

ON HUYGENS'S TRIGONOMETRIC INEQUALITY AND SOME RELATED RESULTS

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1. Introduction The inequalities

$$\sin x < x < \tan x \quad (1)$$

for all $x \in \left(0, \frac{\pi}{2}\right)$ are well-known. They come from the very definition of the trigonometric functions sine and tangent on the unit circle. Note that they can also be written as

$$\cos x < \frac{\sin x}{x} < 1 \quad (1')$$

and, in this form, they hold for any x with $0 < |x| < \frac{\pi}{2}$. By the squeeze theorem they imply the remarkable limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

which is important for computing the derivatives of all the trigonometric functions – it would be nonsensical to try prove these inequalities by using calculus (we would be stuck in a loop of circular reasoning). Nevertheless, in what follows, we will use calculus in order to prove some related but more complicated inequalities. The first one is

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \quad \text{for } 0 < x < \frac{\pi}{2} \quad (A)$$

which is also known (but maybe not so well-known), and which can be written in various equivalent forms, such as

$$\sin x \tan x > x^2 \Leftrightarrow \sin^2 x > x^2 \cos x \Leftrightarrow \sin x > x\sqrt{\cos x}, \quad 0 < x < \frac{\pi}{2}.$$

We will also prove the following improvement of the last form of (A) (actually we will refer to any of these equivalent forms as (A)):

$$\sin x > x\sqrt[3]{\cos x}, \quad \text{for } 0 < x < \frac{\pi}{2} \quad (B)$$

and other inequalities in the same vein (from which the most known is Huygens's inequality mentioned in the title) either using or avoiding calculus if possible. These inequalities (and others) can be found (with different proofs) in Roger Nelsen's article [1] which also provides more interesting references for such problems.

2. The inequalities We start by proving (in two ways) the inequality (A). See [2] for more proofs of (A).

Proposition 1. The following inequality

$$\frac{\tan x}{x} > \frac{x}{\sin x}$$

is true for all x with $0 < |x| < \frac{\pi}{2}$.

First proof. Since both functions from the left-hand side and the right-hand side of the inequality are even, it is enough to prove it for $0 < x < \frac{\pi}{2}$, so it suffices to show that $\sin x \tan x > x^2$ for $0 < x < \frac{\pi}{2}$. Let f be the function defined by

$$f(x) = \sin x \tan x - x^2 \quad \text{for } 0 \leq x < \frac{\pi}{2}.$$

Clearly, f is indefinitely differentiable and its first three derivatives are

$$f'(x) = 2 \sin x + \sin x \tan^2 x - 2x,$$

$$f''(x) = 2 \cos x + 3 \cos x \tan^2 x + 2 \sin x \tan^3 x - 2,$$

and

$$f'''(x) = 4 \sin x + 11 \sin x \tan^2 x + 6 \sin x \tan^2 x$$

for all $x \in \left[0, \frac{\pi}{2}\right)$. (In all the above computations we used the derivative of $\tan x$ in the form $1 + \tan^2 x$, and the obvious fact that $\cos x \tan x = \sin x$.) Clearly, the third derivative is positive on $\left(0, \frac{\pi}{2}\right)$ (and is 0 for $x = 0$), hence the second derivative increases on $\left[0, \frac{\pi}{2}\right)$. It follows that $f''(x) \geq f''(0) = 0$ (with equality only at $x = 0$), therefore f' also increases on $\left[0, \frac{\pi}{2}\right)$. Thus $f'(x) \geq f'(0) = 0$ for all $x \in \left[0, \frac{\pi}{2}\right)$, with equality at 0, which leads to the fact that f also increases on $\left[0, \frac{\pi}{2}\right)$, and, finally, to the desired inequality $f(x) \geq f(0) = 0$ for all $x \in \left[0, \frac{\pi}{2}\right)$ (with equality only at $x = 0$).

Second proof. Here we only use the inequalities between the means of two positive numbers and the second inequality from (1) applied to $\frac{x}{2} \in \left(0, \frac{\pi}{2}\right)$ in the place of x . Indeed, according to the inequality between the geometric and harmonic means and to the mentioned inequality, we have

$$\sqrt{\sin x \cdot \tan x} > \frac{2}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} = \frac{2 \sin x}{1 + \cos x} = 2 \tan \frac{x}{2} > x,$$

so we obtain the desired result.

Alternatively, note that the inequality

$$\sqrt{\sin x \tan x} > 2 \tan \frac{x}{2}, \quad (2)$$

or (squared)

$$\sin x \tan x > 4 \tan^2 \frac{x}{2}$$

is equivalent to

$$1 > 1 - \tan^4 \frac{x}{2}$$

if we use the well-known formulae

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \quad \text{and} \quad \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}.$$

We further use inequality (2) to prove inequality (B).

Proposition 2. We have

$$\sin x > x \sqrt[3]{\cos x}$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Proof. Successively replace x by $\frac{x}{2}$ in (2), in order to obtain:

$$\sqrt{\sin x \tan x} > 2 \tan \frac{x}{2},$$

$$\sqrt{\sin \frac{x}{2} \tan \frac{x}{2}} > 2 \tan \frac{x}{4},$$

$$\sqrt{\sin \frac{x}{4} \tan \frac{x}{4}} > 2 \tan \frac{x}{8},$$

and so on until we get

$$\sqrt{\sin \frac{x}{2^{n-1}} \tan \frac{x}{2^{n-1}}} > 2 \tan \frac{x}{2^n}$$

for some positive integer n (which can be any positive integer n).

Multiplying all these inequalities and cancelling common factors, we get:

$$\sqrt{\sin x \tan x} \cdot \sqrt{\sin \frac{x}{2} \sin \frac{x}{4} \cdots \sin \frac{x}{2^{n-1}}} > 2^n \tan \frac{x}{2^n} \sqrt{\tan \frac{x}{2} \tan \frac{x}{4} \cdots \tan \frac{x}{2^{n-1}}},$$

or

$$\sqrt{\sin x \tan x} > \frac{1}{\sqrt{\cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^{n-1}}}} \cdot \frac{\tan \frac{x}{2^n}}{\frac{x}{2^n}} \cdot x.$$

Repeatedly applying the formula for the sine of the double of an angle we get

$$\cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^{n-1}} = \frac{\sin x}{2^{n-1} \sin \frac{x}{2^{n-1}}} = \frac{\sin x}{x} \cdot \frac{\frac{x}{2^{n-1}}}{\sin \frac{x}{2^{n-1}}}.$$

Now, as n can be any positive integer, we may pass to the limit (for $n \rightarrow \infty$) in the inequality that we previously obtained, and because we have

$$\lim_{n \rightarrow \infty} \frac{\frac{x}{2^{n-1}}}{\sin \frac{x}{2^{n-1}}} = \lim_{n \rightarrow \infty} \frac{\tan \frac{x}{2^n}}{\frac{x}{2^n}} = 1,$$

we thus get

$$\sqrt{\sin x \cdot \tan x} \geq \sqrt{\frac{x}{\sin x}} \cdot x$$

which simplifies to inequality (B):

$$\sin^{3/2} x \geq x^{3/2} \sqrt{\cos x} \Leftrightarrow \sin x \geq x \sqrt[3]{\cos x}.$$

As we see, this approach does not lead directly to the strict inequality. However, if we write it in the form

$$\frac{\sin x}{x} \geq \sqrt[3]{\cos x}$$

and observe that the functions from both sides are strictly decreasing, and that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \sqrt[3]{\cos x} = 1, \text{ while } \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{x} = \frac{2}{\pi} > 0 = \lim_{x \rightarrow \frac{\pi}{2}} \sqrt[3]{\cos x}$$

the conclusion that the inequality is strict for $0 < x < \frac{\pi}{2}$ follows. (In particular, note that the monotony of $x \mapsto \frac{\sin x}{x}$ leads to the inequality

$$\sin x > \frac{2}{\pi} x, \quad 0 < x < \frac{\pi}{2}.$$

An alternate proof can be given (see [2]) by using Hölder's inequality in the form

$$\int_a^b f^3(t) dt \int_a^b g^3(t) dt \int_a^b h^3(t) dt \geq \left(\int_a^b (fgh)(t) dt \right)^3$$

for the functions $f(t) = g(t) = \cos^{\frac{1}{3}} t$ and $h(t) = \frac{1}{\cos^{\frac{2}{3}} t}$ on the interval $[0, x]$. We get

$$\int_0^x \cos t dt \int_0^x \cos t dt \int_0^x \frac{1}{\cos^2 t} dt \geq \left(\int_0^x \left(\cos t \cos t \frac{1}{\cos^2 t} \right)^{\frac{1}{3}} dt \right)^3,$$

that is,

$$\sin^2 x \tan x \geq x^3 \Leftrightarrow \sin x \geq x \sqrt[3]{\cos x},$$

which is actually strict (the conditions for equality not being fulfilled). Finally note that in the form

$$\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x} > 1$$

the inequality (B) holds for all x with $0 < |x| < \frac{\pi}{2}$.

It is time to see now the inequality from the title, which appeared in 1654 in Christiaan Huygens's work *De circuli magnitudine inventa*, but also in 1621, in the book *Cyclometricus* of the astronomer Willebrord Snellius (for which reason this is also called Snell's, or Snellius's inequality). They both used the inequality to find better approximations for the number π .

Proposition 3. The inequality

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 1 \tag{C}$$

holds for $0 < |x| < \frac{\pi}{2}$.

Proof. It is enough to prove that

$$2\sin x + \tan x > 3x$$

for all $0 < x < \frac{\pi}{2}$. But this one follows by applying the inequality between the arithmetic mean and the geometric mean of the three positive numbers $\sin x$, $\sin x$, and $\tan x$, and then the inequality (B):

$$\frac{\sin x + \sin x + \tan x}{3} > \sqrt[3]{\sin^2 x \tan x} > x.$$

We invite the interested reader to find a proof using calculus (for example, by studying the function $x \mapsto 2\sin x + \tan x - 3x$), and we go to another immediate consequence of (B), namely Wilker's inequality (that was proposed in 1989 in *The American Mathematical Monthly* by J. B. Wilker).

Proposition 4. The inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2$$

holds for $0 < |x| < \frac{\pi}{2}$.

Proof. Once again, it is enough to prove the inequality for $0 < x < \frac{\pi}{2}$. We have (by the arithmetic-geometric means inequality, and then by inequality (B)):

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2\sqrt{\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x}} > 2.$$

Note that the inequality between the arithmetic mean and the harmonic mean of $\sin x, \sin x, \tan x$ leads to

$$\frac{2\sin x + \tan x}{3} > \frac{3\sin x}{2 + \cos x},$$

a lower bound that cannot be used for proving Huygens's inequality, since we have the following result:

Proposition 5. The inequality

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}$$

holds for $0 < |x| < \frac{\pi}{2}$.

Proof. This is known as Cusa's inequality and it is equivalent to

$$x > \frac{3\sin x}{2 + \cos x}, \quad 0 < x < \frac{\pi}{2}.$$

With the substitution $x = 2\theta$, we can transform it into

$$\theta \geq \frac{3\tan\theta}{3 + \tan^2\theta}, \quad 0 \leq \theta < \frac{\pi}{4}.$$

Since the function

$$f(\theta) = \theta - \frac{3 \tan \theta}{3 + \tan^2 \theta}$$

has the derivative

$$f'(\theta) = \frac{4 \tan^4 \theta}{(3 + \tan^2 \theta)^2}$$

positive, we get that f is strictly increasing, hence $f(\theta) \geq f(0) = 0$ (with equality only for $\theta = 0$) and thus Cusa's inequality is proved. One other way (and we invite the reader to follow it; probably there are many more other possibilities) is to prove

$$x(2 + \cos x) - 3 \sin x \geq 0, \quad 0 \leq x < \frac{\pi}{2}.$$

3. Final remarks 1) The inequalities

$$\sqrt[3]{\cos x} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}$$

(the first is inequality (B), the second is Cusa's inequality) give better bounds for the ratio $\frac{\sin x}{x}$ than those from (1'), namely

$$\cos x < \sqrt[3]{\cos x} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} < 1 \quad \text{for } 0 < x < \frac{\pi}{2}.$$

We can also show that the inequality

$$\frac{3x - \tan x}{2x} < \frac{\sin x}{x} \quad \text{for } 0 < x < \frac{\pi}{2},$$

which is equivalent to Huygens's inequality (C), is better than the left inequality from (1'), in that

$$\cos x < \frac{3x - \tan x}{2x} \Leftrightarrow 2 \cos x < 3 - \frac{\sin x}{x} \cdot \frac{1}{\cos x},$$

for $x \in \left(0, \frac{\pi}{2}\right)$ such that $\cos x > \frac{1}{3}$ (which means for x in a neighborhood of the origin). Indeed, by using Cusa's inequality, we get

$$3 - \frac{\sin x}{x} \cdot \frac{1}{\cos x} > 3 - \frac{2 + \cos x}{3} \cdot \frac{1}{\cos x} = \frac{6 \cos^2 x + 2(3 \cos x - 1)(1 - \cos x)}{3 \cos x} > \frac{6 \cos^2 x}{3 \cos x} = 2 \cos x.$$

On the other hand, by taking limits for $x \rightarrow \frac{\pi}{2}$ in both sides of the inequality clearly shows that it cannot be true for x close enough to $\frac{\pi}{2}$.

2) For any $t > 0$ we can apply the inequality between the arithmetic mean and geometric mean of three positive numbers, and then inequality (B) (exactly as we did for proving Huygens's inequality) in order to obtain

$$2t \sin x + \frac{1}{t^2} \tan x > 3x, \quad 0 < x < \frac{\pi}{2}. \quad (\text{C}')$$

Nevertheless, we can calculate (for $0 < x < \frac{\pi}{2}$)

$$(2t \sin x + \frac{1}{t^2} \tan x) - (2 \sin x + \tan x) = 2(t - 1) \tan x \left(\cos x - \frac{t + 1}{2t^2} \right)$$

and notice that, in general, this difference is positive. More precisely, for $0 < t < 1$, we have that $t - 1 < 0$ and

$$\cos x < 1 < \frac{t + 1}{2t^2},$$

thus the difference is positive and

$$2t \sin x + \frac{1}{t^2} \tan x > 2 \sin x + \tan x, \quad 0 < x < \frac{\pi}{2},$$

meaning that Huygens's inequality remains sharper than (C'). Whenever $t > 1$, we have $t - 1 > 0$ and $\frac{t+1}{2t^2} < 1$, so the second parenthesis can be negative. However, the second parenthesis is positive, and the difference is also positive for $\cos x > \frac{t+1}{2t^2}$ – that is, in a neighborhood of the origin. Thus, again, (C') does not quite represent an improvement of (C).

3) It would be interesting to see (and we invite the reader to think about this problem) for which positive real numbers a and b the inequality

$$a \sin x + b \tan x > x$$

holds for any $x \in \left(0, \frac{\pi}{2}\right)$. Note that $a+b \geq 1$ is a necessary condition for the inequality to be true in a neighborhood of the origin (divide by x and take limits for $x \rightarrow 0$), and that this becomes Huygens's inequality when $a = \frac{2}{3}$ and $b = \frac{1}{3}$. It would be also nice to know what numbers a_0 and b_0 yield the best such inequality:

$$a \sin x + b \tan x \geq a_0 \sin x + b_0 \tan x > x$$

for all $x \in \left(0, \frac{\pi}{2}\right)$, and for all $a, b \in (0, \infty)$.

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References

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3. Problems 115, 167, *Cruz Mathematicorum* (from vol. 1, no. 1 to vol. 4)