $9 \mid s$. So,

$$\begin{array}{ll} s \equiv 0 \pmod{9} & s \equiv 0 \pmod{9} \\ s \equiv 0 \pmod{11}, & s \equiv 1 \pmod{11}, \\ s \equiv 1 \pmod{9} & s \equiv 1 \pmod{9} \\ s \equiv 0 \pmod{11}, & s \equiv 1 \pmod{11}. \end{array}$$

This together with (1) gives $s \in \{45, 55, 99\}$.

(i) If $s = 45$, then $x = \frac{45 \cdot 44}{99} = 20$ and $y = 25$, which gives $\overline{abcd} = 2025$.

(ii) If $s = 55$, then $x = \frac{55 \cdot 54}{99} = 30$ and $y = 25$, which gives $\overline{abcd} = 3025$.

(iii) If $s = 99$, then $x = \frac{99 \cdot 98}{99} = 98$ and $y = 1$, which gives $\overline{abcd} = 9801$.

In conclusion, $\overline{abcd} \in \{2025, 3025, 9801\}$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $x = \overline{ab}$ and $y = \overline{cd}$, where $10 \leq x \leq 99$ and $0 \leq y \leq 99$. It is given that

$$100x + y = (x + y)^2 \iff x^2 + 2x(y - 50) + y^2 - y = 0 \iff x = 50 - y \pm \sqrt{2500 - 99y}.$$

For integer values of x we must have $y \leq 25$ and

$$99y = (50 - k)(50 + k),$$

where $k \leq 50$ is a positive integer (because $9 \nmid 2500$). If $k = 50$, then $y = 0$, and thus $x = 100$ or $x = 0$, neither of which is acceptable. Let $k < 50$. If $9 \mid (50 - k)$ then $9 \nmid (50 + k)$ because $9 \nmid 100$. Similarly, if $11 \mid (50 - k)$ then $11 \nmid (50 + k)$ because $11 \nmid 100$. The only values of $0 < k < 50$ such that $11 \mid (50 - k)$ are $k \in \{6, 17, 28, 39\}$; in each of these cases we get $9 \nmid (2500 - k^2)$. Thus, $11 \mid (50 + k)$ and therefore $k \in \{5, 16, 27, 38, 49\}$; in this case only for $k = 5, 49$ we get $9 \mid (2500 - k^2)$, and thus the only solutions with

$x \geq 10$ are $(k, x, y) = (5, 30, 25)$, or $(5, 20, 25)$, and $(49, 98, 1)$. Therefore, the only four-digit numbers that satisfy the given condition are $3025 = (30 + 25)^2 = 55^2$, $2025 = (20 + 25)^2 = 45^2$, and $9801 = (98 + 1)^2 = 99^2$.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Elizabeth Standera, SUNY, Brockport, NY, USA; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Juan Silvera, AMMOC- An Int. Math Circle; Kristine Lee, Mount Juliet High School, TN, USA; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Aaditya Bhaumik Sahu, AMMOC, India; Aarnav Sivachandran, AMMOC, USA, CT; Charvi Atreya, AMMOC; Diar Gashi, University of Primorska, Slovenia; E.Madani Harouach, AMMOC; Huy Nguyen, AMMOC; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Medha Shrivastava, AMMOC, India; Parth Andhare, AMMOC; Polyhedra, Polk State College, FL, USA; Prasanna Pawar, AMMOC, Bengaluru, India; Sarthak Dattatray Dhobale, AMMOC, Bengaluru, India; Shrivathsan Krishnan, AMMOC; Sumanyu Nandecha, AMMOC; Winston Ra, Chattahoochee High School in Johns Creek, GA, USA.

J698. Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

$$\frac{a}{2a+bc} + \frac{b}{2b+ca} + \frac{c}{2c+ab} \leq 1.$$

Proposed by Adrian Andreescu, Dallas, USA

Solution 1 by Prodromos Fotiadis, University of Crete, Greece

We observe that

$$\frac{a}{2a+bc} + \frac{b}{2b+ca} + \frac{c}{2c+ab} = \frac{3}{2} - \frac{1}{2} \left(\frac{bc}{2a+bc} + \frac{ca}{2b+ca} + \frac{ab}{2c+ab} \right)$$

So, equivalently, we need to show that

$$\frac{bc}{2a+bc} + \frac{ca}{2b+ca} + \frac{ab}{2c+ab} \geq 1$$

Using Titu's Lemma we obtain

$$\frac{bc}{2a+bc} + \frac{ca}{2b+ca} + \frac{ab}{2c+ab} = \frac{(bc)^2}{2abc + (bc)^2} + \frac{(ca)^2}{2bca + (ca)^2} + \frac{(ab)^2}{2cab + (ab)^2} \geq \frac{(ab+bc+ca)^2}{6abc + (ab)^2 + (bc)^2 + (ca)^2}$$

So, it suffices to prove that

$$(ab+bc+ca)^2 \geq 6abc + (ab)^2 + (bc)^2 + (ca)^2$$

or equivalently

$$2abc(a+b+c) \geq 6abc \iff a+b+c \geq 3$$

This follows immediately from AM-GM:

$$a+b+c \geq 3\sqrt[3]{abc} \geq 3$$

The equality holds if and only if $a = b = c = 1$.

Solution 2 by Nguyen Viet Hung, Hanoi University of Science, Vietnam

The desired inequality is equivalent to the following each inequality

$$\sum_{\text{cyc}} a(2b+ca)(2c+ab) \leq (2a+bc)(2b+ca)(2c+ab),$$

$$\sum_{\text{cyc}} (4abc + 2a^2(b^2 + c^2) + a^3bc) \leq 8abc + \sum_{\text{cyc}} 4a^2b^2 + \sum_{\text{cyc}} 2a^3bc + (abc)^2,$$

$$4abc \leq abc(a^2 + b^2 + c^2) + (abc)^2,$$

$$4 \leq a^2 + b^2 + c^2 + abc.$$

This is true because

$$a^2 + b^2 + c^2 + abc \geq 3\sqrt{(abc)^2} + abc \geq 4.$$

The equality holds if and only if $a = b = c = 1$.

Also solved by Amanda Goodwin, SUNY, Brockport, NY, USA; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Gjiminazi i Special-izuar Matematikor Group, Prishtine, Kosovo; Henry Ricardo, Westchester Area Math Circle, New York,

USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Aarnav Sivachandran, AMMOC; Avyay Singh Chauhan, CMC; Charvi Atreya, AMMOC; Daniel Pascuas, Barcelona, Spain; Debarun Das, CMC, Hyderabad, India; Diar Gashi, University of Primorska, Slovenia; E.Madani Harouach, AMMOC; Huy Nguyen, AMMOC; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Keisha Kwok, AMMOC; Medha Shrivastava, AMMOC, India; Parth Andhare, AMMOC; Polyhedra, Polk State College, FL, USA; Sarthak Dattatray Dhobale, AMMOC, Bengaluru, India; Shrivathsan Krishnan, AMMOC; Sumanyu Nandecha, AMMOC; Winston Ra, Chattahoochee High School in Johns Creek, GA, USA.

J699. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{ab+1}{a+b} + \frac{bc+1}{b+c} + \frac{ca+1}{c+a} \geq \sqrt{6(1-ab-bc-ca)} + 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyhedra, Polk State College, FL, USA

Let $x = 1 - a = b + c$, $y = 1 - b = c + a$, and $z = 1 - c = a + b$. Then

$$\begin{aligned} \left(\frac{ab+1}{a+b} + \frac{bc+1}{b+c} + \frac{ca+1}{c+a} - 3 \right)^2 &= \left(\frac{(1-a)(1-b)}{a+b} + \frac{(1-b)(1-c)}{b+c} + \frac{(1-c)(1-a)}{c+a} \right)^2 \\ &= \left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \right)^2 \geq 3 \left(\frac{xy}{z} \cdot \frac{yz}{x} + \frac{yz}{x} \cdot \frac{zx}{y} + \frac{zx}{y} \cdot \frac{xy}{z} \right) \\ &= 3(y^2 + z^2 + x^2) = 6(a^2 + b^2 + c^2 + ab + bc + ca) \\ &= 6((a+b+c)^2 - ab - bc - ca) = 6(1 - ab - bc - ca). \end{aligned}$$

The equality occurs if and only if $x = y = z = \frac{2}{3}$, i.e. if and only if $a = b = c = \frac{1}{3}$.

Also solved by Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Gjimnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, University of Crete, Greece; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Charvi Atreya, AMMOC; Diar Gashi, University of Primorska, Slovenia; E.Madani Harouach, AMMOC; Sarthak Dattatray Dhobale, AMMOC, Bengaluru, India; Sumanyu Nandecha, AMMOC.

J700. Find all triples (x, y, z) of integers such that

$$(x^2 + y^2)(z^2 + 1) = 4(xyz + 16).$$

Proposed by Titu Andreescu, Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

If $z = 0$, then the equation becomes $x^2 + y^2 = 64$, which has solutions $(x, y) \in \{(\pm 8, 0), (0, \pm 8)\}$. Now, assume that $z \neq 0$. Observe that if (x, y, z) is a solution, then also

$$(-x, -y, z), (y, x, z), (-y, -x, z), (-x, y, -z), (x, -y, -z), (y, -x, -z), (-y, x, -z)$$

are solutions. So, we can assume that $z > 0$ (i.e. $z \geq 1$), $x \geq y$ and $xy \geq 0$. After expanding and factoring we get that the equation can be expressed as

$$(xz - y)^2 + (yz - x)^2 = 64.$$

Clearly, 64 can be written as the sum of two squares as $0^2 + (\pm 8)^2 = 64$. Since $x \geq y$ and $z > 0$, then $xz - y \geq yz - x$. Moreover,

$$(xz - y) + (yz - x) = (z - 1)(x + y) \geq 0,$$

so we conclude that $xz - y = 8$ and $yz - x = 0$. Hence, $x = yz$ and $y(z^2 - 1) = 8$. Since $z^2 - 1$ divides 8, we get $z = 3$ and then $(x, y, z) = (3, 1, 3)$. Restoring the generality of the solutions, we get another 7 solutions, so in total we have the 12 solutions:

$$\{(\pm 8, 0, 0), (0, \pm 8, 0), (\pm 3, 1, \pm 3), (\mp 3, -1, \pm 3), (1, \pm 3, \pm 3), (-1, \mp 3, \pm 3)\}.$$

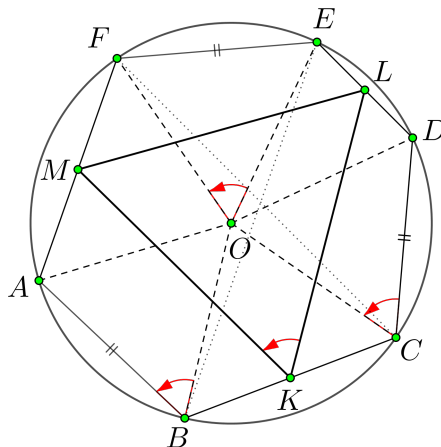
Also solved by Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Diar Gashi, University of Primorska, Slovenia; E.Madani Harouach, AMMOC; Keisha Kwok, AMMOC; Parth Andhare, AMMOC; Polyahedra, Polk State College, FL, USA; Sarthak Dattatray Dhobale, AMMOC, Bengaluru, India; Srijan Sundar, Oxford, UK; Sumanyu Nandecha, AMMOC; Winston Ra, Chattahoochee High School in Johns Creek, GA, USA.

J701. In a circle of radius R , three chords of length R are given. Their ends are joined with segments to obtain a hexagon inscribed in the circle. Show that the midpoints of the new chords are the vertices of an equilateral triangle.

Proposed by Cristian Tudor Popescu, Bucharest, Romania

Solution 1 by Kousik Sett, India

Let O be the center of the given circle of radius R . If three chords are denoted by AB , CD , and EF then by construction, we have triangles OAB , OCD , and OEF are equilateral. Let K , L , and M be the midpoints of BC , DE , and FA respectively. We use vector rotation to prove this result.



We have

$$\overrightarrow{KL} = \frac{\overrightarrow{CD} + \overrightarrow{BE}}{2} = \frac{\overrightarrow{CD} + \overrightarrow{BO} + \overrightarrow{OE}}{2}.$$

We rotate each vector of the above equality by 60° counterclockwise. We define M' be a point such that triangle KLM' is equilateral and after above rotation $\overrightarrow{KL}$ maps to $\overrightarrow{KM'}$. Thus each vector of the above equality transforms to

$$\overrightarrow{KM'} = \frac{\overrightarrow{CO} + \overrightarrow{BA} + \overrightarrow{OF}}{2} = \frac{\overrightarrow{BA} + \overrightarrow{CO} + \overrightarrow{OF}}{2} = \frac{\overrightarrow{BA} + \overrightarrow{CF}}{2} = \overrightarrow{KM},$$

which implies $M' \equiv M$. Therefore, triangle KLM is equilateral and we are done!

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $AB = CD = EF$ be the three chords of length R . Let $\angle BOC = 2\alpha$, $\angle DOE = 2\beta$, and $\angle FOA = 2\gamma$, and let K, L, M be the midpoints of the chords BC, DE, FA , respectively. Then $OK = R \cos \alpha$, $OL = R \cos \beta$, and $OM = R \cos \gamma$. Triangles AOB, COD, EOF are equilateral, and thus $\angle AOB = \angle COD = \angle EOF = 60^\circ$. Therefore, $180^\circ + 2\alpha + 2\beta + 2\gamma = 360^\circ$, and thus $\alpha + \beta + \gamma = 90^\circ$. Using the law of cosines in triangles KOL, KOM we get

$$\begin{aligned} KL^2 &= OK^2 + OL^2 - 2 \cdot OK \cdot OL \cdot \cos(60^\circ + \alpha + \beta), \\ KM^2 &= OK^2 + OM^2 - 2 \cdot OK \cdot OM \cdot \cos(60^\circ + \alpha + \gamma). \end{aligned}$$

Thus, $KL = KM$ if and only if

$$\cos^2 \beta - 2 \cos \alpha \cos \beta \cos(60^\circ + \alpha + \beta) = \cos^2 \gamma - 2 \cos \alpha \cos \gamma \cos(60^\circ + \alpha + \gamma),$$

or

$$\begin{aligned} KL = KM &\iff \cos^2 \beta - \cos^2 \gamma = \cos \alpha [\cos(60^\circ + \alpha + 2\beta) - \cos(60^\circ + \alpha + 2\gamma)] \\ &\iff \frac{\cos(2\beta) - \cos(2\gamma)}{2} = -2 \cos \alpha \sin(60^\circ + \alpha + \beta + \gamma) \sin(\beta - \gamma) \\ &\iff \cos(2\beta) - \cos(2\gamma) = -2 \cos \alpha \sin(\beta - \gamma) = -2 \sin(\beta + \gamma) \sin(\beta - \gamma), \end{aligned}$$

which is obvious. Similarly, we get $KL = LM$, and the triangle KLM is equilateral.

Note: With appropriate signs for the angles α, β, γ , the above holds true whether the hexagon $ABCDEF$ is convex or not.

Solution 3 by Polyhedra, Polk State College, FL, USA

This is well known as the asymmetric propeller theorem with many generalizations and various proofs (Euclidean, vector, and complex-number). It even appeared as Problem B-1 in the 1967 Putnam Competition. For its history, see M. Gardner, The asymmetric propeller, *The College Math. Journal*, January/1999, 18–22. For most recent generalizations, see Q. H. Tran, The asymmetric propeller with squares, and some extensions, *The Math. Gazette*, July/2024, 283–291.

Also solved by Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Charvi Atreya, AMMOC; Debarun Das, CMC, Hyderabad, India; E.Madani Harouach, AMMOC; Keisha Kwok, AMMOC; Parth Andhare, AMMOC; Sarthak Dattatray Dhobale, AMMOC, Bengaluru, India.

Solution 2 by Polyhedra, Polk State College, FL, USA

Since AB and AC are tangents at B_1 and C_1 , La Hire's Theorem gives that B_1 and C_1 lie on the polar of A , so the polar of pole A is B_1C_1 . Similarly, the polar of pole A_1 is BC . Let A_2 be the intersection of BC and B_1C_1 . Then A_2 is the pole of polar AA_1 by duality. Therefore, the perpendicular line from A_2 to AA_1 passes through the incenter I of Γ . Likewise, the perpendicular lines from B_2 to BB_1 and from C_2 to CC_1 pass through I as well.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Keisha Kwok, AMMOC.

Senior problems

S697. Let a, b, c be the side-lengths of a triangle. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 + 15 \geq 16 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

Solution by the authors

First we will prove that if $x, y, z > 0$ such that $xyz = 1$, then

$$(x + y + z)^2 + 7 \geq \frac{8}{3}(x + y + z + xy + yz + zx).$$

Without loss of generality, we may assume that $z = \min\{x, y, z\}$, so it follows that $z \leq 1, xy \geq 1$. Denote by

$$f(x, y, z) = (x + y + z)^2 + 7 - \frac{8}{3}(x + y + z + xy + yz + zx).$$

We notice that

$$\begin{aligned} f(x, y, z) - f(\sqrt{xy}, \sqrt{xy}, z) &= (\sqrt{x} - \sqrt{y})^2 \left[(\sqrt{x} + \sqrt{y})^2 + 2z \right] - \frac{8}{3} (\sqrt{x} - \sqrt{y})^2 (1 + z) \\ &= (\sqrt{x} - \sqrt{y})^2 \left[(\sqrt{x} + \sqrt{y})^2 - \frac{8 + 2z}{3} \right] \\ &\geq (\sqrt{x} - \sqrt{y})^2 \left(4 - \frac{8 + 2z}{3} \right) \geq 0. \end{aligned}$$

Therefore, it remains to prove that $f(\sqrt{xy}, \sqrt{xy}, z) \geq 0$ or, if we denote

$$z = \frac{1}{t^2}, \quad t \geq 1, \quad f\left(t, t, \frac{1}{t^2}\right) \geq 0,$$

that is

$$\frac{(t-1)^2 [(4t^2-3)(t-1)^2+6]}{3t^4} \geq 0,$$

which is true. The equality holds when $x = y = z = 1$.

Returning to our problem, we have

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 + 7 \geq \frac{8}{3} \sum_{cyc} \frac{a+b}{c}.$$

Hence, we need to prove that

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} + 3 \geq 6 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right),$$

or

$$\sum_{cyc} \frac{a+b}{c} - 6 \geq 6 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \right),$$

or

$$\sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} - 2 \right) \geq 6 \left(\frac{\sum_{cyc} (a^3 + b^3) - \sum_{cyc} ab(a+b)}{2(a+b)(b+c)(c+a)} \right),$$

or

$$\sum_{cyc} \frac{(a-b)^2}{ab} \geq \sum_{cyc} \frac{3(a-b)^2}{(a+c)(b+c)} \iff \sum_{cyc} S_c(a-b)^2 \geq 0,$$

where

$$S_c = \frac{1}{ab} - \frac{3}{(a+c)(b+c)}, \quad S_b = \frac{1}{ca} - \frac{3}{(a+b)(b+c)}, \quad S_a = \frac{1}{bc} - \frac{3}{(a+b)(c+a)}.$$

Note that the inequality is symmetric, so without loss of generality, we may assume $a \geq b \geq c$. We have

$$S_a = \frac{a^2 + b(a-c) + c(a-b)}{bc(a+b)(c+a)} \geq 0,$$

$$S_b = \frac{b(b+c) + ab - 2ac}{ca(a+b)(b+c)} > \frac{2ab - 2ac}{ca(a+b)(b+c)} \geq 0.$$

$$\begin{aligned} S_b + S_c &= \frac{a^2(b-c)^2 - 2a^2bc + a(b^3 + c^3) + bc(b+c)^2}{abc(a+b)(b+c)(c+a)} \\ &> \frac{a^2(b-c)^2 - 2a^2bc + abc(b+c) + a^2bc}{abc(a+b)(b+c)(c+a)} \geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{cyc} S_c(a-b)^2 &\geq S_c(a-b)^2 + S_b(a-b)^2 + S_b(b-c)^2 + S_a(b-c)^2 \\ &= (S_b + S_c)(a-b)^2 + (S_a + S_b)(b-c)^2 \geq 0. \end{aligned}$$

The equality holds when $a = b = c$.

Observation. This inequality is stronger than the inequality:

4970. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right)^2 \geq 2(a+b+c) \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right).$$

from Crux Mathematicorum, September 2024.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA.

S698. Let $ABCD$ be a square and Γ the circle with center B and radius BC . Let X be a point on Γ inside the square. The tangent at Γ in X intersects DC in F and AD in E . On the same side as point B a square with side EF and center Y is constructed. The lines BE and AC intersect in H , the lines BF and AC intersect in G , FH and EG intersect in Z . Prove that $\overrightarrow{YZ} = \overrightarrow{YF} + \overrightarrow{YB} + \overrightarrow{YE}$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

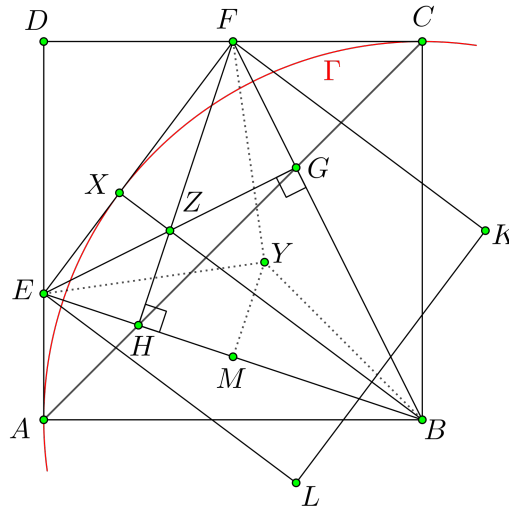
Solution 1 by Kousik Sett, India

Consider triangles EBX and EBA . We have $EX = EA$, $\angle EXB = \angle EAB = 90^\circ$ and EB is the common side. So, $\triangle EBX \cong \triangle EBA$ by *RHS* congruence law. Therefore, $\angle EBX = \angle EBA$, which implies $\angle EBX = \frac{1}{2}\angle ABX$.

Similarly, since $FX = FC$, $\angle FXB = \angle FCB = 90^\circ$, and FB is the common side, we have $\triangle FBX \cong \triangle FBC$. Therefore, $\angle FBX = \angle FBC$, which implies $\angle FBX = \frac{1}{2}\angle CBX$.

Therefore,

$$\angle EBF = \angle EBX + \angle FBX = \frac{1}{2}(\angle ABX + \angle CBX) = \frac{1}{2}\angle ABC = 45^\circ.$$



Since $\angle FCH = \angle FBH = 45^\circ$, we have quadrilateral $FCBH$ is cyclic. Thus, $\angle FHB = 180^\circ - \angle FCB = 180^\circ - 90^\circ = 90^\circ$. Therefore, $FH \perp EB$.

Again, since $\angle EBG = \angle EAG = 45^\circ$, we have quadrilateral $EABG$ is cyclic. Thus, $\angle EGB = 180^\circ - \angle EAB = 180^\circ - 90^\circ = 90^\circ$. Therefore, $EG \perp FB$.

Since $Z = FH \cap EG$, $FH \perp EB$ and $EG \perp FB$, we infer that Z is the orthocenter of triangle BEF . Since $EFKL$ is a square, we have $YE = YF$ and $\angle EYF = 90^\circ = 2\angle EBF$. Therefore, Y is the circumcenter of triangle BEF .

Let M be the midpoint of EB . We know that Z is the orthocenter and Y is the circumcenter of triangle BEF then it is well-known that $FZ = 2YM$.

In triangle YEB , we have $\overrightarrow{YB} + \overrightarrow{YE} = 2\overrightarrow{YM}$. Therefore,

$$\overrightarrow{YF} + \overrightarrow{YB} + \overrightarrow{YE} = \overrightarrow{YF} + 2\overrightarrow{YM} = \overrightarrow{YF} + \overrightarrow{FZ} = \overrightarrow{YZ}.$$

Solution 2 by Theo Koupelis, Clark College, Washington, USA

In the right triangle BAE we have $\angle BAH = \angle EAH = 45^\circ$, and thus $EH/HB = AE/AB = AE/BC$. Similarly, in the right triangle BCF we have $\angle BCG = \angle FCG = 45^\circ$, and thus $BG/GF = BC/FC$. By construction we also have $FX/XE = FC/AE$, because $FX = FC$ and $AE = XE$. Multiplying the above three equations we get

$$\frac{EH}{HB} \cdot \frac{BG}{GF} \cdot \frac{FX}{XE} = 1.$$

Using the converse of Ceva's theorem we get that EG, FH, BX are concurrent at the point Z .

Let $\alpha := \angle ABE = \angle EBX$ and $\beta := \angle XBF = \angle FBC$, where $\alpha + \beta = 45^\circ$. By construction we have $\angle EAX = \angle EXA = \alpha$, and thus $\angle HAX = \angle HXA = 45^\circ - \alpha = \beta$. Similarly, $\angle CXF = \angle XCF = \beta$ and thus $\angle GXC = \angle GCX = 45^\circ - \beta = \alpha$. The quadrilateral $AEXG$ is cyclic because $\angle GAE = \angle GXF = 45^\circ$, and thus $\angle GEX = \angle GAE = \beta$. Thus, $EG \parallel XC$, and therefore $EG \perp BF$. But $AX \perp EF$ and thus $FH \perp BE$ and Z is the orthocenter of triangle BEF .

Let $EFKL$ (with vertices in clockwise direction) be the square of side EF . Let M be the intersection of the line FH with AB . The quadrilateral $AMHE$ is cyclic because $\angle MAE = \angle MHE = 90^\circ$. Thus $\angle HME = \angle HAE = 45^\circ$. Therefore the triangle MHE is an isosceles right triangle and so is the triangle BHF because $\angle FBH = 45^\circ$. Thus, the quadrilateral $MEFB$ is a trapezium. Also, $\angle EMF = \angle ELF = \angle EBF = 45^\circ$, and thus the points M, B are on the circle $(EFKL)$. Therefore, $\angle KBE = 90^\circ$ because KE is a diameter of $(EFKL)$. Thus, $KB \parallel FH$. But we also have $BX \parallel KF$, and therefore $BZFK$ is a parallelogram.

Thus,

$$\overrightarrow{YZ} = \overrightarrow{YF} + \overrightarrow{FZ} = \overrightarrow{YF} + \overrightarrow{KB} = \overrightarrow{YF} + \overrightarrow{KY} + \overrightarrow{YB} = \overrightarrow{YF} + \overrightarrow{YB} + \overrightarrow{YE}.$$

Also solved by Yunyong Zhang, Chinaunicom, Yunnan, China; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam.

S699. Prove that a convex quadrilateral $ABCD$ is cyclic if and only if

$$\frac{AC^2 - AD^2}{AD \cdot CD} + \frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB}{BC} + \frac{CD}{AD}$$

Proposed by Titu Andreescu, Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

By the law of cosine in triangle ABC , we have

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos \angle ABC,$$

so

$$\frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB}{BC} - 2 \cos \angle ABC. \quad (1)$$

By the law of cosine in triangle ADC , we have

$$AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cdot \cos \angle ADC,$$

so

$$\frac{AC^2 - AD^2}{AD \cdot CD} = \frac{CD}{AD} - 2 \cos \angle ADC. \quad (2)$$

Adding (1) and (2) we get

$$\frac{AC^2 - AD^2}{AD \cdot CD} + \frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB}{BC} + \frac{CD}{AD} - 2(\cos \angle ABC + \cos \angle ADC).$$

Quadrilateral $ABCD$ is cyclic if and only if $\angle ABC + \angle ADC = 180^\circ$, that is, if and only if

$$\cos \angle ADC = \cos(180^\circ - \angle ABC) = -\cos \angle ABC.$$

So, quadrilateral $ABCD$ is cyclic if and only if

$$\frac{AC^2 - AD^2}{AD \cdot CD} + \frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB}{BC} + \frac{CD}{AD}.$$

Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Kousik Sett, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Yunyong Zhang, Chinaunicom, Yunnan, China; Diar Gashi, University of Primorska, Slovenia; E.Madani Harouach, AMMOC; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Medha Shrivastava, AMMOC, India; Parth Andhare, AMMOC; Prasanna Pawar, AMMOC, Bengaluru, India; Sarthak Dattatray Dhobale, AMMOC, Bengaluru, India.

S700. Let a, b, c, x, y, z be positive real numbers. Prove that

$$\frac{x}{y+z}(b^3+c^3) + \frac{y}{z+x}(c^3+a^3) + \frac{z}{x+y}(a^3+b^3) \geq 3abc.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution 1 by Arkady Alt, San Jose, California, USA

Since for any $a, b, c, x, y, z > 0$ we have

$$\frac{x(b+c)}{y+z} + \frac{y(c+a)}{z+x} + \frac{z(a+b)}{x+y} \geq \sqrt{3(ab+bc+ca)}$$

(Inequality (F3) in [1]), then by replacing (a, b, c) with (a^3, b^3, c^3) we obtain

$$\frac{x}{y+z}(b^3+c^3) + \frac{y}{z+x}(c^3+a^3) + \frac{z}{x+y}(a^3+b^3) \geq \sqrt{3(a^3b^3+b^3c^3+c^3a^3)}.$$

Moreover $\sqrt{3(a^3b^3+b^3c^3+c^3a^3)} \geq 3abc$, because by the AM-GM Inequality

$$a^3b^3+b^3c^3+c^3a^3 \geq 3(a^3b^3 \cdot b^3c^3 \cdot c^3a^3)^{1/3} = 3a^2b^2c^2.$$

The conclusion follows.

[1] *Variations on theme of one algebraic inequality*, Arkady Alt, Mathematical Reflections n.2, 2009, p.6.

Solution 2 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam

By applying the Cauchy-Schwarz Inequality and the AM-GM Inequality, we have

$$\begin{aligned} & \frac{x}{y+z}(b^3+c^3) + \frac{y}{z+x}(c^3+a^3) + \frac{z}{x+y}(a^3+b^3) \\ &= (x+y+z) \left(\frac{b^3+c^3}{y+z} + \frac{c^3+a^3}{z+x} + \frac{a^3+b^3}{x+y} \right) - 2(a^3+b^3+c^3) \\ &\geq \frac{1}{2}(\sqrt{b^3+c^3} + \sqrt{c^3+a^3} + \sqrt{a^3+b^3})^2 - 2(a^3+b^3+c^3) \\ &= \sum_{cyc} \sqrt{(a^3+b^3)(a^3+c^3)} - \sum_{cyc} a^3 \\ &\geq \sum_{cyc} \sqrt{(a^3+\sqrt{b^3c^3})^2} - \sum_{cyc} a^3 \\ &= \sqrt{a^3b^3} + \sqrt{b^3c^3} + \sqrt{c^3a^3} \\ &\geq 3\sqrt[3]{a^6b^6c^6} = 3abc. \end{aligned}$$

Equality holds if and only if $a = b = c$ and $x = y = z$.

Solution 3 by Theo Koupelis, Clark College, Washington, USA

The desired inequality is equivalent to

$$\frac{x+y+z}{y+z}(b^3+c^3) + \frac{x+y+z}{z+x}(c^3+a^3) + \frac{x+y+z}{x+y}(a^3+b^3) \geq 2(a^3+b^3+c^3) + 3abc,$$

or

$$[(y+z) + (z+x) + (x+y)] \cdot \left[\frac{b^3+c^3}{y+z} + \frac{c^3+a^3}{z+x} + \frac{a^3+b^3}{x+y} \right] \geq 4(a^3+b^3+c^3) + 6abc.$$

Using the Cauchy-Schwarz inequality we get

$$[(y+z) + (z+x) + (x+y)] \cdot \left[\frac{b^3+c^3}{y+z} + \frac{c^3+a^3}{z+x} + \frac{a^3+b^3}{x+y} \right] \geq \left(\sqrt{b^3+c^3} + \sqrt{c^3+a^3} + \sqrt{a^3+b^3} \right)^2.$$

Thus, it suffices to show that

$$\left(\sqrt{b^3+c^3}+\sqrt{c^3+a^3}+\sqrt{a^3+b^3}\right)^2 \geq 4(a^3+b^3+c^3)+6abc,$$

or, after squaring and simplifying

$$\sqrt{a^3+b^3}\sqrt{b^3+c^3}+\sqrt{b^3+c^3}\sqrt{c^3+a^3}+\sqrt{c^3+a^3}\sqrt{a^3+b^3} \geq a^3+b^3+c^3+3abc.$$

Using the Cauchy-Schwarz Inequality we get $\sqrt{a^3+b^3}\sqrt{b^3+c^3} \geq b^3+(ac)^{3/2}$ and cyclic expressions. Thus, it suffices to show that

$$(ab)^{3/2}+(bc)^{3/2}+(ca)^{3/2} \geq 3abc,$$

which is obvious by AM-GM. Equality occurs when $a=b=c$ and $x=y=z$.

Also solved by Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, University of Crete, Greece; Aaditya Bhaumik Sahu, AMMOC, India; E.Madani Harouach, AMMOC; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Parth Andhare, AMMOC; Sumanyu Nandecha, AMMOC.

S701. Let x, y, z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{\sqrt{x^2 + 3}} + \frac{1}{\sqrt{y^2 + 3}} + \frac{1}{\sqrt{z^2 + 3}} \geq \frac{90}{11(x + y + z) + 27}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

The inequality can be rewritten as

$$\frac{1}{\sqrt{(x+y)(x+z)}} + \frac{1}{\sqrt{(y+x)(y+z)}} + \frac{1}{\sqrt{(z+x)(z+y)}} \geq \frac{90}{11(x+y+z) + 9\sqrt{3(xy+yz+zx)}}.$$

There is an acute triangle ABC with sides $a = \sqrt{y+z}$, $b = \sqrt{z+x}$, $c = \sqrt{x+y}$. The inequality becomes

$$\frac{a+b+c}{abc} \geq \frac{180}{11(a^2+b^2+c^2) + 9\sqrt{3(2(a^2b^2+b^2c^2+c^2a^2) - a^4 - b^4 - c^4)}},$$

or

$$\frac{a+b+c}{abc} \geq \frac{180}{11(a^2+b^2+c^2) + 9\sqrt{3(4(ab+bc+ca)^2 - 8abc(a+b+c) - (a^2+b^2+c^2)^2)}},$$

Using the known identities in a triangle

$$abc = 4Rrs, \quad ab + bc + ca = s^2 + r^2 + 4Rr, \quad a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr),$$

after performing the calculations, it becomes

$$\frac{1}{2Rr} \geq \frac{180}{22(s^2 - r^2 - 4Rr) + 9\sqrt{3(16r^2s^2)}},$$

or

$$11s^2 + 18\sqrt{3}rs - 224Rr - 11r^2 \geq 0.$$

Using Walker's Inequality, valid in any acute triangle,

$$s^2 \geq 2R^2 + 8Rr + 3r^2,$$

it is enough to show that

$$22R^2 + 88Rr + 33r^2 + 18r\sqrt{3(2R^2 + 8Rr + 3r^2)} - 224Rr - 11r^2 \geq 0,$$

or

$$22R^2 - 136Rr + 22r^2 + 18r\sqrt{3(2R^2 + 8Rr + 3r^2)} \geq 0,$$

or

$$11R^2 - 68Rr + 11r^2 + 9r\sqrt{3(2R^2 + 8Rr + 3r^2)} \geq 0,$$

or, if we denote $x = \frac{R}{r} \geq 2$,

$$11x^2 - 68x + 11 + 9\sqrt{3(2x^2 + 8x + 3)} \geq 0.$$

This inequality can be rewritten as

$$11(x-2)^2 + 9\sqrt{3(2x^2 + 8x + 3)} - 24x - 33 \geq 0,$$

or

$$11(x-2)^2 - \frac{3[(8x+11)^2 - 27(2x^2 + 8x + 3)]}{8x+11+3\sqrt{3(2x^2 + 8x + 3)}} \geq 0,$$

or

$$11(x-2)^2 - \frac{30(x-2)^2}{8x+11+3\sqrt{3(2x^2 + 8x + 3)}} \geq 0,$$

which is clearly true. The equality holds when $x = y = z$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

S702. Prove that for each positive integer c there are infinitely many positive integers n such that $n! + c$ is not the square of an integer.

Proposed by Mohammadreza Ghelichkhani, Shahed University, Tehran, Iran

Solution 1 by the author

Assume by contradiction that there exists a positive integer m such that for every $n \geq m$, there exists a positive integer x such that $n! + c = x^2$. Choose n such that $n^2 - 1 \geq m$. Then, by our assumption, there exist positive integers x and y such that:

$$\begin{aligned}(n^2)! + c &= x^2 \\ (n^2 - 1)! + c &= y^2.\end{aligned}$$

Since $(n^2)! = n^2(n^2 - 1)!$, we can rewrite the first equation as

$$n^2(n^2 - 1)! + c = x^2.$$

Substituting $(n^2 - 1)! + c = y^2$, we get

$$n^2y^2 - n^2c + c = x^2.$$

Notice that $x^2 < (ny)^2$ (since $n^2y^2 - n^2c + c < n^2y^2$). Therefore

$$n^2y^2 - n^2c + c \leq (ny - 1)^2 = n^2y^2 - 2ny + 1.$$

Simplifying, we obtain $n^2c - c \geq 2ny - 1$, which implies $n^2c \geq 2ny$, and hence

$$nc \geq 2y = 2\sqrt{(n^2 - 1)! + c},$$

which is obviously false for n sufficiently large.

Solution 2 by Alessandro Ventullo, Milan, Italy

We will use the following result (which is just a consequence of Quadratic Reciprocity together with Dirichlet's Theorem on primes in arithmetic progressions).

Useful Lemma. *If a is a fixed non-square integer, then there are infinitely many odd primes p for which a is a quadratic non-residue modulo p , i.e. $\left(\frac{a}{p}\right) = -1$.*

Now, let c be a positive integer.

We have two cases.

(i) If c is not a perfect square, by the lemma there are infinitely many primes $p > c$ with $\left(\frac{c}{p}\right) = -1$. For any such p and any $n \geq p$ we have $p \mid n!$, hence

$$n! + c \equiv c \pmod{p},$$

which is a quadratic non-residue modulo p . But a perfect square is a quadratic residue modulo every odd prime that does not divide it. Therefore $n! + c$ cannot be a square. Since the choice $n \geq p$ gives infinitely many n for each such p , there are infinitely many n with the desired property.

(ii) If c is a perfect square, let $c = m^2$ where m is a positive integer, and let $a = m^2 + 1$. Then a is not a perfect square. By the lemma there are infinitely many primes $p > c$ such that

$$\left(\frac{a}{p}\right) = -1.$$

For such a prime p , take $n = p - 2$. From $(p - 1)! \equiv -1 \pmod{p}$ we get $(p - 2)! \equiv 1 \pmod{p}$, hence

$$n! + c \equiv 1 + m^2 = a \pmod{p}.$$

Then $n! + c \equiv a \pmod{p}$, a quadratic non-residue modulo p . Therefore $n! + c$ cannot be a perfect square. As there are infinitely many such primes p , we obtain infinitely many integers n .

Combining the two cases completes the proof.

Also solved by Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Prodromos Fotiadis, University of Crete, Greece.

Undergraduate problems

U697. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Prove that

$$\left(\int_a^b f(x) dx \right)^4 \leq (b-a)^2 \int_a^b e^x f^2(x) dx \int_a^b \frac{f^2(x)}{e^x} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Henry Ricardo, Westchester Area Math Circle, New York, USA

We can write

$$\int_a^b f^2(x) dx = \int_a^b f(x) e^{x/2} \cdot \frac{f(x)}{e^{x/2}} dx. \quad (*)$$

Now applying the Cauchy-Schwarz Inequality for integrals twice and using (*), we get

$$\left(\int_a^b f(x) \cdot 1 dx \right)^2 \leq \int_a^b f^2(x) dx \int_a^b 1 dx = (b-a) \int_a^b f^2(x) dx,$$

and

$$\begin{aligned} \left(\int_a^b f(x) dx \right)^4 &\leq (b-a)^2 \left(\int_a^b f^2(x) dx \right)^2 \\ &\leq (b-a)^2 \int_a^b f^2(x) e^x dx \cdot \int_a^b \frac{f^2(x)}{e^x} dx. \end{aligned}$$

Also solved by Ashutosh Joshi; Joshua Pit , Cambridge Rindge and Latin School, MA, USA; Kousik Sett, India; Nicu or Zlota, “Traian Vuia” Technical College, Foc ani, Romania; Paolo Perfetti, Universit  degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Yunyong Zhang, Ching-ai-unicom, Yunnan, China; Diar Gashi, University of Primorska, Slovenia; Katherine Merrill, University of Washington, Seattle; Srijan Sundar, Oxford, UK.

U698. Let $f(x + iy) = u(x, y) + iv(x, y)$ be a holomorphic function on $\mathbb{C}$ with real part

$$u(x, y) = 2x^2 - 3xy - 2y^2.$$

Find $\frac{|f'(z)|}{|z|}$ for $z \neq 0$.

Proposed Alessandro Ventullo, Milan, Italy

Solution 1 by Theo Koupelis, Clark College, Washington, USA

Because $f(z)$ is holomorphic, from the Cauchy-Riemann equations we get

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x - 3y \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x + 4y,$$

and thus

$$f'(z) = \frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = (4x - 3y) + i(3x + 4y).$$

Therefore,

$$\frac{|f'(z)|}{|z|} = \sqrt{\frac{(4x - 3y)^2 + (3x + 4y)^2}{x^2 + y^2}} = 5.$$

Solution 2 by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let's find $v(x, y)$ using one of the Cauchy-Riemann equations $u_x = v_y$, that is

$$v_y = 4x - 3y \implies v(x, y) = 4xy - \frac{3y^2}{2} + Q(x).$$

In order to find $Q(x)$ we use the second Cauchy-Riemann equation $v_x = -u_y$, that is

$$4y + Q'(x) = 3x + 4y \implies Q(x) = \frac{3x^2}{2} + c,$$

where c is a constant. Thus

$$f(z) = 2x^2 - 3xy - 2y^2 + i \left(4xy - \frac{3y^2}{2} + \frac{3x^2}{2} \right) + c$$

and

$$f'(z) = 4x - 3y + i(4y + 3x) = (4 + 3i)(x + iy) = (4 + 3i)z.$$

Hence $\frac{|f'(z)|}{|z|} = \sqrt{16 + 9} = 5$.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, New York, USA; Kousik Sett, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Yunyong Zhang, Chinaunicom, Yunnan, China; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Katherine Merrill, University of Washington, Seattle; Srijan Sundar, Oxford, UK.

- U699. (a) Prove that if $r \in \{7, 11, 13, 17\}$, for each prime p such that $p \equiv r \pmod{24}$, there is a positive integer n such that $p \mid 2^n + 3^n$.
- (b) Prove that for no prime $p \equiv 23 \pmod{24}$ there exists a positive integer n such that $p \mid 2^n + 3^n$.
- (c) Prove that for infinitely many primes $p \equiv 1 \pmod{24}$, there is a positive integer n such that $p \mid 2^n + 3^n$.

Proposed by Mohammadreza Ghlichkhani, Shahed University, Tehran, Iran

Solution by the author

- (a) If $p \equiv 7, 17 \pmod{24}$, then $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{3}{p}\right) = -1$. If $p \equiv 11, 13 \pmod{24}$, then $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{3}{p}\right) = 1$. Taking $n = (p-1)/2$, by Euler's Criterion we have $2^n \equiv \left(\frac{2}{p}\right) \pmod{p}$ and $3^n \equiv \left(\frac{3}{p}\right) \pmod{p}$, so

$$2^n + 3^n \equiv 0 \pmod{p}.$$

- (b) If $p \equiv 23 \pmod{24}$, then $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = 1$, then $2^n + 3^n \equiv a^2 + b^2 \pmod{p}$, where obviously $\gcd(a, b) = 1$, so if such a prime p divides $2^n + 3^n$ then $p \equiv 1 \pmod{4}$, contradiction.
- (c) Assume by contradiction that only finitely many such primes exist. Let $S = \{p_1, p_2, \dots, p_t\}$ be the set of primes $p \equiv 1 \pmod{24}$ for which there is n with $p \mid 2^n + 3^n$. Take

$$n = 12(p_1 - 1)(p_2 - 1) \cdots (p_t - 1).$$

Then for each $p_i \in S$, Fermat's Little Theorem gives $2^n \equiv 3^n \equiv 1 \pmod{p_i}$, hence $2^n + 3^n \equiv 2 \pmod{p_i}$. By definition of S , this means that no prime congruent to 1 modulo 24 divides $2^n + 3^n$.

Now we prove that every odd prime divisor of $N = 2^n + 3^n$ is congruent to 1 modulo 24, reaching a contradiction.

Since n is divisible by 4, then

$$N = (2^{\frac{n}{4}})^4 + (3^{\frac{n}{4}})^4 = a^4 + b^4,$$

where $a = 2^{\frac{n}{4}}$ and $b = 3^{\frac{n}{4}}$. Since $\gcd(a, b) = 1$, all of the prime divisors p of N satisfy $p \equiv 1 \pmod{8}$.

Furthermore,

$$N = (2^{\frac{n}{3}})^3 + (3^{\frac{n}{3}})^3 = a^3 + b^3,$$

where $a = 2^{\frac{n}{3}}$ and $b = 3^{\frac{n}{3}}$. Since $\gcd(a, b) = 1$ and $3 \nmid a + b$, then $\gcd(a + b, a^2 - ab + b^2) = 1$. Since $p \mid N = (a + b)(a^2 - ab + b^2)$, but $p \nmid a + b$, then $p \mid a^2 - ab + b^2$ and so $p \equiv 1 \pmod{3}$.

Thus, $p \equiv 1 \pmod{24}$ and we are done.

U700. Let a, b, c, d be nonnegative real numbers such that $ab + ac + ad + bc + bd + cd = 1$. Prove that

$$\frac{1}{a+b+c} + \frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} \geq 3.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, Romania

Solution by the author

Let $p = a + b + c + d$. Assume that $a = \max\{a, b, c, d\}$, and write the inequality as $E \geq 3$. For fixed a and d , we may assume that $b \geq c$, and b and E are functions of c . By differentiating the equality constraint, we get

$$(p-b)b' + p - c = 0, \quad -b' = \frac{p-c}{p-b}.$$

Let

$$A = \frac{1}{(p-a)^2} + \frac{1}{(p-d)^2} > \frac{1}{(p-a)^2} \geq \frac{1}{(p-b)(p-c)}.$$

We have

$$\begin{aligned} E'(c) &= \left(A + \frac{1}{(p-c)^2} \right) (-b') - A - \frac{1}{(p-b)^2} = A(-b' - 1) - \frac{b'}{(p-c)^2} - \frac{1}{(p-b)^2} \\ &= \frac{(b-c)A}{p-b} - \frac{b-c}{(p-b)^2(p-c)} \geq \frac{b-c}{(p-b)^2(p-c)} - \frac{b-c}{(p-b)^2(p-c)} = 0. \end{aligned}$$

Therefore, $E(c)$ is increasing and has the minimum value when c is minimum possible (to respect the constraint $a \geq b \geq c \geq 0$), hence when $c = 0$ or $b = a$ (because $b(c)$ is a decreasing function). So, abandoning the condition $b \geq c$, it suffices to prove the required inequality for the case when one of b and c is equal to 0 or a . By extending this result to any two numbers from b, c, d , it suffices to consider the case when at most one of b, c, d is different from 0 or a . So, we may assume the cases $(a, b, c, d) = (a, x, 0, 0)$, $(a, b, c, d) = (a, a, x, 0)$ and $(a, b, c, d) = (a, a, a, x)$, where $x \leq a$.

Case 1: $(a, b, c, d) = (a, x, 0, 0)$. We need to show that $ax = 1$ implies

$$\frac{2}{a+x} + \frac{1}{a} + \frac{1}{x} \geq 3.$$

The inequality is equivalent to

$$\begin{aligned} \frac{2a}{a^2+1} + \frac{1}{a} + a &\geq 3, \\ (a-1)^2(a^2-a+1) &\geq 0. \end{aligned}$$

Case 2: $(a, b, c, d) = (a, a, x, 0)$. We need to show that $a^2 + 2ax = 1$ implies

$$\frac{1}{2a+x} + \frac{2}{a+x} + \frac{1}{2a} \geq 3,$$

which is equivalent to

$$\begin{aligned} \frac{2a}{3a^2+1} + \frac{4a}{a^2+1} + \frac{1}{2a} &\geq 3, \\ 1 - 6a + 16a^2 - 24a^3 + 31a^4 - 18a^5 &\geq 0, \\ (1-a) [a^2(18a^2 - 13a + 4) + (7a^2 - 5a + 1)] &\geq 0. \end{aligned}$$

Since $1 = a^2 + 2ax \geq a^2$, hence $a \leq 1$, and the quantities in parentheses are positive, the last inequality is true.

Case 3: $(a, b, c, d) = (a, a, a, x)$. We need to show that $3a^2 + 3ax = 1$ implies

$$\frac{1}{3a} + \frac{3}{2a+x} \geq 3.$$

From $1 = 3a^2 + 3ax \geq 3a^2$ and $1 = 3a^2 + 3ax \leq 3a^2 + 3a^2$, we get $\frac{1}{\sqrt{6}} \leq a \leq \frac{1}{\sqrt{3}}$. The inequality is equivalent to

$$\begin{aligned} \frac{1}{3a} + \frac{9a}{3a^2+1} &\geq 3, \\ 1 - 9a + 30a^2 - 27a^3 &\geq 0. \end{aligned}$$

We have

$$1 - 9a + 30a^2 - 27a^3 > 1 - 9a + 30a^2 - 28a^3 = (2a-1)^2(5-7a) + 2(2-3a)(3a-1) > 0,$$

since $5-7a > 0$, $2-3a > 0$ and $3a-1 > 0$.

The proof is completed. The equality occurs when two of a, b, c, d are equal to 1 and all the others to 0.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

U701. Find all finite groups G of order $3n$ having at least two distinct subgroups H_1 and H_2 of order n such that $H_1 \cap H_2 = \{1\}$.

Proposed Alessandro Ventullo, Milan, Italy

Solution by the author

Clearly, $n > 1$. Since H_1 and H_2 are subgroups of G , then H_1H_2 is a subset of G (not necessarily a subgroup) of size

$$|H_1H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|} = n^2.$$

So, $n^2 \leq 3n$, which gives $n \leq 3$.

If $n = 2$ it is easy to see that $G \cong S_3$, the symmetric group on a set of 3 elements.

If $n = 3$, then $|G| = 9 = 3^2$, so G is abelian. The only possibilities are $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\mathbb{Z}_9$ has only one subgroup of order 3 and $\mathbb{Z}_3 \times \mathbb{Z}_3$ has exactly 4 distinct subgroups of order 3, we conclude that $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Also solved by Prodromos Fotiadis, University of Crete, Greece; Yunyong Zhang, Chinaunicom, Yunnan, China; Keisha Kwok, AMMOC; Srijan Sundar, Oxford, UK.

U702. Let a and b be real numbers. Let $(x_n)_{n \geq 0}$ be a sequence of real numbers such that $2x_n \leq x_{n-1} + x_{n+1}$, for all $n \geq 1$ and $\lim_{n \rightarrow \infty} (x_n - an - b) = 0$. Prove that $x_n \geq an + b$, for all $n \geq 0$.

Proposed by Dan-Ştefan Marinescu, Hunedoara, Romania

Solution by Daniel Pascuas, Barcelona, Spain

Let $y_n = an + b$, for all $n \geq 0$. Then it is clear that $2y_n = y_{n-1} + y_{n+1}$, for all $n \geq 1$, and so $z_n = x_n - y_n$ satisfy that $2z_n \leq z_{n-1} + z_{n+1}$, for all $n \geq 1$, and $\lim_{n \rightarrow \infty} z_n = 0$. We want to prove that $z_n \geq 0$, for all $n \geq 0$. In order to do that, note that the condition $2z_n \leq z_{n-1} + z_{n+1}$ can be written as $z_n - z_{n-1} \leq z_{n+1} - z_n$. It follows that $u_n = z_{n+1} - z_n$ defines a non-decreasing sequence, so $\sup_{n \geq 0} u_n = \lim_{n \rightarrow \infty} u_n = 0$, which shows that $u_n \leq 0$, for all $n \geq 0$. Therefore $(z_n)_{n \geq 0}$ is a non-increasing sequence so $\inf_{n \geq 0} z_n = \lim_{n \rightarrow \infty} z_n = 0$, and hence $z_n \geq 0$, for all $n \geq 0$.

Also solved by Joshua Pit , Cambridge Rindge and Latin School, MA, USA; Paolo Perfetti, Universit  degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA; Diar Gashi, University of Primorska, Slovenia; Srijan Sundar, Oxford, UK.

Olympiad problems

O697. Let a, b, c, d be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$. Prove that

$$\sqrt{\frac{1}{a^2} + \sqrt{a} + 1} + \sqrt{\frac{1}{b^2} + \sqrt{b} + 1} + \sqrt{\frac{1}{c^2} + \sqrt{c} + 1} + \sqrt{\frac{1}{d^2} + \sqrt{d} + 1} \leq \frac{\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}}{2} + 3.$$

Proposed by Titu Andreescu, Dallas, USA

Solution 1 by the author

Let $\sqrt{a} = s, \sqrt{b} = t, \sqrt{c} = u, \sqrt{d} = v$. We have

$$\frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2} + \frac{1}{v^2} = 1,$$

and we need to prove

$$\frac{\sqrt{s^5 + s^4 + 1}}{s^2} + \frac{\sqrt{t^5 + t^4 + 1}}{t^2} + \frac{\sqrt{u^5 + u^4 + 1}}{u^2} + \frac{\sqrt{v^5 + v^4 + 1}}{v^2} \leq \frac{s + t + u + v}{2} + 3.$$

Because $x^5 + x^4 + 1 = (x^3 - x + 1)(x^2 + x + 1)$, by the AM-GM Inequality we get

$$\frac{\sqrt{x^5 + x^4 + 1}}{x^2} \leq \frac{1}{2} \cdot \frac{x^3 - x + 1 + x^2 + x + 1}{x^2} = \frac{x + 1}{2} + \frac{1}{x^2},$$

with equality if and only if $x = 2$.

Hence, the left-hand side of the desired inequality is less than or equal to

$$\frac{s + t + u + v + 4}{2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2} + \frac{1}{v^2} = \frac{s + t + u + v + 4}{2} + 1,$$

as desired.

The equality holds if and only if $a = b = c = d = 4$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

We have

$$\begin{aligned} \sum_c \sqrt{\frac{1}{a^2} + \sqrt{a} + 1} &= \sum_c \frac{1}{a} \sqrt{a^{5/2} + a^2 + 1} \leq \sum_c \frac{a^{3/2} + a + 2}{2a} \\ &= \frac{\sum_c \sqrt{a}}{2} + 2 + \sum_c \frac{1}{a} = \frac{\sum_c \sqrt{a}}{2} + 3, \end{aligned}$$

because for positive real numbers x we get, after squaring and simplifying,

$$2\sqrt{x^{5/2} + x^2 + 1} \leq x^{3/2} + x + 2 \iff x(\sqrt{x} - 2)^2(\sqrt{x} + 1)^2 \geq 0,$$

which is obvious. Equality occurs when $a = b = c = d = 4$.

Also solved by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam.

O698. Find the smallest positive constant k such that

$$\left(\frac{ka + b + c + d + e}{k + 4} \right)^2 \geq \frac{ab + bc + cd + de + ef + fa}{6}$$

whenever $a \geq b \geq c \geq d \geq e \geq f \geq 0$.

Proposed by Vasile Cîrtoaje, Oil-Gas University of Ploiești, Romania

Solution by the author

Setting $a = b = 1$ and $d = e = f = 0$, which involves $c \in (0, 1)$, the inequality becomes

$$\left(\frac{k + 1 + c}{k + 4} \right)^2 \geq \frac{c + 1}{6}.$$

Moreover, for $c + 1 = k$, we get $k(k^2 - 16k + 16) \leq 0$, hence

$$k \geq k_0 = 4(2 - \sqrt{3}) \approx 1.0718.$$

For $k = k_0$, we have $c = k_0 - 1 \in (0, 1)$. So, we claim that k_0 is the smallest positive value of k . To show this, we only need to prove the desired inequality for $k = k_0$. Since

$$ab + bc + cd + de + ef + fa \leq ab + bc + cd + de + e^2 + ea,$$

it suffices to prove that $F(a, b, c, d, e) \geq 0$, where

$$F(a, b, c, d, e) = 6(ka + b + c + d + e)^2 - (k + 4)^2(ab + bc + cd + de + ea + e^2).$$

We will show that

$$F(a, b, c, d, e) \geq F(a, b, c, s, s) \geq F(S, S, c, s, s) \geq 0,$$

where

$$s = \frac{d + e}{2}, \quad S = \frac{a + b}{2}, \quad S \geq c \geq s.$$

We have

$$\begin{aligned} F(a, b, c, d, e) - F(a, b, c, s, s) &= (k + 4)^2[(ab + bc + cs + s^2 + sa + s^2) - (ab + bc + cd + de + ea + e^2)] \\ &= (k + 4)^2[c(s - d) + (s^2 - de) + a(s - e) + (s^2 - e^2)] \geq (k + 4)^2[c(s - d) + a(s - e)] \\ &= \frac{1}{2}(k + 4)^2(d - e)(a - c) \geq 0 \end{aligned}$$

and

$$\begin{aligned} F(a, b, c, s, s) - F(S, S, c, s, s) &= (k + 4)^2[(S^2 + Sc + cs + s^2 + sS + s^2) - (ab + bc + cs + s^2 + sa + s^2)] \\ &= (k + 4)^2[(S^2 - ab) + c(S - b) + s(S - a)] \geq (k + 4)^2[c(S - b) + s(S - a)] = \frac{1}{2}(k + 4)^2(a - b)(c - s) \geq 0. \end{aligned}$$

Next, for fixed S and c , we write the inequality $F(S, S, c, s, s) \geq 0$ as $f(s) \geq 0$, where

$$f(s) = 6(kS + S + c + 2s)^2 - (k + 4)^2(S^2 + Sc + cs + 2s^2 + sS).$$

Since $f(s)$ is concave and $s \in [0, c]$, it suffices to show that $f(0) \geq 0$ and $f(c) \geq 0$. We have

$$f(0) = 6(kS + S + c)^2 - (k + 4)^2(S^2 + Sc) = (5k^2 + 4k - 10)S^2 - (k^2 - 4k + 4)^2Sc + 6c^2.$$

Since $k^2 = 16k - 16$, we have

$$5k^2 + 4k - 10 = 6k^2 - k^2 + 4k - 10 = 6k^2 - (16k - 16) + 4k - 10 = 6(k - 1)^2,$$

$$k^2 - 4k + 4 = (16k - 16) - 4k + 4 = 12(k - 1),$$

hence

$$f(0) = 6[(k - 1)S - c]^2 \geq 0.$$

Also,

$$f(c) = 6(kS + S + 3c)^2 - (k + 4)^2(S^2 + 2Sc + 3c^2) = 6(S - c)[(14k - 15)S + 3(4k - 3)c] \geq 0.$$

For $k = k_0$, the equality occurs when $a = b = c = d = e = f$, and also when $a = b = (7 + 4\sqrt{3})c$ and $d = e = f = 0$.

Also solved by Theo Koupelis, Clark College, Washington, USA.

O699. Prove that the equation

$$y(x^2 - z^2 - y + 1) = x^2 + 3$$

has no integer solutions.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let $t = x^2 - z^2 - y + 1$. Then, $yt = x^2 + 3$. Hence,

$$(t - 1)(y - 1) = ty - t - y + 1 = (x^2 + 3) - (x^2 - z^2 - y + 1) - y + 1$$

that is

$$(t - 1)(y - 1) = z^2 + 3.$$

It is clear that y and t have the same sign and we may assume that they are positive. Letting $(y_0, t_0) = (1 - y, 1 - t)$ leads to the same system of positive variables. First assume that both y and t are even. Because $x^2 + 3$ is not divisible by 8 this is only possible if $y \equiv t \equiv 2 \pmod{4}$. Then $t - 1$ and $y - 1$ are both congruent to 1 modulo 4, and so their product $z^2 + 3$, impossible.

Analogously, if both y and t are odd then $y - 1$ and $t - 1$ are both even which would only be possible if they both are congruent to 2 modulo 4. Then, y and t are both congruent to 3 modulo 4 and their product is 1 modulo 4. But their product is $x^2 + 3$, a contradiction.

Finally, assume that y is even and t is odd (the case when y is odd and t is even is similar). Then $x^2 + 3$ is even and must therefore be congruent to 4 modulo 8, yielding $y = 4a$ for some odd a . Analogously, $t = 1 + 4b$ for some odd b . Then we would have $4b(4a - 1) = z^2 + 3$ and $(4b + 1)4a = x^2 + 3$. Thus, primes dividing a , b , $4a - 1$ and $4b + 1$ must either be 3 or of the form $3m + 1$, implying that none of them can be congruent to 2 modulo 3. This is possible only if $a \equiv 1 \pmod{3}$, $b \equiv 0 \pmod{3}$. But in this case, $4b$ and $4a - 1$ are both divisible by 3, hence their product $z^2 + 3$ is divisible by 9, a contradiction.

Also solved by Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; E.Madani Harouach, AM-MOC.

O700. Let $a_1, \dots, a_n$ be positive real numbers such that $a_1 \dots a_n = 1$. Prove that

$$\sum_{i=1}^n \frac{a_i^5}{a_1 + \dots + a_{i-1} + a_i^2 + a_{i+1} + \dots + a_n} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Prodromos Fotiadis, University of Crete, Greece;

Let

$$A_k := \sum_{i=1}^n a_i^k, \quad k = 1, 2, 3$$

Then

$$\sum_{i=1}^n \frac{a_i^5}{a_1 + \dots + a_{i-1} + a_i^2 + a_{i+1} + \dots + a_n} = \sum_{i=1}^n \frac{a_i^5}{A_1 + a_i^2 - a_i} = \sum_{i=1}^n \frac{a_i^6}{a_i A_1 + a_i^3 - a_i^2}$$

Using Titu's Lemma:

$$\sum_{i=1}^n \frac{a_i^6}{a_i A_1 + a_i^3 - a_i^2} \geq \frac{A_3^2}{A_1^2 + A_3 - A_2}$$

so it suffices to show the following

$$A_3^2 \geq A_1^2 + A_3 - A_2 \iff A_3^2 - A_3 \geq A_1^2 - A_2$$

According to the Power Mean inequality we have

$$\sqrt[3]{\frac{A_3}{n}} \geq \sqrt{\frac{A_2}{n}} \geq \frac{A_1}{n} \geq \sqrt[n]{a_1 \dots a_n} = 1$$

This implies

$$A_3 \geq n \implies A_3^2 - A_3 \geq (n-1)A_3$$

and

$$nA_2 \geq A_1^2 \implies A_1^2 - A_2 \leq (n-1)A_2$$

Therefore, it suffices to prove that $A_3 \geq A_2$. To show this, notice that

$$\frac{A_3}{n} \geq 1 \iff \frac{A_3}{n} \geq \left(\frac{A_3}{n}\right)^{2/3} \geq \frac{A_2}{n} \implies A_3 \geq A_2$$

This completes the proof. The equality occurs if and only if $a_1 = a_2 = \dots = a_n = 1$.

Also solved by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Clark College, Washington, USA.

O701. Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

Prove that

$$8(a^3 + b^3 + c^3 + d^3) + 49abcd \geq 81.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

We have the following inequalities:

$$(a + b + c + d)^4 + 32abcd \geq 3(a + b + c + d)^2(ab + ac + ad + bc + bd + cd), \quad (1)$$

$$3(a + b + c + d)^3 + 18(abc + bcd + cda + dab) \geq 11(a + b + c + d)(ab + ac + ad + bc + bd + cd). \quad (2)$$

Proof. We can consider $a \geq b \geq c \geq d$ and $a + b + c = 1$. Denoting $t = abc$, by the AM-GM Inequality it follows that $t \leq \frac{1}{27}$. Obviously we also have $d \leq \frac{1}{3}$.

For (1) we need to show that

$$(d + 1)^4 + 32td \geq 3(d + 1)^2(ab + bc + ca + d).$$

By Schur's Inequality, i.e.

$$(a + b + c)^2 + \frac{9abc}{a + b + c} \geq 4(ab + bc + ca),$$

it follows that

$$ab + bc + ca \leq \frac{9t + 1}{4}. \quad (3)$$

It suffices to show that

$$(d + 1)^4 + 32td \geq 3(d + 1)^2 \left(\frac{9t + 1}{4} + d \right),$$

or

$$4d^4 + 4d^3 - 3d^2 - 2d + 1 - t(27d^2 - 74d + 27) \geq 0.$$

But $27t \leq 1$ so it remains to show that

$$27(4d^4 + 4d^3 - 3d^2 - 2d + 1) - 27d^2 + 74d - 27 \geq 0,$$

or

$$4d(3d - 1)^2(3d + 5) \geq 0,$$

clearly true.

For (2) we need to show that

$$3(d + 1)^3 + 18(t + d(ab + bc + ca)) \geq 11(d + 1)(d + ab + bc + ca),$$

or

$$3(d + 1)^3 - 11d(d + 1) + 18t - (11 - 7d)(ab + bc + ca) \geq 0.$$

It suffices to show that

$$3(d + 1)^3 - 11d(d + 1) + 18t - \frac{(11 - 7d)(9t + 1)}{4} \geq 0,$$

or

$$12d^3 - 8d^2 - d + 1 - 9t(3 - 7d) \geq 0.$$

Since $3 - 7d > 0$ and $27t \leq 1$ it remains to show that

$$36d^3 - 24d^2 - 3d + 3 - 3 + 7d \geq 0,$$

or

$$4d(3d - 1)^2 \geq 0,$$

clearly true. □

Also we have

$$a^3 + b^3 + c^3 + d^3 = (a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 3 \sum_{cyc} abc,$$

therefore our inequality becomes

$$8(a + b + c + d)^3 - 24 \sum_{cyc} a \sum_{sym} ab + 24 \sum_{cyc} abc + 49abcd \geq 81. \quad (4)$$

If we denote $t = a + b + c + d$ from inequalities (1), (2) and the condition of the hypothesis, we get

$$abcd \geq \frac{t^3(9 - 2t)}{64},$$

$$abc + bcd + cda + dab \geq \frac{t^2(11 - 2t)}{12}.$$

Also by Maclaurin's Inequality, we have

$$\frac{a + b + c + d}{4} \geq \sqrt{\frac{ab + bc + cd + da + ac + bd}{6}} \implies t \geq 4.$$

We have two cases:

1. If $4 \leq t \leq \frac{9}{2}$, then returning to inequality (4) and using these results, we have to prove the following inequality

$$8t^3 - 36t^2 + 2t^2(11 - 2t) + \frac{49t^3(9 - 2t)}{64} - 81 \geq 0,$$

or after we make the calculations

$$(t - 4)(9 - 2t)(49t^2 + 68t + 144) \geq 0,$$

clearly true.

2. If $t > \frac{9}{2}$, then by Cauchy-Schwarz Inequality, we have

$$\begin{aligned} (a^3 + b^3 + c^3 + d^3)(a + b + c + d) &\geq (a^2 + b^2 + c^2 + d^2)^2 \\ &= ((a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd))^2 \\ &= (a + b + c + d)^2(a + b + c + d - 3)^2 \end{aligned}$$

which means

$$a^3 + b^3 + c^3 + d^3 \geq t(t - 3)^2 > \frac{9}{2} \left(\frac{9}{2} - 3 \right)^2 = \frac{81}{8},$$

hence

$$8(a^3 + b^3 + c^3 + d^3) + 49abcd > 81$$

The equality holds when $a = b = c = d = 1$ or $a = b = c = \frac{3}{2}$, $d = 0$ and its cyclic permutations.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA.

O702. Let ABC be a triangle with $AB \neq AC$. Let H be its orthocenter, O be its circumcenter and I be its incenter. Let D be the reflection of A in O , let F be the midpoint of AH , and J be the intersection point of AI and BC . Let Q be the midpoint of the arc BAC . Let D_B, D_C are the feet of the perpendiculars dropped from the vertices B, C to the lines CD, BD respectively and E be the intersection point of $D_B D_C$ and BC . Knowing that $DE \parallel JO$, prove that line FQ is also parallel to them.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution 1 by the author

Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle ABC .

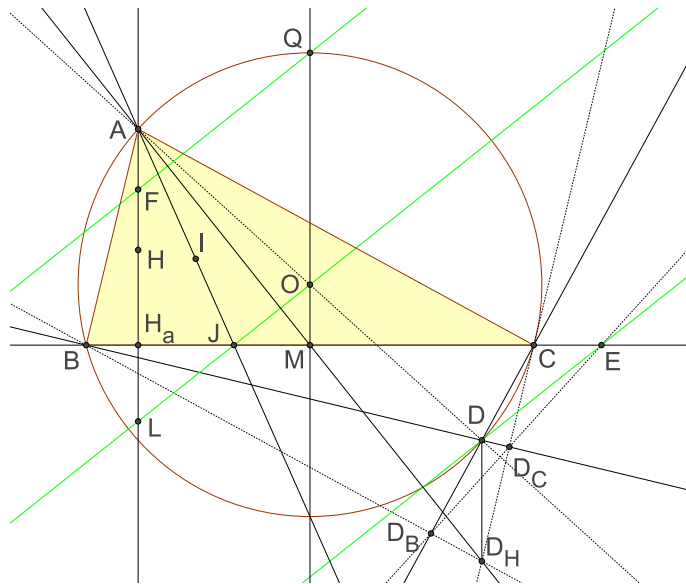


Figure 1:

Let H_a be the foot of the perpendicular dropped from A to the line BC . Denote with $h_a = AH_a$ the altitude and with $R = AO$ the radius of the circumcircle of the triangle ABC . Let S be the area of the $\triangle ABC$.

$$S = \frac{1}{2}ah_a = \frac{abc}{4R} = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

$$R = \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}$$

Denote with D_H the orthocenter of the triangle $\triangle DBC$. It is well known that $2AF = AH = 2OM = DD_H$, and $AH \parallel OM \parallel DD_H \perp BC$. In the triangle $\triangle ADD_H$, $AO = OD$ and $DD_H = 2OM, DD_H \parallel OM$, so that OM is a midsegment and A, M, D_H are collinear and $AM = MD_H$.

Since $\angle BD_B C = \angle BD_C C = 90^\circ$, $BD_B D_C C$ is a cyclic quadrilateral centered at M . From Brocard's Theorem, M is the orthocenter of $\triangle DED_H$, so $AM \perp DE$ and $AM \perp OJ$. Without loss of generality we may assume that $b > c$, hence M is between C and the points H_a, J . So $\angle A < 90^\circ$ and

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc} > 0; \quad -a^2 + b^2 + c^2 > 0$$

$AM \perp OJ, OM \perp H_a M$, hence $\triangle AH_a M \sim \triangle JMO$ and

$$\frac{AH_a}{JM} = \frac{H_a M}{OM}$$

$$\begin{aligned}
H_a M &= H_a C - MC = b \cos C - \frac{a}{2} = \frac{a^2 + b^2 - c^2}{2a} - \frac{a}{2} = \frac{b^2 - c^2}{2a} \\
\frac{BJ}{JC} &= \frac{c}{b}; \quad BJ = \frac{ac}{b+c} \\
JM &= BM - BJ = \frac{a}{2} - \frac{ac}{b+c} = \frac{a(b+c) - 2ac}{2(b+c)} = \frac{a(b-c)}{2(b+c)}
\end{aligned}$$

From the Pythagorean theorem for $\triangle OMC$:

$$\begin{aligned}
OM^2 &= OC^2 - CM^2 = R^2 - \frac{a^2}{4} = \frac{a^2(-a^2 + b^2 + c^2)^2}{4(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\
AH_a^2 &= h_a^2 = \frac{4S^2}{a^2} = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2}
\end{aligned}$$

Now

$$\begin{aligned}
0 &= AH_a \cdot OM - JM \cdot H_a M = \\
&= \frac{-a^2 + b^2 + c^2}{4} - \frac{(b-c)^2}{4} = \frac{1}{4}(-a^2 + 2bc)
\end{aligned}$$

The condition that $DE \parallel JO$ is equivalent with

$$a^2 = 2bc. \quad (1)$$

Let L be the intersect point of AH and OJ . $\triangle OMJ \sim LH_a J$ and

$$\begin{aligned}
H_a L^2 &= OM^2 \frac{H_a J^2}{JM^2} = OM^2 \frac{(H_a M - JM)^2}{JM^2} = \\
&= \frac{a^2(-a^2 + b^2 + c^2)^2}{4(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \cdot \frac{\left(\frac{b^2-c^2}{2a} - \frac{a(b-c)}{2(b+c)}\right)^2}{\frac{a^2(b-c)^2}{4(b+c)^2}} = \\
&= \frac{(-a+b+c)(a+b+c)(-a^2 + b^2 + c^2)^2}{4a^2(a+b-c)(a-b+c)} \stackrel{(1)}{=} -\frac{(b-c)^4(b^2 + c^2)}{8bc(b^2 - 4bc + c^2)} \\
AH_a^2 &\stackrel{(1)}{=} -\frac{(b^2 + c^2)(b^2 - 4bc + c^2)}{8bc} \\
OM^2 &\stackrel{(1)}{=} -\frac{bc(b-c)^4}{2(b^2 + c^2)(b^2 - 4bc + c^2)} \\
FL &= AH_a + H_a L - AF = AH_a + H_a L - OM = \\
&\stackrel{(1)}{=} \sqrt{-\frac{(b^2 + c^2)(b^2 - 4bc + c^2)}{8bc}} + \frac{(b-c)^4}{\sqrt{-8bc(b^2 + c^2)(b^2 - 4bc + c^2)}} = \\
&= \sqrt{2bc} \sqrt{-\frac{bc}{(b^2 + c^2)(b^2 - 4bc + c^2)}}
\end{aligned}$$

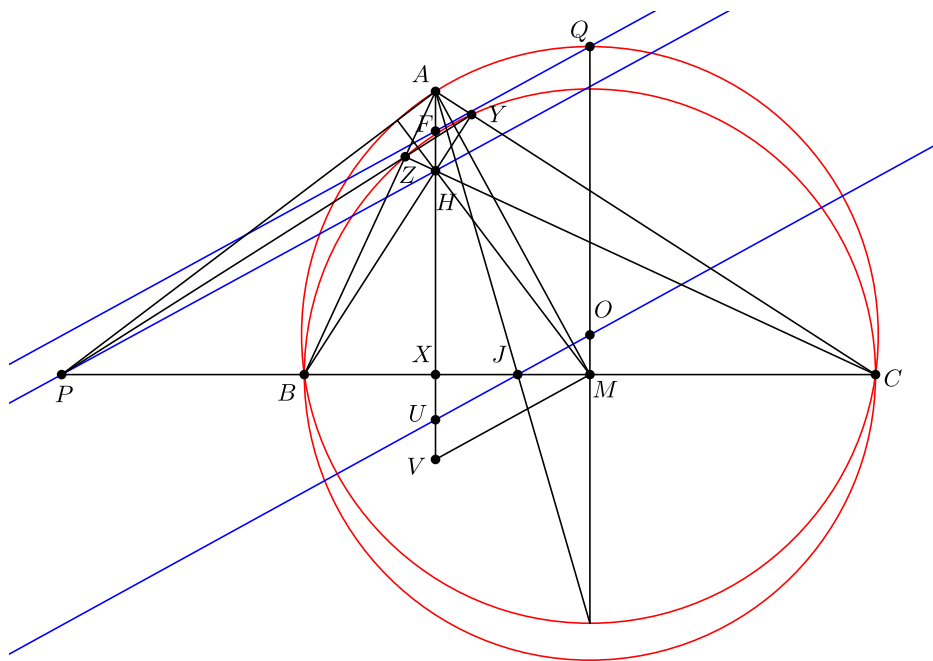
But

$$OQ = R \stackrel{(1)}{=} \sqrt{2bc} \sqrt{-\frac{bc}{(b^2 + c^2)(b^2 - 4bc + c^2)}} = FL$$

But Q, O, M are collinear and A, F, H, L are collinear, so $OQ \parallel FL$. Hence $OQFL$ is a parallelogram, and so $FQ \parallel OJ$ if $DE \parallel OJ$.

Solution 2 by Li Zhou, Polk State College, FL, USA

Let M be the midpoint of BC and X, Y, Z be the feet of altitudes on BC, CA, AB respectively. Suppose YZ intersects BC at P . Since D, D_B, D_C are reflections of H, Y, Z across M respectively, $PH \parallel JO$. Notice that B, C, Y, Z lie on a circle Γ with center M and AP is the polar of H with respect to Γ , so $AP \perp MH$, thus H is also the orthocenter of $\triangle APM$. Hence $PH \perp AM$.



Assume $b > c$ and the circumradius of ABC is R . Since $JO \perp AM$, $\triangle JMO \sim \triangle AXM$, so $AX \cdot OM = JM \cdot XM$. Plugging in $OM = R \cos A$, $AX = \frac{bc}{2R}$, $JM = \frac{ab}{b+c} - \frac{a}{2} = \frac{a(b-c)}{2(b+c)}$, and $XM = \frac{a}{2} - c \cos B = \frac{b^2 - c^2}{2a}$, we get $AX \cdot OM = \frac{b^2 + c^2 - a^2}{4}$ and $JM \cdot XM = \frac{(b-c)^2}{4}$, so $a^2 = 2bc$.

Next, suppose that OJ intersects AX at U and the line through M and parallel to OJ intersects AX at V . Then $\frac{2b^2 + 2c^2 - a^2}{4} = AM^2 = AX \cdot AV = AX(FU + 2OM) = AX \cdot FU + \frac{b^2 + c^2 - a^2}{2}$, so $AX \cdot FU = \frac{a^2}{4}$, thus $FU = \frac{a^2}{4AX} = \frac{a^2 R}{2bc} = R = QO$. Therefore $FUOQ$ is a parallelogram, completing the proof.

Also solved by Theo Koupelis, Clark College, Washington, USA.