

AN ANALYTIC GEOMETRY APPROACH TO SOLVING CERTAIN PLANE GEOMETRY PROBLEMS

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Abstract

This article showcases a selection of compelling examples that underscore the significance and elegance of employing analytic geometry in the solution of plane geometry problems. The problems discussed originate either from the authors' own creations or from the International Mathematical Olympiad (IMO), serving to illustrate the versatility and strength of coordinate-based approaches in mathematical problem-solving.

1 Introduction

The advent of analytic geometry marked a revolutionary turning point in the history of mathematics, forging a profound connection between two seemingly independent branches: geometry and algebra. In 1637, the French philosopher and mathematician René Descartes (1596–1650) published his work *La Géométrie* as an appendix to *Discours de la méthode*. In this work, Descartes introduced a completely novel method for investigating geometric objects by expressing them through algebraic equations in a coordinate plane, now known as the Cartesian coordinate system.

This method not only enabled precise descriptions of curves and planar figures using algebraic language, but also laid the foundation for the development of calculus, differential geometry, classical mechanics, and many other modern mathematical disciplines. By translating geometric problems into algebraic ones, Descartes inaugurated a new perspective on geometry—shifting from pure intuition to logical reasoning and computation.

The historical significance of analytic geometry lies not only in its groundbreaking mathematical insights but also in its far-reaching influence on the natural sciences and engineering. Even today, the coordinate system and the fundamental concepts of analytic geometry remain foundational tools in mathematics education, research, and application across the globe.

Analytic geometry stands as a symbol of the unification between algebra and geometry. It vividly demonstrates the superior power of algebra in tackling geometric problems. In practice, for a wide range of specific geometric problems—including those involving loci, geometric extrema, relative positions of conics, or coordinate geometry in space—analytic geometry continues to be the most effective, and in many cases, irreplaceable method.

More than just a computationally feasible and general approach, analytic geometry opens new ways of thinking. It allows us to explore geometric problems with greater depth and broader vision, closely tied to analytic spaces and modern geometry. Representing geometric entities such as points, lines, curves, and planes through algebraic equations introduces a unified language that transcends the limits of visual intuition.

To further demonstrate the strength and practical utility of this method, we present several illustrative examples in the following sections. These examples not only highlight the efficiency of analytic geometry in solving complex geometric problems but also underscore why this approach remains central to modern mathematics to this day.

2 Illustrative Examples

In this section, we present a number of examples to illustrate the power of the analytic geometry method.

Problem 1. Let $ABCD$ be a quadrilateral with the diagonals AC and BD perpendicular and intersecting at T . Let M, N, P, Q lie on the sides AB, BC, CD, DA , respectively, such that $MNPQ$ is a rectangle with sides parallel to the two diagonals of the quadrilateral. QM and QP intersect DC and AB at E and F , respectively. Let ω_q be the circle with diameter EF . Similarly, we have circles ω_m , ω_n , and ω_p . (See figure 1.)

- 1) Prove that the circles ω_m , ω_n , ω_p , and ω_q have a common radical center O .
- 2) Let X be the intersection of the perpendicular bisectors of the two diagonals AC and BD . Let Y be the intersection of the perpendicular bisectors of the sides AD and BC . Let Z be the intersection of the perpendicular bisectors of the sides AB and CD . Construct the parallelogram $XYWZ$. Prove that the lines TW and SO are parallel.
- 3) Prove that there exist two circles Ω and ω that are both tangent to all four circles ω_m , ω_n , ω_p , and ω_q .
- 4) Let S be the center of the rectangle $MNPQ$. Prove that the line SO passes through the centers of the two circles Ω and ω .

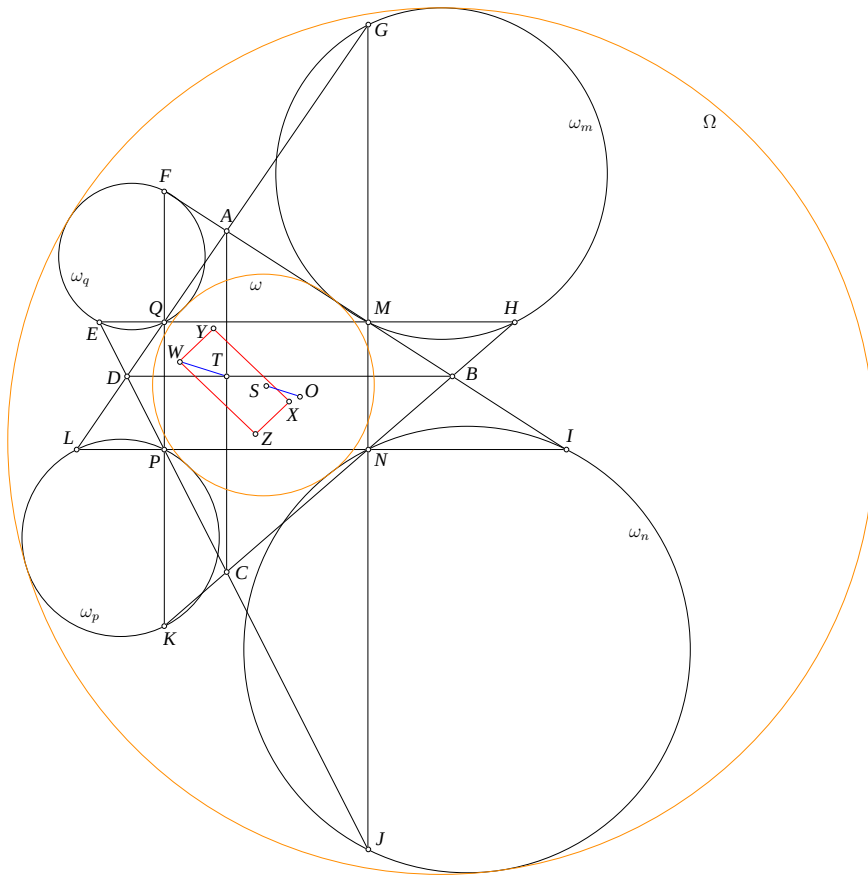


Figure 1: Illustration for the first example.

Proof. (See Figure 1.) Consider this configuration in a Cartesian plane with BD and AC being the x -axis and the y -axis respectively. Let $A = (0, a)$, $B = (b, 0)$, $C = (0, c)$, $D = (d, 0)$,

$T = (0, 0)$. Let

$$\frac{\overline{AM}}{\overline{AB}} = \frac{\overline{CN}}{\overline{CB}} = \frac{\overline{CP}}{\overline{CD}} = \frac{\overline{AQ}}{\overline{AD}} = t,$$

so that we have

$$M = (1 - t)A + tB = ((1 - t)b, ta)$$

$$N = (1 - t)C + tB = ((1 - t)b, tc)$$

$$P = (1 - t)C + tD = ((1 - t)d, tc)$$

$$Q = (1 - t)A + tD = ((1 - t)d, ta).$$

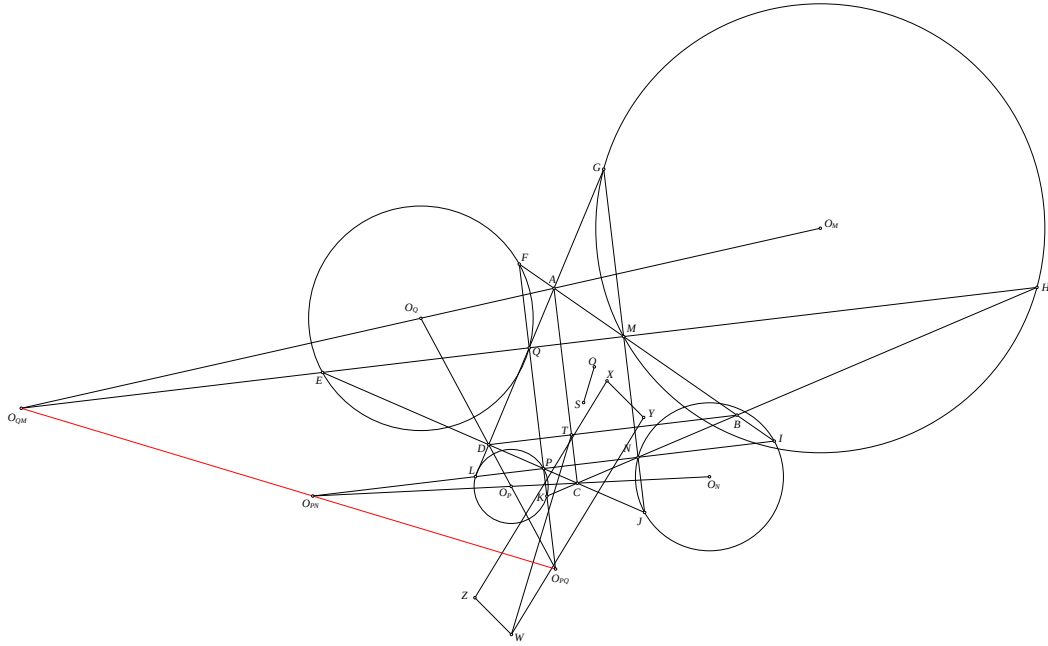


Figure 2: Illustration for the first part.

We also have the equations of the lines AB , BC , CD and DA

$$AB : y = \frac{-a}{b}x + a$$

$$BC : y = \frac{-c}{b}x + c$$

$$CD : y = \frac{-c}{d}x + c$$

$$DA : y = \frac{-a}{d}x + a,$$

therefore

$$E = \left(\frac{cd - tad}{c}, ta \right)$$

$$F = \left((1 - t)d, \frac{ab - ad(1 - t)}{b} \right).$$

1) Let O_M , O_N , O_P , O_Q be the centers of the circles ω_m , ω_n , ω_p , ω_q respectively. Then

$$O_Q = \frac{E + F}{2} = \left(\frac{(2 - t)cd - tad}{2c}, \frac{(t + 1)ab - (1 - t)ad}{2b} \right).$$

Let

$$O = \left(\frac{(tac + (t+1)bd)(1-t)(b+d)}{2bd}, \frac{t(a+c)(2ac+bd-(ac+bd)t)}{2ac} \right).$$

We can calculate the power of the point O with respect to the circle ω_q

$$\text{Pow}(O, \omega_q) = OO_Q^2 - QO_Q^2 = \frac{\gamma_4 t^4 + \gamma_3 t^3 + \gamma_2 t^2 + \gamma_1 t + \gamma_0}{4a^2 c^2 b^2 d^2} = f(d, a, b, c)$$

with

$$\begin{aligned}\gamma_4 &= (ac + bd)^2 (a^2 c^2 (b+d)^2 + b^2 d^2 (a+c)^2) \\ \gamma_3 &= -2(ac + bd) (a^3 c^3 (b+d)^2 + b^2 d^2 (a+c)^2 (2ac + bd)) \\ \gamma_2 &= a^4 c^4 (b+d)^2 + b^4 d^4 (a+c)^2 + 2acbd(ac + bd) (2bd(a+c)^2 - ac(b^2 + d^2)) \\ \gamma_1 &= 2a^3 c^3 bd(b-d)^2 \\ \gamma_0 &= a^2 c^2 b^2 d^2 (b-d)^2\end{aligned}$$

Since

$$f(d, a, b, c) = f(b, a, d, c) = f(d, c, b, a) = f(b, c, d, a)$$

we obtain that O is the common radical center of the circles ω_m , ω_n , ω_p , and ω_q .

2) Because X is the intersection of the perpendicular bisectors of the two diagonals AC and BD , we have

$$X = \left(\frac{b+d}{2}, \frac{a+c}{2} \right)$$

The equation of the perpendicular bisector of AB is

$$x^2 + (y-a)^2 = (x-b)^2 + y^2.$$

or

$$2bx - 2ay = b^2 - a^2$$

Similarly, the equation of the perpendicular bisector of BC is

$$2bx - 2cy = b^2 - c^2,$$

the equation of the perpendicular bisector of CD is

$$2dx - 2cy = d^2 - c^2,$$

and the equation of the perpendicular bisector of DA is

$$2dx - 2ay = d^2 - a^2.$$

All these lead to

$$\begin{aligned}Y &= \left(\frac{a^2 c + b^2 a - c^2 a - d^2 c}{2(ab - cd)}, \frac{a^2 b + b^2 d - d^2 b - c^2 d}{2(ab - cd)} \right) \\ Z &= \left(\frac{a^2 c - c^2 a + d^2 a - b^2 c}{2(ad - bc)}, \frac{a^2 d - b^2 d - c^2 b + d^2 b}{2(ad - bc)} \right).\end{aligned}$$

Since $XYWZ$ is a parallelogram, we have

$$W = Y + Z - X = \left(\frac{ac(b+d)(a+b-c-d)(a+d-b-c)}{2(ab-cd)(ad-bc)}, \right.$$

$$\frac{bd(a+c)(a+b-c-d)(a+d-b-c)}{2(ab-cd)(ad-bc)}\bigg),$$

and, since S is the center of the rectangle $MNPQ$, we have

$$S = \frac{M+P}{2} = \left(\frac{(1-t)(b+d)}{2}, \frac{t(a+c)}{2} \right)$$

Now

$$\overrightarrow{OS} = S - O = \left(\frac{t(t-1)(b+d)(ac+bd)}{2bd}, \frac{t(t-1)(a+c)(ac+bd)}{2ac} \right)$$

thus, because

$$\frac{x_W}{y_W} = \frac{x_S - x_O}{y_S - y_O} = \frac{ac(b+d)}{bd(a+c)},$$

we conclude that $TW \parallel SO$.

3) Using the law of sines, we have

$$\frac{QE}{QF} = \frac{QE}{QD} \cdot \frac{QA}{QF} \cdot \frac{QD}{QA} = \frac{\sin \angle QDE}{\sin \angle QED} \cdot \frac{\sin \angle QFA}{\sin \angle QAF} \cdot \frac{QD}{QA} = \frac{\sin \angle ADC}{\sin \angle CDB} \cdot \frac{\sin \angle CAB}{\sin \angle DAB} \cdot \frac{QD}{QA}$$

and

$$\frac{MH}{MG} = \frac{MH}{MB} \cdot \frac{MA}{MG} \cdot \frac{MB}{MA} = \frac{\sin \angle MBH}{\sin \angle MHB} \cdot \frac{\sin \angle MGA}{\sin \angle MAG} \cdot \frac{MB}{MA} = \frac{\sin \angle ABC}{\sin \angle DBC} \cdot \frac{\sin \angle DAC}{\sin \angle DAB} \cdot \frac{MB}{MA}.$$

Since $QM \parallel DB$, we have

$$\frac{QD}{QA} = \frac{MB}{MA},$$

therefore

$$\frac{QE}{QF} \div \frac{MH}{MG} = \frac{\sin \angle CDA}{\sin \angle CAD} \cdot \frac{\sin \angle CAB}{\sin \angle CBA} \cdot \frac{\sin \angle CBD}{\sin \angle CDB} = \frac{CA}{CD} \cdot \frac{CB}{CA} \cdot \frac{CD}{CB} = 1.$$

Since $\angle FQE = \angle GMH = 90^\circ$, we have $\triangle FQE \sim \triangle GMH$, thus $O_Q Q \parallel O_M H$

Because $FQ \parallel MG$, we have that A is the internal homothetic center of ω_q and ω_m , and then O_{QM} —the external homothetic center of ω_q and ω_m —lies on QM , we get

$$O_{QM} = \left(\frac{bd(2c - ta - tc)}{c(b+d)}, ta \right).$$

Let O_{PN} be the external homothetic center of ω_p and ω_n , and let O_{PQ} be the external homothetic center of ω_p and ω_q . We analogously obtain

$$O_{PN} = \left(\frac{bd(2a - ta - tc)}{a(b+d)}, tc \right)$$

and

$$O_{PQ} = \left((1-t)d, \frac{ac(bt + dt + b - d)}{b(a+c)} \right).$$

Now O_{QM} , O_{PN} , O_{PQ} are collinear because they all belong to the line with the equation

$$y = \frac{-ac(b+d)}{bd(a+c)}x + \frac{2ac}{a+c}.$$

Applying Monge's circle theorem to every three of the four circles ω_m , ω_n , ω_p , ω_q , we have that ω_m , ω_n , ω_p , ω_q have the same axis of similitude, and we also know that ω_m , ω_n , ω_p , ω_q have one

common radical center O , so by Gergonnes's Solution to Apollonius' Problem, we obtain the existence of two circles Ω and ω that are both tangent to $\omega_m, \omega_n, \omega_p, \omega_q$.

4) From part 3), we see that we cannot choose two among the circles $\omega_m, \omega_n, \omega_p, \omega_q$ such that one circle contains another circle.

Let r_m, r_n, r_p, r_q be the radius of $\omega_m, \omega_n, \omega_p, \omega_q$. We solve the system of equations for (x, y) and r

$$\begin{cases} (x - x_{O_M})^2 + (y - y_{O_M})^2 = (r + r_m)^2 \\ (x - x_{O_N})^2 + (y - y_{O_N})^2 = (r + r_n)^2 \\ (x - x_{O_P})^2 + (y - y_{O_P})^2 = (r + r_p)^2 \\ (x - x_{O_Q})^2 + (y - y_{O_Q})^2 = (r + r_q)^2 \end{cases} \quad (1)$$

to find the center (x, y) and the radius r of the circle ω externally tangent to the four circles $\omega_m, \omega_n, \omega_p, \omega_q$. If we allow the case $r < 0$ we also get from the above system the center and the radius of Ω .

The equation of the line SO is

$$y = \frac{bd(a+c)}{ac(b+d)}x + \frac{(a+c)(tac + tbd - bd)}{2ac}.$$

Solving the system of equations (1) gives us that the centers of Ω and ω are

$$\left(x, \frac{bd(a+c)}{ac(b+d)}x + \frac{(a+c)(tac + tbd - bd)}{2ac} \right)$$

with

$$x = \frac{\alpha \pm \sqrt{\Delta}}{2\beta},$$

where we define α, β, Δ as follows

$$\begin{aligned} \alpha &= -4bd(\alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0) \\ \alpha_3 &= (a+c)^2(ac+bd)(a^2 c^2(b-d)^2 + b^2 d^2(a-c)^2) \\ \alpha_2 &= -3a^3 c^3(a^2 + c^2)(b-d)^2 - b^3 d^3(a^4 + c^4) \\ &\quad - 2a^2 c^2 [3a^2 c^2(b-d)^2 + bd(2(a^2 + c^2)(b^2 - bd + d^2) + 2ac(b^2 + d^2) - b^2 d^2 - 8acbd)] \\ \alpha_1 &= a^2 c^2(b-d)^2(2ac(b^2 + 4bd + d^2) + 3bd(a+c)^2) \\ \alpha_0 &= -4a^3 c^3(b-d)^2(ac+bd) \\ \beta &= \beta_2 t^2 + \beta_1 t + \beta_0 \\ \beta_2 &= 4(a^2 c^2(b-d)^2 + b^2 d^2(a-c)^2)(a^2 c^2(b+d)^2 + b^2 d^2(a+c)^2) \\ \beta_1 &= -8a^2 c^2(b-d)^2(a^2 c^2(b+d)^2 + b^2 d^2(a+c)^2) \\ \beta_0 &= 4a^3 c^3(b-d)^2(ac(b+d)^2 + 4b^2 d^2) \\ \Delta &= -64a^3 c^3 bd(ac+bd)(tab - tcd - bc + cd)(tab - tcd - ab + ad) \\ &\quad \cdot (tad - tbc + ab - ad)(tad - tbc + cb - cd)(\Delta_2 t^2 + \Delta_1 t + \Delta_0) \\ \Delta_2 &= a^2 c^2(b-d)^2 + b^2 d^2(a-c)^2 \\ \Delta_1 &= -2a^2 c^2(b-d)^2 \\ \Delta_0 &= a^2 c^2(b-d)^2 \end{aligned}$$

Therefore SO passes through the centers of the two circles Ω and ω . □

Problem 2 (Problem 2197 from *Mathematics Magazine* (Volume 97, 2024 - Issue 4)). Let $ABCD$ be a rectangle and let P be an arbitrary point in the same plane, distinct from

the center of $ABCD$. Let X, Y, Z, W, S, T be the orthogonal projections of P on the lines AB, BC, CD, DA, BD, AC , respectively. Let the perpendicular bisectors of XY and WZ intersect at Q , and let the perpendicular bisectors of YZ and XW intersect at R . Prove that the lines QR and ST are parallel or coincide.

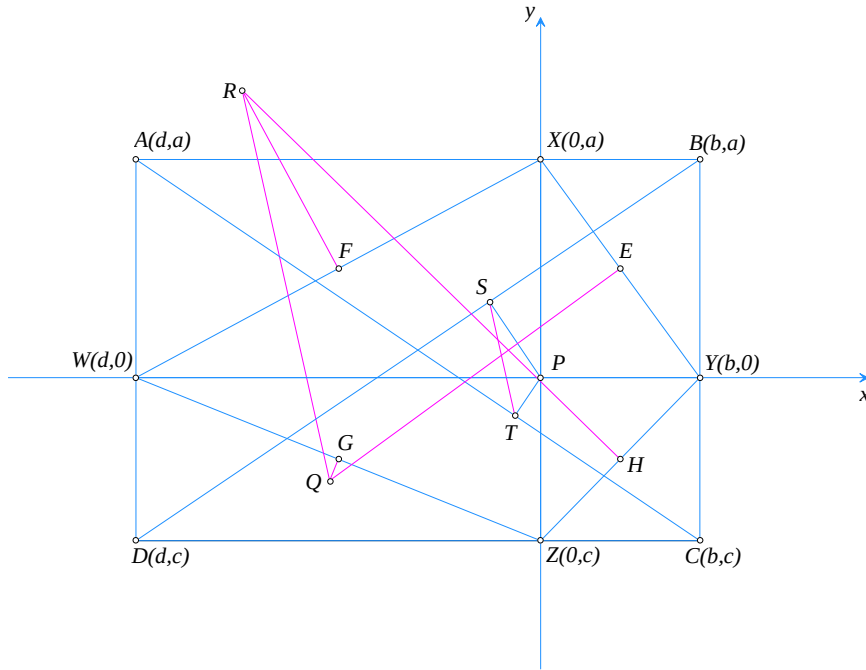


Figure 3: Illustration for the second example.

Proof. (See Figure 3.) Consider this configuration in a Cartesian plane, such that

$$P = (0, 0), \quad X = (0, a), \quad Y = (b, 0), \quad Z = (0, c), \quad W = (d, 0).$$

We then have

$$A = (d, a), \quad B = (b, a), \quad C = (b, c), \quad D = (d, c).$$

Now, we let E, F, G, H be the midpoints of XY, YZ, ZW, WX , respectively, therefore

$$E = \frac{X+Y}{2} = \left(\frac{b}{2}, \frac{a}{2}\right), \quad F = \frac{Y+Z}{2} = \left(\frac{b}{2}, \frac{c}{2}\right),$$

$$G = \frac{Z+W}{2} = \left(\frac{d}{2}, \frac{c}{2}\right), \quad H = \frac{W+X}{2} = \left(\frac{d}{2}, \frac{a}{2}\right).$$

The equations of the lines AC and BD are

$$AC : y = \frac{c-a}{b-d} \cdot x + \frac{ab-cd}{b-d}, \quad BD : y = \frac{c-a}{d-b} \cdot x + \frac{ad-bc}{d-b}.$$

Note that, since $SP \perp BD$ and $S \in BD$, we have

$$\begin{cases} \vec{PS} \cdot \vec{BD} = 0 \\ S \in BD \end{cases} \implies S = \left(\frac{(ad-bc)(a-c)}{(a-c)^2 + (b-d)^2}, \frac{(ad-bc)(d-b)}{(a-c)^2 + (b-d)^2} \right).$$

Using the same reason, we also have

$$T = \left(\frac{(ab-cd)(a-c)}{(a-c)^2 + (b-d)^2}, \frac{(ab-cd)(b-d)}{(a-c)^2 + (b-d)^2} \right),$$

so the equation of the line ST is

$$y = \frac{b+d}{a+c} \cdot x - \frac{2(ad-bc)(ab-cd)}{(a+c)((a-c)^2 + (b-d)^2)}.$$

From the fact that $RH \perp WX$ and $RF \perp YZ$, we have

$$\begin{cases} \overrightarrow{HR} \cdot \overrightarrow{WX} = 0 \\ \overrightarrow{FR} \cdot \overrightarrow{YZ} = 0 \end{cases} \implies R = \left(\frac{a(b^2 - c^2) + c(a^2 - d^2)}{2(ab - cd)}, \frac{b(a^2 - d^2) + d(b^2 - c^2)}{2(ab - cd)} \right).$$

A similar argument yields

$$Q = \left(\frac{a(d^2 - c^2) + c(a^2 - b^2)}{2(ad - bc)}, \frac{d(a^2 - b^2) + b(d^2 - c^2)}{2(ad - bc)} \right),$$

thus the equation of the line QR is

$$y = \frac{b+d}{a+c} \cdot x + \frac{a^2 + c^2 - b^2 - d^2}{2(a+c)}.$$

Because the two lines QR and ST have the same slope, we conclude that they are parallel or coincide, as desired. $\square$

Problem 3. Let $ABCD$ be a square, and let M and N be the midpoints of AB and AD , respectively. CM cuts BN at P . The circle ω passes through D and the two Fermat points of $\triangle APN$. The radical axis of ω and (APN) meets AD at X . Prove that $AD = 8NX$.

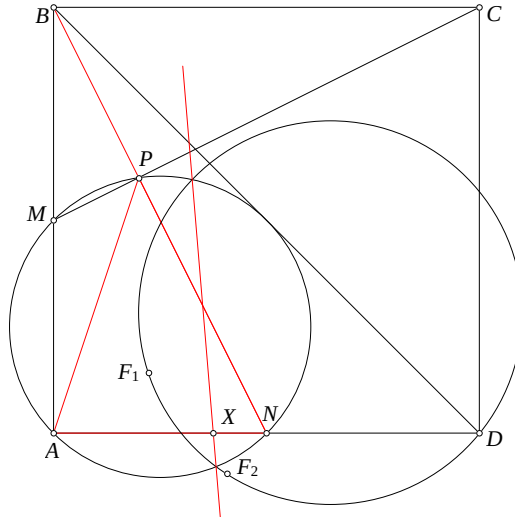


Figure 4: Illustration for the third example.

Proof. (See Figure 4.) Consider this configuration in a Cartesian plane such that

$$A = (0, 0), B = (0, 2), C = (2, 2), D = (2, 0),$$

hence we have

$$M = (0, 1), N = (1, 0)$$

and the equations of the lines BN and CM are

$$BN : y = -2x + 2, CM : y = \frac{1}{2}x + 1.$$

Since CM cuts BN at P , we get

$$P = \left(\frac{2}{5}, \frac{6}{5} \right).$$

Note that

$$PA^2 = \left(\frac{2}{5} \right)^2 + \left(\frac{6}{5} \right)^2 = \frac{8}{5}, \quad PN^2 = \left(\frac{3}{5} \right)^2 + \left(\frac{6}{5} \right)^2 = \frac{9}{5}, \quad AN^2 = 1, \quad S_{APN} = \frac{3}{5}.$$

Now let F_1 and F_2 be the two Fermat points of the triangle APN . Using the barycentric coordinates of Fermat points ($X(13)$ and $X(14)$) in the Encyclopedia of Triangle Centers [7], we obtain the coordinates of F_1 and F_2 as follows

$$F_1 = \frac{\alpha_1 P + \alpha_2 A + \alpha_3 N}{\alpha_1 + \alpha_2 + \alpha_3}, \quad F_2 = \frac{\alpha_4 P + \alpha_5 A + \alpha_6 N}{\alpha_4 + \alpha_5 + \alpha_6}$$

with

$$\begin{aligned} \alpha_1 &= AN^4 - 2(PA^2 - PN^2)^2 + AN^2(PA^2 + PN^2 + 4\sqrt{3}S_{APN}) = \frac{108 + 60\sqrt{3}}{25} \\ \alpha_2 &= PN^4 - 2(PA^2 - AN^2)^2 + PN^2(PA^2 + AN^2 + 4\sqrt{3}S_{APN}) = \frac{180 + 108\sqrt{3}}{25} \\ \alpha_3 &= PA^4 - 2(AN^2 - PN^2)^2 + PA^2(AN^2 + PN^2 + 4\sqrt{3}S_{APN}) = \frac{144 + 96\sqrt{3}}{25} \\ \alpha_4 &= AN^4 - 2(PA^2 - PN^2)^2 + AN^2(PA^2 + PN^2 - 4\sqrt{3}S_{APN}) = \frac{108 - 60\sqrt{3}}{25} \\ \alpha_5 &= PN^4 - 2(PA^2 - AN^2)^2 + PN^2(PA^2 + AN^2 - 4\sqrt{3}S_{APN}) = \frac{180 - 108\sqrt{3}}{25} \\ \alpha_6 &= PA^4 - 2(AN^2 - PN^2)^2 + PA^2(AN^2 + PN^2 - 4\sqrt{3}S_{APN}) = \frac{144 - 96\sqrt{3}}{25}. \end{aligned}$$

Thus we have

$$F_1 = \left(\frac{41 - 7\sqrt{3}}{65}, \frac{3 + 9\sqrt{3}}{65} \right), \quad F_2 = \left(\frac{41 + 7\sqrt{3}}{65}, \frac{3 - 9\sqrt{3}}{65} \right).$$

The equation of the circle (DF_1F_2) is

$$x^2 + y^2 - \frac{13}{5} \cdot x - \frac{17}{15} \cdot y + \frac{6}{5} = 0,$$

and the equation of the circle (APN) is

$$x^2 + y^2 - x - y = 0,$$

therefore the equation of the radical axis of the circles ω and (APN) is

$$-\frac{8}{5} \cdot x - \frac{2}{15} \cdot y + \frac{6}{5} = 0.$$

We conclude that

$$X = \left(\frac{3}{4}, 0 \right),$$

which immediately implies $AD = 8NX$, as desired. □

Problem 4 (IMO 2023 Problem 6). Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ$$

Let BC_1 and CB_1 meet at A_2 , let CA_1 and AC_1 meet at B_2 , and let AB_1 and BA_1 meet at C_2 . Prove that if triangle $A_1B_1C_1$ is scalene, then the Euler lines of the triangles AB_2C_2 , BC_2A_2 , and CA_2B_2 are concurrent.

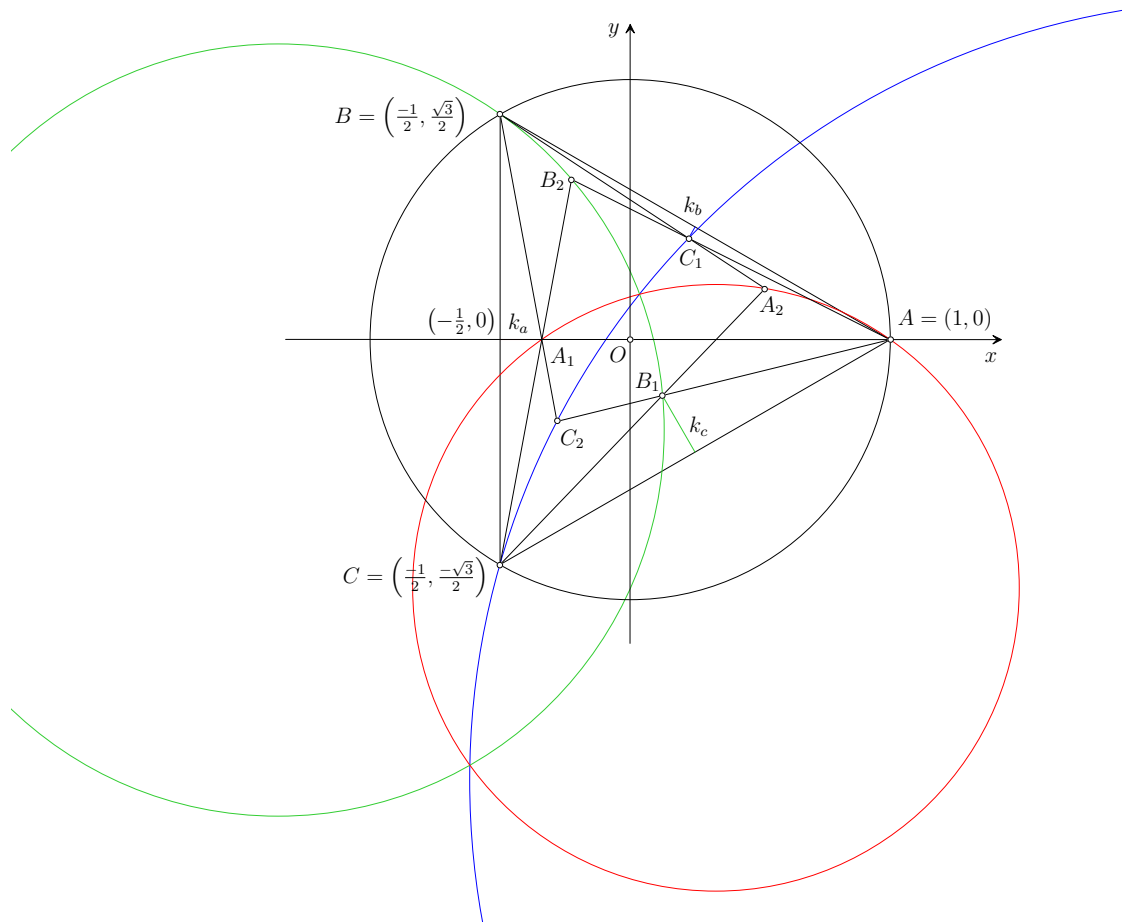


Figure 5: Illustration for the fourth example.

Proof. (See Figure 5.) Consider this configuration in a Cartesian plane such that

$$A = (1, 0), \quad B = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad C = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Let k_a, k_b and k_c be the distances from A_1, B_1 and C_1 to the lines BC, CA , and AB respectively. Since A_1, B_1, C_1 are interior points of triangle ABC and $\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ$, we get the relation

$$k_a = \frac{3 - 4k_bk_c - 6(k_b + k_c)}{2(3 - 4k_bk_c + 2(k_b + k_c))}.$$

Thus we can assume that

$$A_1 = \left(\frac{3 - 4k_bk_c - 6(k_b + k_c)}{2(3 - 4k_bk_c + 2(k_b + k_c))} - \frac{1}{2}, 0 \right) = \left(-\frac{4k_b + 4k_c}{4k_bk_c - 2k_b - 2k_c - 3}, 0 \right),$$

$$B_1 = \left(\frac{-2k_c + 1}{4}, \frac{\sqrt{3}(2k_c - 1)}{4} \right),$$

and

$$C_1 = \left(\frac{-2k_b + 1}{4}, \frac{\sqrt{3}(-2k_b + 1)}{4} \right).$$

The intersection A_2 of the lines BC_1 and CB_1 is

$$A_2 = \left(\frac{-4k_b k_c + 4k_b + 4k_c - 3}{4k_b k_c - 2k_b - 2k_c - 3}, \frac{2\sqrt{3}(k_b - k_c)}{4k_b k_c - 2k_b - 2k_c - 3} \right).$$

The intersection B_2 of the lines CA_1 and AC_1 is

$$B_2 = \left(\frac{8k_b^2 k_c + 4k_b^2 - 24k_b - 18k_c + 3}{16k_b^2 + 12}, \frac{\sqrt{3}(8k_b^2 k_c - 12k_b^2 - 16k_b k_c + 6k_c + 3)}{16k_b^2 + 12} \right).$$

The intersection C_2 of the lines AB_1 and BA_1 is

$$C_2 = \left(\frac{8k_b k_c^2 - 18k_b + 4k_c^2 - 24k_c + 3}{16k_c^2 + 12}, \frac{\sqrt{3}(-8k_b k_c^2 + 16k_b k_c - 6k_b + 12k_c^2 - 3)}{16k_c^2 + 12} \right).$$

The circumcenter O_a of the triangle AB_2C_2 is the intersection of the perpendicular bisectors of the segments AB_2 and AC_2 , therefore:

$$O_a = \left(\frac{4k_b^2 k_c + 4k_b k_c^2 - 6k_b^2 - 8k_b k_c - 6k_c^2 + 7k_b + 7k_c - 3}{2(4k_b k_c + 2k_b + 2k_c - 3)}, \frac{-\sqrt{3}(4k_b^2 k_c - 4k_b k_c^2 - 6k_b^2 + 6k_c^2 - 3k_b + 3k_c)}{6(4k_b k_c + 2k_b + 2k_c - 3)} \right).$$

Similarly, the circumcenter O_b of triangle BC_2A_2 is the intersection of the perpendicular bisectors of the segments BC_2 and BA_2 :

$$O_b = \left(\frac{4k_b k_c - 6k_b - 6k_c + 1}{4(2k_c + 1)}, \frac{-\sqrt{3}(16k_b^2 k_c^2 - 32k_b^2 k_c - 64k_b k_c^2 + 12k_b^2 + 48k_b k_c + 60k_c^2 - 24k_b + 9)}{12(8k_b k_c^2 - 4k_c^2 - 2k_b - 8k_c - 3)} \right),$$

and the circumcenter O_c of the triangle CA_2B_2 is the intersection of the perpendicular bisectors of the segments CA_2 and CB_2 :

$$O_c = \left(\frac{4k_b k_c - 6k_b - 6k_c + 1}{4(2k_b + 1)}, \frac{\sqrt{3}(16k_b^2 k_c^2 - 64k_b^2 k_c - 32k_b k_c^2 + 60k_b^2 + 48k_b k_c + 12k_c^2 - 24k_c + 9)}{12(8k_b k_c^2 - 4k_c^2 - 2k_b - 8k_c - 3)} \right).$$

The coordinates of the centroid G_a of triangle AB_2C_2 are

$$G_a = \left(\frac{(4k_b k_c + 6k_b + 6k_c - 3)(4k_b^2 k_c + 4k_b k_c^2 - 6k_b^2 - 6k_c^2 + 3k_b + 3k_c - 9)}{6(4k_b^2 + 3)(4k_c^2 + 3)}, \right. \\ \left. \frac{-\sqrt{3}(4k_b k_c - 2k_b - 2k_c - 3)(4k_b k_c - 6k_b - 6k_c - 3)(k_b - k_c)}{6(4k_b^2 + 3)(4k_c^2 + 3)} \right),$$

the coordinates of the centroid G_b of the triangle BC_2A_2 are

$$G_b = \left(\frac{32k_b^2 k_c^3 - 16k_b^2 k_c^2 - 96k_b k_c^3 - 72k_b^2 k_c - 48k_b k_c^2 + 72k_c^3 + 36k_b^2 + 24k_b k_c + 12k_c^2 + 108k_b + 126k_c - 27}{12(4k_c^2 + 3)(4k_b k_c - 2k_b - 2k_c - 3)}, \right.$$

$$\frac{-\sqrt{3}(32k_b^2k_c^3 - 80k_b^2k_c^2 - 96k_bk_c^3 + 56k_b^2k_c + 16k_bk_c^2 + 72k_c^3 - 12k_b^2 + 24k_bk_c + 60k_c^2 - 36k_b + 30k_c + 9)}{12(4k_c^2 + 3)(4k_bk_c - 2k_b - 2k_c - 3)} \Bigg),$$

and the coordinates of the centroid G_c of the triangle CA_2B_2 are

$$G_c = \left(\frac{32k_b^3k_c^2 - 96k_b^3k_c - 16k_b^2k_c^2 + 72k_b^3 - 48k_b^2k_c - 72k_bk_c^2 + 12k_b^2 + 24k_bk_c + 36k_c^2 + 126k_b + 108k_c - 27}{12(4k_bk_c - 2k_b - 2k_c - 3)(4k_b^2 + 3)}, \right. \\ \left. \frac{\sqrt{3}(32k_b^3k_c^2 - 96k_b^3k_c - 80k_b^2k_c^2 + 72k_b^3 + 16k_b^2k_c + 56k_bk_c^2 + 60k_b^2 + 24k_bk_c - 12k_c^2 + 30k_b - 36k_c + 9)}{12(4k_bk_c - 2k_b - 2k_c - 3)(4k_b^2 + 3)} \right).$$

Consequently, the equations of the Euler lines of the triangles AB_2C_2 , BC_2A_2 and CA_2B_2 are:

$$O_aG_a : u_1x + v_1y + 1 = 0, \quad O_bG_b : u_2x + v_2y + 1 = 0, \quad O_cG_c : u_3x + v_3y + 1 = 0$$

with

$$u_1 = \frac{48(k_b + k_c)}{16k_b^2k_c^2 - 48k_b^2k_c - 48k_bk_c^2 + 36k_b^2 + 48k_bk_c + 36k_c^2 - 12k_b - 12k_c + 9}$$

$$u_2 = \frac{12(4k_bk_c - 2k_b - 2k_c - 3)(-8k_bk_c^2 + 8k_b^2 + 16k_bk_c + 12k_c^2 - 6k_b + 3)}{(8k_b^2k_c - 4k_b^2 - 8k_bk_c - 12k_b - 6k_c + 3)(16k_b^2k_c^2 - 48k_b^2k_c - 48k_bk_c^2 + 36k_b^2 + 48k_bk_c + 36k_c^2 - 12k_b - 12k_c + 9)}$$

$$u_3 = \frac{-12(4k_bk_c - 2k_b - 2k_c - 3)(8k_b^2k_c - 12k_b^2 - 16k_bk_c - 8k_c^2 + 6k_c - 3)}{(8k_bk_c^2 - 8k_bk_c - 4k_c^2 - 6k_b - 12k_c + 3)(16k_b^2k_c^2 - 48k_b^2k_c - 48k_bk_c^2 + 36k_b^2 + 48k_bk_c + 36k_c^2 - 12k_b - 12k_c + 9)}$$

$$v_1 = \frac{2\sqrt{3}(16k_b^2k_c^2 - 16k_b^2k_c - 16k_bk_c^2 + 12k_b^2 + 12k_c^2 - 12k_b - 12k_c + 9)}{(16k_b^2k_c^2 - 48k_b^2k_c - 48k_bk_c^2 + 36k_b^2 + 48k_bk_c + 36k_c^2 - 12k_b - 12k_c + 9)(k_b - k_c)}$$

$$v_2 = \frac{8\sqrt{3}(8k_b^2k_c^2 - 8k_b^2k_c - 12k_bk_c^2 + 6k_b^2 + 3k_b + 6k_c)(4k_bk_c - 2k_b - 2k_c - 3)}{(8k_b^2k_c - 4k_b^2 - 8k_bk_c - 12k_b - 6k_c + 3)(16k_b^2k_c^2 - 48k_b^2k_c - 48k_bk_c^2 + 36k_b^2 + 48k_bk_c + 36k_c^2 - 12k_b - 12k_c + 9)}$$

$$v_3 = \frac{-8\sqrt{3}(8k_b^2k_c^2 - 12k_b^2k_c - 8k_bk_c^2 + 6k_c^2 + 6k_b + 3k_c)(4k_bk_c - 2k_b - 2k_c - 3)}{(8k_bk_c^2 - 8k_bk_c - 4k_c^2 - 6k_b - 12k_c + 3)(16k_b^2k_c^2 - 48k_b^2k_c - 48k_bk_c^2 + 36k_b^2 + 48k_bk_c + 36k_c^2 - 12k_b - 12k_c + 9)}$$

Now, since we have

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{pmatrix} = 0,$$

the conclusion in the statement of the problem follows. $\square$

Problem 5 (IMO 2009 Problem 2). Let ABC be a triangle with circumcenter O . The points P and Q are interior points of the sides CA and AB respectively. Let K , L and M be the midpoints of the segments BP , CQ and PQ respectively, and let Γ be the circle passing through K , L , and M . Suppose that the line PQ is tangent to the circle Γ . Prove that $OP = OQ$.

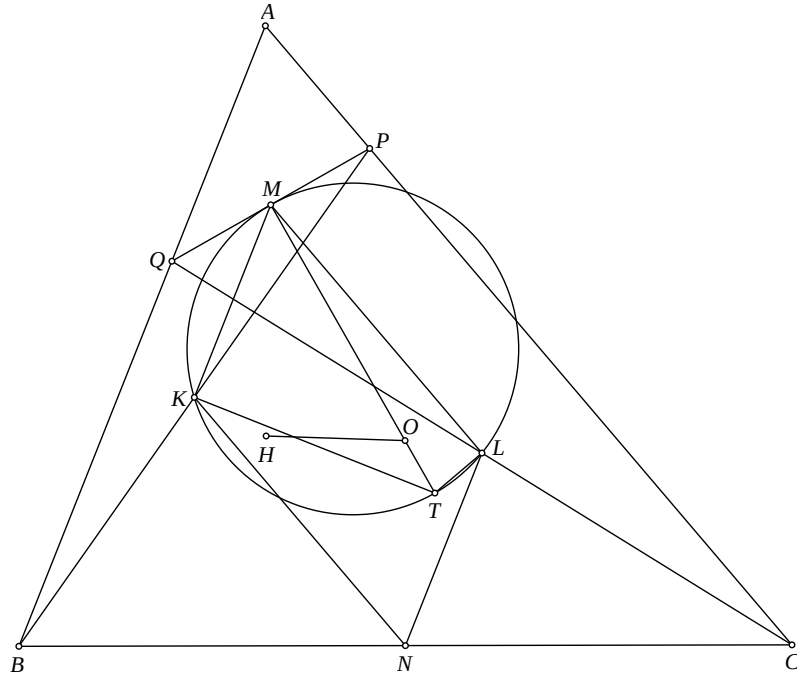


Figure 6: Illustration for the fifth example.

Proof. (See Figure 6.) Note that PQ is tangent to Γ at M (as M belongs to both PQ and Γ). Consider this configuration in a Cartesian plane and let

$$A = (0, a), \quad B = (b, 0), \quad C = (c, 0).$$

Let H be the orthocenter of the $\triangle ABC$, and let N be the midpoint of BC , so that

$$H = \left(0, -\frac{bc}{a}\right), \quad N = \left(\frac{b+c}{2}, 0\right).$$

Since $\overrightarrow{AH} = 2 \cdot \overrightarrow{ON}$, we have

$$H - A = 2 \cdot (N - O) \implies O = \frac{A - H + 2N}{2} = \left(\frac{b+c}{2}, \frac{a^2 + bc}{2a}\right).$$

The equations of the lines AB and AC are:

$$AB : ax + by - ab = 0, \quad AC : ax + cy - ac = 0,$$

so we have

$$P = (cd, a - ad), \quad Q = (be, a - ae)$$

for some real numbers d and e . Since K , L , M are the midpoints of the segments BP , CQ and PQ , we have

$$K = \left(\frac{b+cd}{2}, \frac{a(1-d)}{2}\right), \quad L = \left(\frac{be+c}{2}, \frac{a(1-e)}{2}\right), \quad M = \left(\frac{be+cd}{2}, \frac{a(2-d-e)}{2}\right).$$

Let MT be the diameter (passing through M) of the circle (KLM) , hence $KT \perp KM$ and $LT \perp LM$.

Since $KM \parallel AB$ and $LM \parallel AC$, we have

$$\begin{cases} \overrightarrow{KT} \cdot \overrightarrow{AB} = 0, \\ \overrightarrow{LT} \cdot \overrightarrow{AC} = 0 \end{cases} \implies T = \left(\frac{(a^2 + bc)(d - e) + b^2 - c^2}{2(b - c)}, \frac{(a^2 + bc)(cd - be) + (b - c)(a^2 + bc)}{2a(b - c)} \right)$$

Observe that

$$\overrightarrow{TM} \cdot \overrightarrow{PQ} = \overrightarrow{OM} \cdot \overrightarrow{PQ} = \frac{(a^2 + b^2)(e^2 - e) - (a^2 + c^2)(d^2 - d)}{2}.$$

Since the line PQ is tangent to the circle Γ at M , we have $TM \perp PQ$, thus

$$\overrightarrow{OM} \cdot \overrightarrow{PQ} = \overrightarrow{TM} \cdot \overrightarrow{PQ} = 0 \implies OM \perp PQ,$$

and because M is the midpoint of PQ the conclusion $OP = OQ$ follows. $\square$

Problem 6 (A generalization of IMO 2012 Problem 5). Let two circles (O_1) and (O_2) intersect at A, B . The points C, D are on the line O_1O_2 such that $AC \perp O_1A$ and $AD \perp O_2A$, and P is a point on AB . CP cuts (O_1) at L such that C, L are not on the same side of AB . DP cuts (O_2) at K such that D, K are not on the same side of AB . LO_1 intersects KO_2 at M . Prove that $MK = ML$.

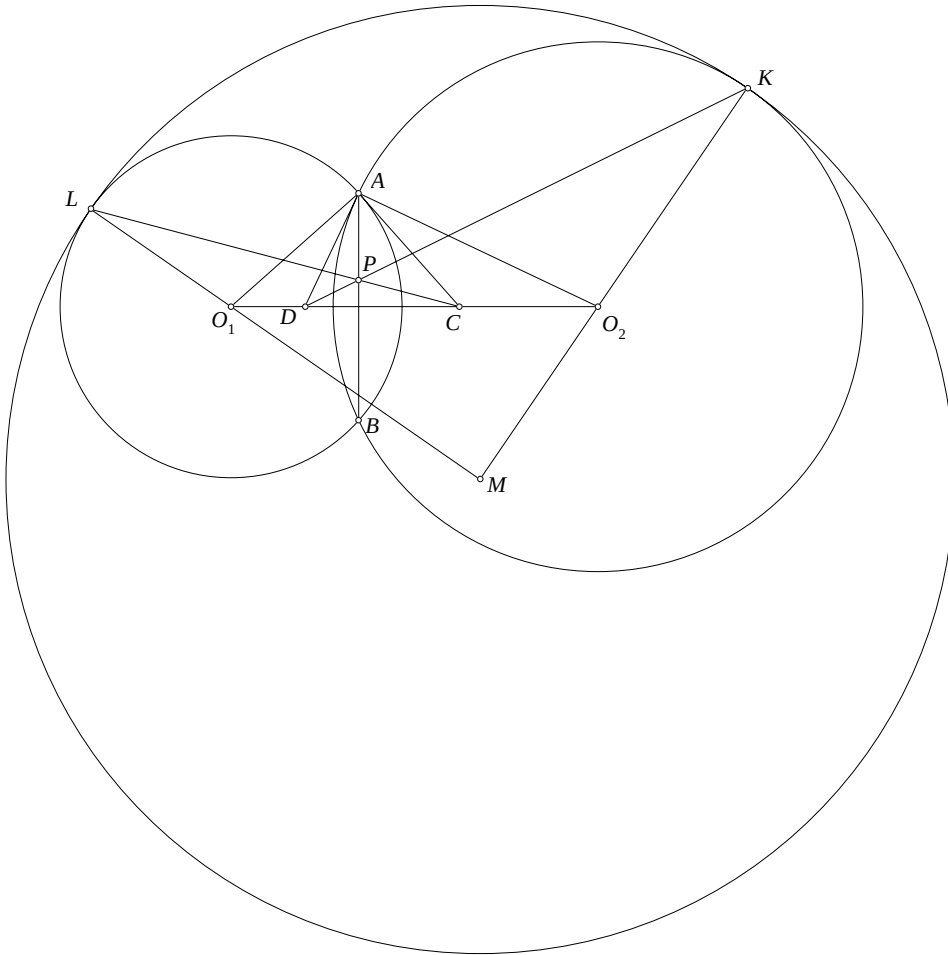


Figure 7: Illustration for the sixth example.

Proof. (See Figure 7.) Consider this configuration in a Cartesian plane so that we have

$$A = (0, a), \quad B = (0, -a), \quad O_1 = (z, 0), \quad O_2 = (t, 0), \quad P = (0, p), \quad a > 0.$$

The equations of the lines AB and O_1O_2 are

$$AB : x = 0, \quad O_1O_2 : y = 0$$

and we have

$$C = \left(-\frac{a^2}{z}, 0\right), \quad D = \left(-\frac{a^2}{t}, 0\right),$$

hence the equations of the lines CP and DP are

$$CP : pzx - a^2y + a^2p = 0, \quad DP : ptx - a^2y + a^2p = 0.$$

The equations of the circles (O_1) and (O_2) are

$$(O_1) : x^2 + y^2 - 2xz - a^2 = 0, \quad (O_2) : x^2 + y^2 - 2xt - a^2 = 0.$$

Then the coordinates of the points K and L are

$$K = (\lambda_1 a^2, pt\lambda_1 + p), \quad L = (\lambda_2 a^2, pz\lambda_2 + p)$$

with

$$(a^4 + p^2 t^2) \cdot \lambda_1^2 - 2t(a^2 - p^2) \cdot \lambda_1 - (a^2 - p^2) = 0,$$

and

$$(a^4 + p^2 z^2) \cdot \lambda_2^2 - 2z(a^2 - p^2) \cdot \lambda_2 - (a^2 - p^2) = 0.$$

Thus the intersection of the lines LO_1 and KO_2 is

$$M = \left(\frac{a^2 \lambda_1 \lambda_2 (t^2 - z^2) + a^2 (\lambda_2 t - \lambda_1 z) + \lambda_2 t z^2 - \lambda_1 t^2 z}{a^2 \lambda_1 \lambda_2 (t - z) - (a^2 + zt)(\lambda_1 - \lambda_2)}, \frac{(\lambda_1 t + 1)(\lambda_2 z + 1)(t - z)p}{a^2 \lambda_1 \lambda_2 (t - z) - (a^2 + zt)(\lambda_1 - \lambda_2)} \right).$$

We then have

$$MK^2 = \frac{(a^4 \lambda_1^2 + \lambda_1^2 p^2 t^2 - 2a^2 \lambda_1 t + 2\lambda_1 p^2 t + p^2 + t^2)(a^2 \lambda_1 \lambda_2 t - a^2 \lambda_1 \lambda_2 z - a^2 \lambda_1 + a^2 \lambda_2 - \lambda_1 t z + \lambda_2 z^2)^2}{(a^2 \lambda_1 \lambda_2 t - a^2 \lambda_1 \lambda_2 z - a^2 \lambda_1 + a^2 \lambda_2 - \lambda_1 t z + \lambda_2 t z + t - z)^2}$$

and

$$ML^2 = \frac{(a^4 \lambda_2^2 + \lambda_2^2 p^2 z^2 - 2a^2 \lambda_2 z + 2\lambda_2 p^2 z + p^2 + z^2)(a^2 \lambda_1 \lambda_2 t - a^2 \lambda_1 \lambda_2 z - a^2 \lambda_1 + a^2 \lambda_2 - \lambda_1 t^2 + \lambda_2 t z)^2}{(a^2 \lambda_1 \lambda_2 t - a^2 \lambda_1 \lambda_2 z - a^2 \lambda_1 + a^2 \lambda_2 - \lambda_1 t z + \lambda_2 t z + t - z)^2}$$

Note that

$$a^4 \lambda_1^2 + \lambda_1^2 p^2 t^2 - 2a^2 \lambda_1 t + 2\lambda_1 p^2 t + p^2 + t^2 = a^2 + t^2$$

and

$$a^4 \lambda_2^2 + \lambda_2^2 p^2 z^2 - 2a^2 \lambda_2 z + 2\lambda_2 p^2 z + p^2 + z^2 = a^2 + z^2,$$

therefore

$$MK^2 - ML^2 = \frac{a^2(t - z)^2}{(a^2 \lambda_1 \lambda_2 t - a^2 \lambda_1 \lambda_2 z - a^2 \lambda_1 + a^2 \lambda_2 - \lambda_1 t z + \lambda_2 t z + t - z)^2} \cdot \lambda_3$$

with

$$\lambda_3 = a^2 \lambda_1^2 \lambda_2^2 t^2 - a^2 \lambda_1^2 \lambda_2^2 z^2 - 2a^2 \lambda_1^2 \lambda_2 z + 2a^2 \lambda_1 \lambda_2^2 t - 2\lambda_1^2 \lambda_2 t^2 z + 2\lambda_1 \lambda_2^2 t z^2 - a^2 \lambda_1^2 + a^2 \lambda_2^2 - \lambda_1^2 t^2 + \lambda_2^2 z^2.$$

Since

$$\begin{aligned} \lambda_3 &= a^2 \lambda_1^2 \lambda_2^2 t^2 - a^2 \lambda_1^2 \lambda_2^2 z^2 - 2a^2 \lambda_1^2 \lambda_2 z + 2a^2 \lambda_1 \lambda_2^2 t - 2\lambda_1^2 \lambda_2 t^2 z + 2\lambda_1 \lambda_2^2 t z^2 - a^2 \lambda_1^2 + a^2 \lambda_2^2 - \lambda_1^2 t^2 + \lambda_2^2 z^2 \\ &= (a^2 \lambda_2^2 t^2 - a^2 \lambda_2^2 z^2 - 2a^2 \lambda_2 z - 2\lambda_2 t^2 z - a^2 - t^2) \lambda_1^2 + (2a^2 \lambda_2^2 t + 2\lambda_2^2 t z^2) \lambda_1 + a^2 \lambda_2^2 + \lambda_2^2 z^2 \\ &= \frac{(a^2 + t^2)(a^4 \lambda_2^2 + \lambda_2^2 p^2 z^2 - 2a^2 \lambda_2 z + 2\lambda_2 p^2 z - a^2 + p^2)(2\lambda_1 t + 1)}{a^4 + p^2 t^2} = 0 \end{aligned}$$

(because $(a^4 + p^2 z^2) \cdot \lambda_2^2 - 2z(a^2 - p^2) \cdot \lambda_2 - (a^2 - p^2) = 0$), we get the desired conclusion. $\square$

Problem 7. Let ABC be a triangle with altitude AD ($D \in BC$), circumcenter O , and let P be a point on AD . Let E and F be the orthogonal projections of P on CA and AB , respectively. Let the line DE intersect AB at M , and let DF intersect AC at N . Let Q be the midpoint of MN . The line through P perpendicular to MN intersects AQ and AO at points T and S , respectively. Let R be the midpoint of PS . Prove that the orthocenter of the triangle ART lies on the line EF .

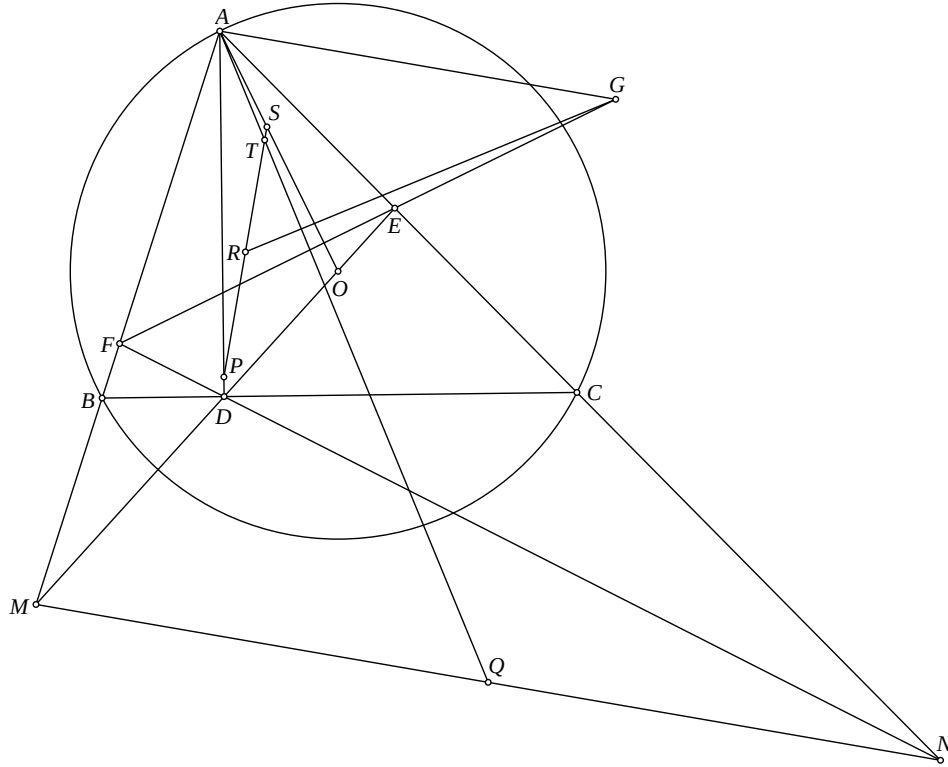


Figure 8: Illustration for the seventh example.

Proof. (See Figure 8.) Consider this configuration in the Cartesian plane such that

$$D = (0, 0), \quad A = (0, a), \quad B = (b, 0), \quad C = (c, 0), \quad P = (0, p).$$

The equations of the lines AB and AC are

$$AB : y = \frac{-a}{b}x + a, \quad AC : y = \frac{-a}{c}x + a.$$

We have

$$O = \left(\frac{b+c}{2}, \frac{a^2+bc}{2a} \right), \quad E = \left(\frac{ac(a-p)}{a^2+c^2}, \frac{a(ap+c^2)}{a^2+c^2} \right), \quad F = \left(\frac{ab(a-p)}{a^2+b^2}, \frac{a(ap+b^2)}{a^2+b^2} \right).$$

Since AB meets DE at M , we have

$$M = \left(\frac{abc(a-p)}{a^2c+abp-acp+bc^2}, \frac{ab(ap+c^2)}{a^2c+abp-acp+bc^2} \right).$$

Since AC meets DF at N , we have

$$N = \left(\frac{abc(a-p)}{a^2b-abp+acp+b^2c}, \frac{ac(ap+b^2)}{a^2b-abp+acp+b^2c} \right).$$

Since Q is the midpoint of segment MN , we have

$$Q = \frac{M + N}{2} = \left(\frac{acb(a-p)(b+c)(a^2+bc)}{2(a^2c+abp-acp+b^2c^2)(a^2b-abp+acp+b^2c)}, \right. \\ \left. \frac{a(a^3b^2p+a^3c^2p+2a^2b^2c^2-a^2b^2p^2+2a^2bcp^2-a^2c^2p^2+2ab^3cp-2ab^2c^2p+2abc^3p+2b^3c^3)}{2(a^2c+abp-acp+b^2c^2)(a^2b-abp+acp+b^2c)} \right).$$

Now, if X is the midpoint of AP , then we have $XE = XF = XA$. Note that $BCEF$ is a cyclic quadrilateral, so $AO \perp EF$. Because XR is the midline opposite P of the triangle PAS , we have that $XR \parallel AS$, that is, $XR \parallel AO$, and since $AO \perp EF$, we get $XR \perp EF$. Combining with the fact that $XE = XF$, we see that R lies on the perpendicular bisector of EF , so

$$R = \left(\frac{(b+c)(a-p)a^2p}{2(a^2+bc)(ap-bc)}, -\frac{a^3bc-2a^3p^2-a^2bcp+ab^2c^2+b^2c^2p}{2(a^2+bc)(ap-bc)} \right).$$

Let G be the orthocenter of the triangle ART , then $AG \parallel MN$ and $RG \perp AQ$, thus

$$G = \left(-\frac{abc(a^3-3a^2p+abc+2ap^2-bcp)}{(a^3p-a^2bc+abcp-c^2b^2)(b+c)}, -\frac{a(a^2bc-a^2p^2-abcp+c^2b^2)}{a^3p-a^2bc+abcp-c^2b^2} \right)$$

Since $\overrightarrow{GE} \cdot \overrightarrow{AO} = 0$ and $\overrightarrow{EF} \cdot \overrightarrow{AO} = 0$, we have that both GE and EF are perpendicular to AO , so G lies on the line EF . $\square$

Problem 8 (IMO 2024 Problem 4). Let ABC be a triangle with $AB < AC < BC$. Let the incenter and incircle of triangle ABC be I and ω , respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω . Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω . Let AI intersect the circumcircle of triangle ABC at $P \neq A$. Let K and L be the midpoints of AC and AB , respectively. Prove that $\angle KIL + \angle XPY = 180^\circ$.

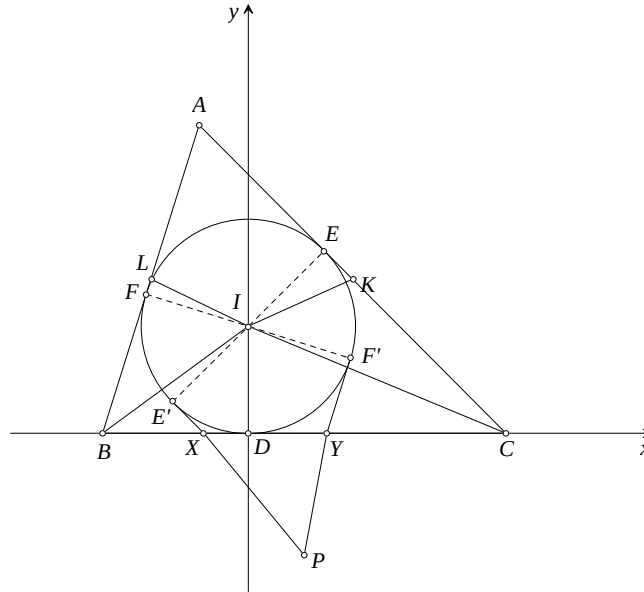


Figure 9: Illustration for the eighth example.

Proof. (See Figure 9.) Suppose the incircle ω touches BC, CA, AB at D, E, F respectively. Let E' and F' be the reflections with respect to I of E and F respectively.

Consider a Cartesian coordinate system such that $D(0,0)$, $B(-b,0)$, $C(c,0)$ ($b, c > 0$), and $I(0,1)$. Then E and F are the reflections of D with respect to IC and IB respectively. We obtain

$$E = \left(\frac{2c}{c^2 + 1}, \frac{2c^2}{c^2 + 1} \right)$$

and

$$F = \left(\frac{-2b}{b^2 + 1}, \frac{2b^2}{b^2 + 1} \right).$$

Since E' and F' are the reflections with respect to I of E and F respectively, we obtain

$$E' = \left(\frac{-2c}{c^2 + 1}, \frac{2}{c^2 + 1} \right)$$

and

$$F' = \left(\frac{2b}{b^2 + 1}, \frac{2}{b^2 + 1} \right).$$

Then X is the intersection of BC and the line through E' parallel to EC , and Y is the intersection of BC and the line through F' parallel to FB , so we get

$$X = \left(\frac{-1}{c}, 0 \right)$$

and

$$Y = \left(\frac{1}{b}, 0 \right).$$

We have that A is the intersection of the lines CE and BF , thus

$$A = \left(\frac{b - c}{bc - 1}, \frac{2bc}{bc - 1} \right).$$

Because K and L are the midpoints of CA and AB respectively, we have

$$K = \left(\frac{bc^2 + b - 2c}{2bc - 2}, \frac{bc}{bc - 1} \right)$$

and

$$L = \left(\frac{-b^2c + 2b - c}{2bc - 2}, \frac{bc}{bc - 1} \right).$$

As P is the circumcenter of the triangle IBC that is, the intersection of the perpendicular bisectors of BC and IB , we get

$$P = \left(\frac{c - b}{2}, \frac{1 - bc}{2} \right).$$

Then we have the equations of the lines

$$PX : y = \frac{bc^2 - c}{-c^2 + bc - 2}x + \frac{bc - 1}{-c^2 + bc - 2},$$

$$PY : y = \frac{b^2c - b}{b^2 - bc + 2}x + \frac{-bc + 1}{b^2 - bc + 2},$$

$$IK : y = \frac{2}{bc^2 + b - 2c}x + 1$$

and

$$IL : y = \frac{-2}{b^2c - 2b + c}x + 1.$$

We obtain

$$\tan \angle(PX, PY) = \frac{\frac{bc^2-c}{-c^2+bc-2} - \frac{b^2c-b}{b^2-bc+2}}{1 + \frac{bc^2-c}{-c^2+bc-2} \cdot \frac{b^2c-b}{b^2-bc+2}} = \frac{-2(b+c)(bc-1)}{b^3c^3 + b^3c - 4b^2c^2 - 2b^2 + bc^3 + 5bc - 2c^2 - 4}$$

and

$$\tan \angle(IK, IL) = \frac{\frac{2}{bc^2+b-2c} - \frac{-2}{b^2c-2b+c}}{1 + \frac{2}{bc^2+b-2c} \cdot \frac{-2}{b^2c-2b+c}} = \frac{2(b+c)(bc-1)}{b^3c^3 + b^3c - 4b^2c^2 - 2b^2 + bc^3 + 5bc - 2c^2 - 4},$$

therefore,

$$\tan \angle KIL = -\tan \angle XPY$$

meaning that

$$\angle KIL + \angle XPY = 180^\circ,$$

as desired. □

3 Conclusion

Analytic geometry, pioneered by René Descartes in 1637 through his work *La Géométrie*, established a powerful bridge between algebra and geometry. By representing points, lines, and curves via algebraic equations in a Cartesian plane, this approach transformed geometric reasoning from intuitive constructions into systematic computation.

The impact of analytic geometry extends far beyond its original scope: it laid the groundwork for calculus, differential geometry, and modern physics, and it remains an essential tool in mathematical research and education. In particular, for problems involving loci, conic sections, extremal configurations, or spatial geometry, the analytic method often provides the most direct and illuminating path to a solution.

We tried above to illustrate these advantages through selected examples, demonstrating how coordinate techniques yield clear and concise proofs of nontrivial geometric facts.

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