

## Junior problems

J703. Find all triples  $(x, y, z)$  of real numbers greater than 1 such that

$$\frac{x + \frac{1}{x}}{9} = \frac{y + \frac{1}{y}}{17} = \frac{z + \frac{1}{z}}{24}$$

and

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} + 4.$$

*Proposed by Titu Andreescu, Dallas, USA*

*Solution 1 by Kousik Sett, India*

Let

$$\frac{x + \frac{1}{x}}{9} = \frac{y + \frac{1}{y}}{17} = \frac{z + \frac{1}{z}}{24} = k,$$

where  $k$  is a positive real number. Therefore,

$$x + \frac{1}{x} = 9k, \quad y + \frac{1}{y} = 17k, \quad \text{and} \quad z + \frac{1}{z} = 24k.$$

Let

$$\sqrt{x} - \frac{1}{\sqrt{x}} = a, \quad \sqrt{y} - \frac{1}{\sqrt{y}} = b, \quad \text{and} \quad \sqrt{z} - \frac{1}{\sqrt{z}} = c.$$

Then the second relation becomes  $a + b + c = 4$ .

Since  $x > 1$ , we have  $\sqrt{x} > \frac{1}{\sqrt{x}}$ , i.e.,  $\sqrt{x} - \frac{1}{\sqrt{x}} > 0$ . Similarly,  $\sqrt{y} - \frac{1}{\sqrt{y}} > 0$  and  $\sqrt{z} - \frac{1}{\sqrt{z}} > 0$ .

So,  $a, b, c$  are all positive real numbers.

We have

$$a^2 = 9k - 2, \quad b^2 = 17k - 2, \quad \text{and} \quad c^2 = 24k - 2.$$

Since  $a + b + c = 4$ , we have the equation

$$\sqrt{9k - 2} + \sqrt{17k - 2} + \sqrt{24k - 2} = 4,$$

which can be written as

$$2\sqrt{9k - 2} - 1 + 2\sqrt{17k - 2} - 3 + 2\sqrt{24k - 2} - 4 = 0,$$

which implies

$$\frac{4(9k - 2) - 1}{2\sqrt{9k - 2} + 1} + \frac{4(17k - 2) - 9}{2\sqrt{17k - 2} + 3} + \frac{4(24k - 2) - 16}{2\sqrt{24k - 2} + 4} = 0,$$

which gives

$$(4k - 1) \left( \frac{9}{2\sqrt{9k - 2} + 1} + \frac{17}{2\sqrt{17k - 2} + 3} + \frac{24}{2\sqrt{24k - 2} + 4} \right) = 0.$$

Hence,  $4k - 1 = 0$ , i.e.,  $k = \frac{1}{4}$ . This is the only real value obtained as the second factor of the above expression is non-zero.

Therefore, we obtain

$$x + \frac{1}{x} = \frac{9}{4}, \quad y + \frac{1}{y} = \frac{17}{4}, \quad \text{and} \quad z + \frac{1}{z} = 6,$$

which give

$$x = \frac{9 \pm \sqrt{17}}{8}, \quad y = 4, \frac{1}{4}, \quad \text{and} \quad z = 3 \pm 2\sqrt{2}.$$

Since the real numbers  $x$ ,  $y$ , and  $z$  are all greater than 1, the only real solution that satisfy both the given conditions is

$$(x, y, z) = \left( \frac{9 + \sqrt{17}}{8}, 4, 3 + 2\sqrt{2} \right).$$

*Solution 2 by Polyhedra, Polk State College, FL, USA*

Let  $a = \sqrt{x} - \frac{1}{\sqrt{x}}$ ,  $b = \sqrt{y} - \frac{1}{\sqrt{y}}$ , and  $c = \sqrt{z} - \frac{1}{\sqrt{z}}$ . Then  $a, b, c > 0$  and

$$\frac{a^2 - \frac{1}{4}}{9} = \frac{b^2 - \frac{9}{4}}{17} = \frac{c^2 - 4}{24} = k$$

for some  $k$ . Therefore,

$$0 = a - \frac{1}{2} + b - \frac{3}{2} + c - 2 = k \left( \frac{9}{a + \frac{1}{2}} + \frac{17}{b + \frac{3}{2}} + \frac{24}{c + 2} \right),$$

so  $k = 0$ . Hence  $a^2 = \frac{1}{4}$ ,  $b^2 = \frac{9}{4}$ , and  $c^2 = 4$ , from which we easily obtain  $x = \frac{9 + \sqrt{17}}{8}$ ,  $y = 4$ , and  $z = 3 + 2\sqrt{2}$ . Finally, it is routine to verify that this triple indeed satisfies the conditions.

*Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, VA, USA; Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Hannah Yoon, Gwinnett School of Mathematics, Science, and Technology, GA, USA; Jonathan Lee, Buford High School Buford, GA, USA; Jung Hoon Song, Champsmath Academy, GA, USA; Kevin Soto Palacios, Huarmey, Peru; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Daniel Pascuas, Barcelona, Spain; Keisha Kwok, AMMOC; Nguyen Tuan Minh; Parth Andhare, AMMOC; Taes Padhihary, IIT Guwahati, India; Winston Ra, Chattahoochee High School in Johns Creek, GA, USA.*

J704. Let  $a, b, c$  be positive real numbers such that  $a + b + c = 6$ . Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{2(a^2+b^2+c^2)} \geq \frac{7}{8}.$$

*Proposed by Mihaela Berindeanu, Bucharest, Romania*

*Solution 1 by Polyhedra, Polk State College, FL, USA*

Writing each 1 in the numerators as  $\frac{a+b+c}{6}$  and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{2(a^2+b^2+c^2)} &= \frac{1}{2} + \frac{c^2}{6c(a+b)} + \frac{a^2}{6a(b+c)} + \frac{b^2}{6b(c+a)} + \frac{9}{6(a^2+b^2+c^2)} \\ &\geq \frac{1}{2} + \frac{(c+a+b+3)^2}{6(2ca+2ab+2bc+a^2+b^2+c^2)} = \frac{1}{2} + \frac{9^2}{6(a+b+c)^2} = \frac{7}{8}. \end{aligned}$$

The equality holds if and only if  $a = b = c = 2$ .

*Solution 2 by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

Among three numbers  $a, b, c$  there exists two numbers  $\geq 2$  or  $\leq 2$ . Without loss of generality we can assume that

$$(b-2)(c-2) \geq 0,$$

i.e.

$$bc + 4 \geq 2(b+c).$$

This leads us to

$$(b+c)^2 + 8 \geq 4(b+c) + b^2 + c^2,$$

or

$$b^2 + c^2 \leq (6-a)^2 - 4(6-a) + 8 = a^2 - 8a + 20.$$

Hence

$$\begin{aligned} \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{2(a^2+b^2+c^2)} &\geq \frac{1}{b+c} + \frac{4}{2a+b+c} + \frac{3}{2(2a^2-8a+20)} \\ &= \frac{1}{6-a} + \frac{4}{a+6} + \frac{3}{4(a^2-4a+10)} \\ &= \frac{-12a^3 + 165a^2 - 600a + 1308}{4(36-a^2)(a^2-4a+10)}. \end{aligned}$$

It suffices to show that

$$\frac{-12a^3 + 165a^2 - 600a + 1308}{4(36-a^2)(a^2-4a+10)} \geq \frac{7}{8}.$$

This is equivalent to

$$\begin{aligned} 7a^4 - 52a^3 + 148a^2 - 192a + 96 &\geq 0, \\ (a-2)^2(7a^2 - 24a + 24) &\geq 0, \\ (a-2)^2[a^2 + 6(a-2)^2] &\geq 0 \end{aligned}$$

which is clearly true and we are done.

*Solution 3 by Theo Koupelis, Clark College, Washington, USA*

Homogenizing the desired inequality, we get

$$\sum_c \frac{1}{a+b} + \frac{a+b+c}{4[(a+b+c)^2 - 2(ab+bc+ca)]} \geq \frac{21}{4(a+b+c)}.$$

Let  $p = a + b + c = 6$ ,  $q = ab + bc + ca$ , and  $r = abc$ . Substituting into the desired inequality, we get the equivalent expression

$$\frac{p^2 + q}{pq - r} + \frac{p}{4(p^2 - 2q)} \geq \frac{21}{4p}.$$

Clearing denominators we get

$$r(10p^2 - 21q) \geq 12p^3q - 2p^5 - 17pq^2.$$

By AM-GM we get  $p^2 \geq 3q$ , and thus the left-hand-side of the above inequality is positive. By Schur's inequality we get  $r \geq \frac{4pq - p^3}{9}$ , and thus it suffices to show that

$$\frac{4pq - p^3}{9} \geq \frac{12p^3q - 2p^5 - 17pq^2}{10p^2 - 21q} \iff p(8p^2 - 23q)(p^2 - 3q) \geq 0,$$

which is obvious. Equality occurs when  $p^2 = 3q$  or  $a = b = c = 2$ .

*Also solved by Arkady Alt, San Jose, California, USA; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Aaditya Bhaumik Sahu, AMMOC, India; Daniel Pascuas, Barcelona, Spain; Kristine Lee, Mount Juliet High School, TN, USA; Medha Shrivastava, AMMOC, India; Prasanna Pawar, AMMOC, Bengaluru, India; Winston Ra, Chattahoochee High School in Johns Creek, GA, USA.*

J705. Sally and Tommy play the following game: they start with a monic quadratic trinomial  $f(x) = x^2 + bx + c$ , where both  $b$  and  $c$  are initially undefined integers. They take turns assigning integer values to  $b$  and  $c$  under the following rules:

- (i) Sally and Tommy take turns, with Sally going first.
- (ii) On each turn, a player must assign an integer value between  $-N$  and  $N$  (inclusive) to either  $b$  or  $c$  that hasn't been assigned yet.
- (iii) Once a coefficient ( $b$  or  $c$ ) has been assigned a value, it cannot be changed.
- (iv) The game ends when both  $b$  and  $c$  have been assigned.
- (v) The player who assigns the last coefficient wins if the resulting quadratic trinomial  $f(x) = x^2 + bx + c$  factors over the integers; otherwise, this player loses.

Determine for which positive integers  $N$  Sally has a winning strategy, and for which ones Tommy has a winning strategy, assuming both play optimally.

*Proposed by Alessandro Ventullo, Milan, Italy*

*Solution by Theo Koupelis, Clark College, Washington, USA*

With her first move, Sally should not assign a value to  $b$ . If she were to set  $b = n$ , where  $n$  is an integer such that  $-N \leq n \leq N$ , then Tommy can assign  $c = 0$  and win the game because in this case  $f(x)$  factors over the integers as

$$f(x) = x(x + n).$$

Thus, with her first move, Sally should assign a value to  $c$ . If Sally were to set  $c = m$ , where  $m$  is an integer such that  $-N + 1 \leq m \leq N - 1$ , then Tommy can set  $b = -(m + 1)$  and win the game because in this case  $f(x)$  factors over the integers as

$$f(x) = (x - m)(x - 1).$$

Therefore, with her first move, Sally's options are to assign  $c = N$  (or  $c = -N$ ). If  $N$  is a composite number, such that  $N = k\ell$ , where  $k, \ell$  are positive integers with  $2 \leq k \leq \ell$ , then Tommy wins the game by choosing  $b = k + \ell$  (or  $b = k - \ell$ ) because in this case  $f(x)$  factors over the integers as

$$f(x) = (x + k)(x \pm \ell).$$

Tommy's choice is allowed because  $k \pm \ell \leq N = k\ell \iff k \leq \ell(k \mp 1)$ , which is obvious because  $2 \leq k \leq \ell$ . If  $N = 1$  or  $N = p$ , a prime, then Sally should not choose  $c = -N$ , because Tommy will win the game by choosing  $b = N - 1$ ; in this case

$$f(x) = (x - 1)(x + N).$$

From the above we get that Sally can win the game only when  $N = 1$  or  $N = p$ , a prime, and by assigning  $c = N$  with her first move; in this case, for an integer  $x_0$  such that  $f(x_0) = x_0^2 + bx_0 + N = 0$ , we must have  $x_0 \mid N$ , and thus  $x_0 = \pm 1$  or  $x_0 = \pm N$ , and therefore  $b = -(N + 1)$  or  $b = N + 1$ ; neither case is acceptable because both values are outside the range  $[-N, N]$ .

So, Sally wins when  $N = 1$  or  $N$  is prime, and Tommy wins when  $N$  is composite.

*Also solved by Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Aaditya Bhaumik Sahu, AMMOC, India; Aarnav Sivachandran, AMMOC, USA, CT; Keisha Kwok, AMMOC; Polyhedra, Polk State College, FL, USA; Prasanna Pawar, AMMOC, Bengaluru, India.*

J706. Find all pairs  $(p, q)$  of primes such that  $pq + 1$  is a perfect cube.

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution by Theo Koupelis, Clark College, Washington, USA*

Without loss of generality, let  $p \leq q$ . Let  $pq + 1 = m^3$ , where  $m$  is a positive integer such that  $m \geq 3$  (it is easy to see that  $m \in \{1, 2\}$  gives no solutions). We have

$$pq = m^3 - 1 = (m - 1)(m^2 + m + 1).$$

Because  $m \geq 3$ , we get  $m - 1 \geq 2$ , and thus  $p = m - 1$  and  $q = m^2 + m + 1 = p^2 + 3p + 3$ . Therefore, all pairs  $(p, q) = (p, p^2 + 3p + 3)$ , where  $p$  is a prime such that  $p^2 + 3p + 3$  is also a prime, satisfy the statement of the problem, namely that  $pq + 1 = (p + 1)^3$  is a perfect cube. Examples of such primes are  $p = 2, 5, 7, 11, 13, 19, 23, 37$ , etc. Examples of primes  $p$  such that  $q$  is not a prime include  $p = 3, 17, 29, 31, 41, 43, 47$ , etc.

*Also solved by Anita Tufisi, Romania; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania; Gjimnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Kalkin Gandhi, AMMOC Bengaluru, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Aaditya Bhaumik Sahu, AMMOC, India; Aarnav Sivachandran, AMMOC, USA, CT; Keisha Kwok, AMMOC; Medha Shrivastava, AMMOC, India; Polyhedra, Polk State College, FL, USA; Sergei Makarevich, AMMOC; Taes Padhihary, IIT Guwahati, India.*

J707. Find all pairs  $(m, n)$  of nonnegative integers such that

$$m^2 - 16mn + n^2 = 9.$$

*Proposed by Adrian Andreescu, Dallas, USA*

*Solution by Kousik Sett, India*

Solving the given equation  $m^2 - 16mn + n^2 - 9 = 0$  as quadratic of  $m$ , we obtain

$$m = \frac{16n \pm \sqrt{252n^2 + 36}}{2} = 8n \pm 3\sqrt{7n^2 + 1}.$$

For,  $m$  to be an integer,  $7n^2 + 1$  must be a perfect square. Let  $7n^2 + 1 = t^2$  for some integer  $t$ . Then  $t^2 - 7n^2 = 1$ , which is a Pell's equation.

Its fundamental nontrivial solution is  $t_1 = 8, n_1 = 3$  because  $8^2 - 7 \cdot 3^2 = 64 - 63 = 1$ . Hence all integer solutions  $(t, n)$  are given by

$$t + n\sqrt{7} = (8 + 3\sqrt{7})^k, \quad k = 0, 1, 2, 3 \dots$$

So,

$$(t_0, n_0) = (1, 0), \quad (t_1, n_1) = (8, 3), \quad (t_2, n_2) = (127, 48), \quad (t_3, n_3) = (2024, 765) \text{ and so on.}$$

For each such  $(t, n)$ , the quadratic formula gives

$$m = 8n \pm 3t.$$

So the integer solutions  $(m, n)$  are exactly the pairs

$$(m, n) = (8n_k + 3t_k, n_k) \text{ or } (8n_k - 3t_k, n_k)$$

for  $k = 0, 1, 2, 3 \dots$ , where  $(t_k, n_k)$  is the  $k$ -th Pell solution from  $(t_k + n_k\sqrt{7}) = (8 + 3\sqrt{7})^k$ .

Considering only the non-negative values of  $m$ , we list the smallest solutions below.

$$(m, n) = \{(3, 0), (0, 3), (48, 3), (3, 48), (765, 48), (48, 765), (12192, 765), (765, 12192), \dots\}$$

They occur in symmetric pairs because the equation is symmetric in  $(m, n)$ . So, there are infinitely many solutions and they are exactly the ones produced by the Pell parametrization above.

*Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; G. C. Greubel, Newport News, VA, USA; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Kalkin Gandhi, AMMOC Bengaluru, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Thomas McCormick, AMMOC; Aaditya Bhaumik Sahu, AMMOC, India; Kristine Lee, Mount Juliet High School, TN, USA; Parth Andhare, AMMOC; Polyhedra, Polk State College, FL, USA; Sergei Makarevich, AMMOC; Taes Padhihary, IIT Guwahati, India.*

J708. Let  $ABC$  be a triangle with incenter  $I$ . Prove that

(a)  $AI \cdot BI \cdot CI = 4Rr^2$ .

(b)  $\frac{AI^2 + BI^2 + CI^2}{r} = \frac{b + c - a}{\sin A} + \frac{c + a - b}{\sin B} + \frac{a + b - c}{\sin C}$ .

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution 1 by the author*

(a) We have

$$AI = \frac{r}{\sin \frac{A}{2}} = \frac{s - a}{\cos \frac{A}{2}}.$$

It follows that

$$AI \cdot BI \cdot CI = \frac{r^3}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{4Rr^3}{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 4Rr^2.$$

(b) From the previous part we get

$$AI^2 = \frac{r(s - a)}{\sin \frac{A}{2} \cos \frac{A}{2}} = \frac{2r(s - a)}{\sin A}.$$

This yields

$$\frac{AI^2}{r} = \frac{b + c - a}{\sin A}.$$

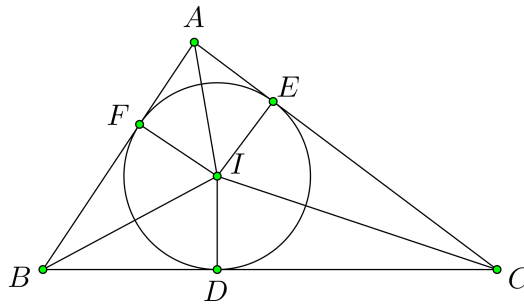
Similarly

$$\begin{aligned} \frac{BI^2}{r} &= \frac{c + a - b}{\sin B}, \\ \frac{CI^2}{r} &= \frac{a + b - c}{\sin C}. \end{aligned}$$

Summing up these relations we obtain immediately the desired result.

*Solution 2 by Kousik Sett, India*

Let the incircle of triangle  $ABC$  touch  $BC$  at  $D$ ,  $CA$  at  $E$  and  $AB$  at  $F$ , respectively. Then  $ID = IE = IF = r$ . Moreover,  $AE = AF = s - a$ ,  $BD = BF = s - b$ , and  $CD = CE = s - c$ , where  $BC = a$ ,  $CA = b$ ,  $AB = c$ , and  $s$  is the semi-perimeter of triangle  $ABC$ , i.e.,  $2s = a + b + c$ .



We will use well-known formulae  $r = \frac{\Delta}{s}$ , and  $abc = 4R\Delta$ , where  $\Delta$  and  $R$  denote the area and circumradius of triangle  $ABC$ , respectively. Moreover,  $\Delta = \sqrt{s(s - a)(s - b)(s - c)} = rs$ . Then

$$\begin{aligned} AI^2 &= IF^2 + AF^2 = r^2 + (s - a)^2 \\ &= \frac{s(s - a)(s - b)(s - c)}{s^2} + (s - a)^2 \\ &= (s - a) \left[ s - (b + c) + \frac{bc}{s} + s - a \right] = \frac{bc(s - a)}{s}. \end{aligned}$$

Similarly,

$$BI^2 = \frac{ca(s-b)}{s} \quad \text{and} \quad CI^2 = \frac{ab(s-c)}{s}.$$

Therefore,

$$\begin{aligned} AI^2 \cdot BI^2 \cdot CI^2 &= \frac{bc(s-a) \cdot ca(s-b) \cdot ab(s-c)}{s^3} \\ &= \frac{a^2b^2c^2 \cdot s(s-a)(s-b)(s-c)}{s^4} \\ &= \frac{a^2b^2c^2 \cdot \Delta^2}{s^4} = \frac{16R^2\Delta^2 \cdot \Delta^2}{s^4} = 16R^2 \cdot r^2 \cdot r^2. \end{aligned}$$

Thus

$$AI \cdot BI \cdot CI = 4Rr^2.$$

Again,

$$\frac{AI^2}{r} = \frac{bc(s-a)}{rs} = \frac{bc(s-a)}{\Delta} = \frac{bc(s-a)}{\frac{abc}{4R}} = \frac{s-a}{\frac{a}{4R}} = \frac{b+c-a}{\frac{a}{2R}} = \frac{b+c-a}{\sin A}.$$

Similarly,

$$\frac{BI^2}{r} = \frac{c+a-b}{\sin B} \quad \text{and} \quad \frac{CI^2}{r} = \frac{a+b-c}{\sin C}.$$

Thus

$$\frac{AI^2 + BI^2 + CI^2}{r} = \frac{b+c-a}{\sin A} + \frac{c+a-b}{\sin B} + \frac{a+b-c}{\sin C}.$$

*Also solved by Arkady Alt, San Jose, California, USA; Daniel Văcaru, National Economic College “Maria Teiuleanu”, Pitești, Romania; Gjimnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Miquel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Telemachus Baltsavias, Keramies Junior High School, Kefalonia, Greece; Theo Koupelis, Clark College, Washington, USA; Nguyen Tuan Minh; Polyhedra, Polk State College, FL, USA; Sergei Makarevich, AMMOC.*

## Senior problems

S703. Solve in real numbers the system of equations

$$x^2(y - z) = 2025, \quad y^2(z - x) = 275, \quad z^2(x - y) = -320.$$

*Proposed by Adrian Andreescu, Dallas, USA*

*Solution 1 by Theo Koupelis, Clark College, Washington, USA*

Clearly  $xyz \neq 0$  and  $x \neq y \neq z \neq x$ . Let  $s = x/z$  and  $t = y/z$ . The given system of equations is equivalent to

$$z^3 s^2 (t - 1) = 2025, \tag{1}$$

$$z^3 t^2 (1 - s) = 275, \tag{2}$$

$$z^3 = \frac{320}{t - s}. \tag{3}$$

Substituting (3) into (1) and (2) we get

$$s^2 (t - 1) = 2025 \cdot \frac{t - s}{320}, \tag{4}$$

$$t^2 (1 - s) = 275 \cdot \frac{t - s}{320}. \tag{5}$$

Adding (4) and (5) and dividing by  $(t - s) \neq 0$ , we get

$$st - (s + t) = -\frac{115}{16} \iff t = \frac{16s - 115}{16(s - 1)}. \tag{6}$$

Substituting (6) into (4) and simplifying the resulting quadratic in  $s$  we get

$$45(4s - 15)(4s - 345) = 0,$$

and thus, using (6) we get the solutions  $(s, t) = \left(\frac{15}{4}, -\frac{5}{4}\right)$  or  $(s, t) = \left(\frac{345}{4}, \frac{115}{124}\right)$ . From (3) and the definitions of  $s, t$  we get the only two solutions

$$(x, y, z) = (-15, 5, -4) \quad \text{and} \quad (x, y, z) = \left(-15\sqrt[3]{713}, -5\sqrt[3]{23/961}, -4\sqrt[3]{31/529}\right).$$

*Solution 2 by Theo Koupelis, Clark College, Washington, USA*

From the given system of equations we get  $y > z > x$ . Let  $z = x + a$  and  $y = x + a + b$ , where  $a, b$  are positive real numbers. Substituting, we get the system of equations

$$x^2 \cdot b = 2025 \tag{1}$$

$$(x + a + b)^2 \cdot a = 275 \tag{2}$$

$$(x + a)^2 \cdot (a + b) = 320 \tag{3}$$

Adding (1), (2), and (3) we get

$$(a + b) \cdot [2(x + a)^2 + ab] = 2620. \tag{4}$$

Substituting the expression for  $(x + a)^2$  from (3) into (4) and simplifying we get

$$ab(a + b) = 1980 \implies a = \frac{\sqrt{b^4 + 7920b} - b^2}{2b}. \tag{5}$$

From (1) we get

$$x = \pm \frac{45}{\sqrt{b}}. \quad (6)$$

Substituting (5) and (6) into (3) and simplifying we get

$$\frac{\sqrt{b^4 + 7920b} + b^2}{2b} \cdot \left( \pm \frac{45}{\sqrt{b}} + \frac{\sqrt{b^4 + 7920b} - b^2}{2b} \right)^2 = 320.$$

Setting  $b = k^2$ , where  $k$  is a positive real number, expanding and simplifying the above equation we get

$$801\sqrt{k^6 + 7920} = 119k^3 \mp 71280. \quad (7)$$

Squaring and simplifying we get

$$880(713k^3 \pm 27)(k \pm 3)(k^2 \mp 3k + 9) = 0. \quad (8)$$

Clearly  $k^2 \pm 3k + 9 > 0$ . From (6), (7), and (8) we see that there is no solution for  $x > 0$ . For  $x < 0$ , we get  $k = 3$  or  $k^3 = 27/713$ . From the former we get the solution  $(x, y, z) = (-15, 5, -4)$ . From the latter we get the solution  $(x, y, z) = \left(-15\sqrt[3]{713}, -5\sqrt[3]{23/961}, -4\sqrt[3]{31/529}\right)$ . Both solutions satisfy the given system of equations.

*Also solved by Anita Tufisi, Romania; G. C. Greubel, Newport News, VA, USA; Gjinnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Kousik Sett, India; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Aaditya Bhaumik Sahu, AMMOC, India; Madani Harouach, AMMOC; Hoang-Bao Le, Dublin, Ireland; Keisha Kwok, AMMOC; Prasanna Pawar, AMMOC, Bengaluru, India.*

S704. Let  $x, y, z \geq 0$  such that  $x^2 + y^2 + z^2 + xyz = 4$ . Prove that

$$(x + y + z - 2)(x + y + z)^2 \leq \frac{18 - 7\sqrt{2}}{8}(x + y)(y + z)(z + x) + 7\sqrt{2} - 9.$$

*Proposed by Marius Stănean, Zalău, Romania*

*Solution by the author*

This condition leads us to the following substitutions

$$x = 2 \cos A, \quad y = 2 \cos B, \quad z = 2 \cos C$$

where  $ABC$  is a nonobtuse triangle, including the possibility of having a degenerate triangle.

By the known identity in a triangle:

$$x + y + z = 2(\cos A + \cos B + \cos C) = \frac{2(R + r)}{R},$$

$$xy + yz + zx = 4 \sum_{cyc} \cos A \cos B = \frac{s^2 + r^2 - 4R^2}{R^2},$$

$$xyz = 8 \cos A \cos B \cos C = \frac{2(s^2 - r^2 - 4Rr - 4R^2)}{R^2},$$

$$(x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) - xyz = \frac{2r(s^2 + 2Rr + r^2)}{R^3},$$

the inequality becomes

$$\frac{8r(R + r)^2}{R^3} \leq \frac{(18 - 7\sqrt{2})r(s^2 + 2Rr + r^2)}{4R^3} + 7\sqrt{2} - 9.$$

If we denote  $t^2 = 1 - \frac{2r}{R} \in [0, 1)$ , then by Blundon's Inequality, i.e.

$$s^2 \geq 2R^2 + 10Rr - r^2 - 2\sqrt{R(R - 2r)^3}$$

becomes

$$\frac{s^2}{R^2} \geq 2 + 5(1 - t^2) - \frac{(1 - t^2)^2}{4} - 2t^3 = \frac{(1 - t)(t + 3)^3}{4}. \quad (1)$$

On the other hand, we have  $xyz \geq 0$  which means

$$\frac{s^2}{R^2} \geq \frac{r^2}{R^2} + \frac{4r}{R} + 4 = \frac{(5 - t^2)^2}{4}. \quad (2)$$

In terms of  $t$  it suffices to prove that

$$(1 - t^2)(3 - t^2)^2 \leq \frac{(18 - 7\sqrt{2})(1 - t^2)}{8} \left( \frac{s^2}{R^2} + 1 - t^2 + \frac{(1 - t^2)^2}{4} \right) + 7\sqrt{2} - 9.$$

We have two cases:

**Case 1.**  $\frac{(1 - t)(t + 3)^3}{4} \leq \frac{(5 - t^2)^2}{4}$

$$\iff (t + 1)^2(t^2 + 2t - 1) \geq 0 \iff t \in [\sqrt{2} - 1, 1),$$

so it suffices to prove that

$$(1 - t^2)(3 - t^2)^2 \leq \frac{(18 - 7\sqrt{2})(1 - t^2)}{8} \left( \frac{(5 - t^2)^2}{4} + 1 - t^2 + \frac{(1 - t^2)^2}{4} \right) + 7\sqrt{2} - 9.$$

or

$$\left[ t^2 - (\sqrt{2} - 1)^2 \right] \left[ 47t^4 - 2(125 - 9\sqrt{2})t^2 + 106\sqrt{2} + 131 \right] \geq 0,$$

clearly true for  $t \in [\sqrt{2} - 1, 1)$ .

**Case 2.**  $\frac{(1-t)(t+3)^3}{4} > \frac{(5-t^2)^2}{4}$

$$\iff (t+1)^2(t^2+2t-1) < 0 \iff t \in [0, \sqrt{2}-1),$$

so it suffices to prove that

$$(1-t^2)(3-t^2)^2 \leq \frac{(18-7\sqrt{2})(1-t^2)}{8} \left( \frac{(1-t)(t+3)^3}{4} + 1 - t^2 + \frac{(1-t^2)^2}{4} \right) + 7\sqrt{2} - 9.$$

or

$$t^2(\sqrt{2}-1-t) \left[ -4t^3 - (14-3\sqrt{2})t^2 - (6-4\sqrt{2})t + 32 - 17\sqrt{2} \right] \geq 0$$

which is true for  $t \in [0, \sqrt{2}-1)$ .

The equality holds when  $t = 0$  which means  $x = y = z = 1$  or  $t = \sqrt{2} - 1$  which means  $x = 0, y = z = \sqrt{2}$  and its cyclic permutations.

*Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India.*

S705. Prove that there are infinitely many triples  $(a, b, c)$  of distinct positive integers such that  $a^3 + b^3 + c^3 - 3abc$  is a perfect square.

*Proposed by Titu Andreescu, USA and Marian Tetiva, Romania*

*Solution 1 by the authors*

We choose

$$a = \frac{1}{3}(2m^2 - 2mn - n^2 + p^2), \quad b = a - m^2 + n^2 = \frac{1}{3}(-m^2 - 2mn + 2n^2 + p^2),$$

and

$$c = a - m^2 + 2mn = \frac{1}{3}(-m^2 + 4mn - n^2 + p^2),$$

where  $m, n, p$  are integers congruent to 1 modulo 3—which ensures that  $a, b, c$  are integers, too. Moreover, if (once  $m$  and  $n$  are chosen) we consider  $p$  to be big enough,  $a, b, c$  are, clearly, positive. Also, if we impose the conditions  $m^2 \neq n^2$ ,  $m^2 \neq 2mn$  and  $n^2 \neq 2mn$ , then any two of  $a, b, c$  are distinct - and we still remain with infinitely many such triples  $(a, b, c)$ .

We have, for such  $a, b, c$ ,

$$a + b + c = p^2$$

and

$$\begin{aligned} a^2 + b^2 + c^2 - ab - ac - bc &= (a - b)^2 - (a - b)(a - c) + (a - c)^2 = \\ &= (m^2 - n^2)^2 - (m^2 - n^2)(m^2 - 2mn) + (m^2 - 2mn)^2 = (m^2 - mn + n^2)^2. \end{aligned}$$

Consequently,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = (p(m^2 - mn + n^2))^2$$

is a perfect square for infinitely many triples  $(a, b, c)$  of mutually distinct integers, as we intended to show.

*Remark.* The problem is much easier if we do not ask for  $a, b, c$  to be distinct: the triples for which  $b = c$  (thus having the form  $(a, b, b)$ ), and for which  $a + 2b$  is a perfect square do have the required property, since, for any such triple,

$$a^3 + b^3 + c^3 - 3abc = a^3 + 2b^3 - 3ab^2 = (a + 2b)(a - b)^2$$

is obviously a perfect square.

*Solution 2 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam*

We use the classical identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Consider an arithmetic progression

$$(a, b, c) = (k - d, k, k + d) \quad (k > d \geq 1).$$

A direct computation gives

$$a + b + c = 3k,$$

$$a^2 + b^2 + c^2 - ab - bc - ca = (k - d)^2 + k^2 + (k + d)^2 - ((k - d)k + k(k + d) + (k + d)(k - d)) = 3d^2.$$

Hence

$$a^3 + b^3 + c^3 - 3abc = (3k)(3d^2) = 9kd^2.$$

If  $k$  is a square, say  $k = t^2$  with  $t \geq 2$ , then

$$9kd^2 = 9t^2d^2 = (3td)^2$$

is a perfect square. Since  $k - d, k, k + d$  are positive and pairwise distinct for  $k > d \geq 1$ , we obtain infinitely many solutions by taking

$$(a, b, c) = (t^2 - d, t^2, t^2 + d) \quad \text{with} \quad t \geq 2, 1 \leq d \leq t^2 - 1.$$

*Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Diar Gashi, University of Primorska, Slovenia; Keisha Kwok, AMMOC; Medha Shrivastava, AMMOC, India; Prasanna Pawar, AMMOC, Bengaluru, India; Sergei Makarevich, AMMOC; Taes Padhiary, IIT Guwahati, India; Winston Ra, Chattahoochee High School in Johns Creek, GA, USA.*

S706. Let  $ABC$  be a triangle with incenter  $I$  and let  $\omega$  be the center of its nine-point circle. Let  $\omega_A, \omega_B, \omega_C$  be the centers of nine-point circles of triangles  $IBC, ICA, IAB$ , respectively. Prove that

$$\overrightarrow{I\omega_A} + \overrightarrow{I\omega_B} + \overrightarrow{I\omega_C} = 2\overrightarrow{I\omega}$$

*Proposed by Mihaela Berindeanu, Bucharest, Romania*

*Solution by Theo Koupelis, Clark College, Washington, USA*

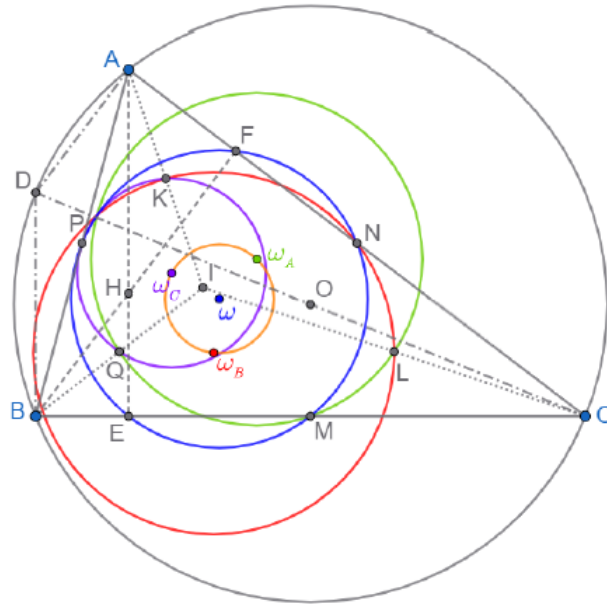
*Lemma:* In a triangle  $ABC$ , let  $H, O$  be the orthocenter and circumcenter, respectively. Then  $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HO}$ .

*Proof:* Let  $D$  be the point of intersection of the lines from vertices  $A, B$ , parallel to the lines  $BH, AH$ , respectively. We have  $DA \perp AC$ , because  $BH \perp AC$ , and  $DB \perp BC$ , because  $AH \perp BC$ . Thus,  $ADBC$  is cyclic, because  $\angle DAC = \angle DBC = 90^\circ$ ; therefore, point  $D$  is on the circumcircle of the triangle, and  $DC$  is a diameter. We have:

$$\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HD} + \overrightarrow{HC} = 2\overrightarrow{HO},$$

because  $O$  is the midpoint of the side  $DC$  of triangle  $DHC$ .

It is well-known (see, e.g., Nguyen Thanh Dung, "The Feuerbach Point and the Fuhrmann Triangle," *Forum Geometricorum*, **16**, 299-311, 2016) that the centers  $\omega_A, \omega_B, \omega_C$  are concyclic; the center of the circle  $(\omega_A\omega_B\omega_C)$  is  $\omega$ , and the point  $I$  is the orthocenter of triangle  $(\omega_A\omega_B\omega_C)$ . According to the lemma above, the desired equation holds.



*Also solved by Buzatu Rareș, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Gjimnazi i Specializuar Matematikor Group, Prishtine, Kosovo.*

S707. Prove that

$$\binom{2n+1}{1}3^n + \binom{2n+1}{3}3^{n-1} + \dots + \binom{2n+1}{2n+1}$$

is fully divisible by  $2^n$ .

*Proposed by Titu Andreescu, Dallas, USA*

*Solution 1 by Joshua Pit , Cambridge Rindge and Latin School, MA*

Let

$$S_n := \sum_{k=0}^n \binom{2n+1}{2k+1} 3^{n-k}.$$

Observing that

$$(\sqrt{3}+1)^{2n+1} - (\sqrt{3}-1)^{2n+1} = \sum_{j=0}^{2n+1} \binom{2n+1}{j} (\sqrt{3})^{2n+1-j} (1 - (-1)^j) = 2 \sum_{k=0}^n \binom{2n+1}{2k+1} 3^{n-k},$$

we get

$$2S_n = (\sqrt{3}+1)^{2n+1} - (\sqrt{3}-1)^{2n+1}.$$

Set  $a = \sqrt{3}+1$  and  $b = \sqrt{3}-1$ . Then

$$2S_n = a^{2n+1} - b^{2n+1} = a(a^2)^n - b(b^2)^n.$$

Let  $r = a^2 = 4 + 2\sqrt{3}$  and  $s = b^2 = 4 - 2\sqrt{3}$ . Any sequence  $X_n = \alpha r^n + \beta s^n$  satisfies

$$X_{n+1} = (r+s)X_n - rsX_{n-1}.$$

Here  $r+s=8$  and  $rs=4$ , so

$$2S_{n+1} = 8(2S_n) - 4(2S_{n-1}) \implies S_{n+1} = 8S_n - 4S_{n-1}.$$

We now prove by strong induction that  $2^n \mid S_n$  for all  $n \geq 0$ .

The base cases are  $S_0 = \binom{1}{1} = 1$  and  $S_1 = \binom{3}{1}3 + \binom{3}{3} = 10$ , divisible by 2.

Assume  $2^n \mid S_n$  and  $2^{n-1} \mid S_{n-1}$ . Since  $8S_n$  is divisible by  $2^{n+3}$  and  $4S_{n-1}$  by  $2^{n+1}$ , their difference is divisible by  $2^{n+1}$ . Thus  $2^{n+1} \mid S_{n+1}$ , completing the induction, and this completes the proof.

*Solution 2 by Anita Tufisi, Romania*

Consider the identity

$$(x+1)^{2n+1} - (x-1)^{2n+1} = 2 \sum_{k=0}^n \binom{2n+1}{2k+1} x^{2n-2k}.$$

For  $x = \sqrt{3}$ ,

$$2S = (\sqrt{3}+1)^{2n+1} - (\sqrt{3}-1)^{2n+1}.$$

Let  $a = \sqrt{3}+1$ , and  $b = \sqrt{3}-1$ . Then

$$S = \frac{a^{2n+1} - b^{2n+1}}{2} = \frac{1}{2}(a-b) \sum_{j=0}^{2n} a^{2n-j} b^j = \sum_{j=0}^{2n} a^{2n-j} b^j.$$

Note that  $ab = (\sqrt{3}+1)(\sqrt{3}-1) = 2$ , hence

$$a^{2n-j} b^j = (ab)^n a^{n-j} b^{j-n} = 2^n a^{n-j} b^{j-n}.$$

Thus

$$S = 2^n \cdot U, \quad U = \sum_{j=0}^{2n} a^{n-j} b^{j-n} = \sum_{k=-n}^n r^k,$$

where  $r = \frac{b}{a} = 2 - \sqrt{3}$ . Since  $r$  is an algebraic integer (indeed a unit in  $\mathbb{Z}[\sqrt{3}]$ ), each  $r^k$  is an algebraic integer. Hence,  $U$  is an algebraic integer. Moreover, the conjugation  $\sqrt{3} \mapsto -\sqrt{3}$  sends  $r \mapsto r' = r^{-1}$ , so

$$U' = \sum_{k=-n}^n (r')^k = \sum_{k=-n}^n r^{-k} = U,$$

i.e.  $U$  is fixed by the Galois action and thus  $U \in \mathbb{Q}$ .

Being both a rational number and an algebraic integer,  $U \in \mathbb{Z}$ . Therefore,  $S = 2^n \cdot U$ , with  $U \in \mathbb{Z}$ , so  $2^n \mid S$ .

*Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; G. C. Greubel, Newport News, VA, USA; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Madani Harouach, AMMOC; Sergei Makarevich, AMMOC.*

S708. Let  $a, b, c$  be positive real numbers such that at most one of them is less than 1 and  $a + b + c = 3$ .  
Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 6(ab + bc + ca) \geq 21.$$

*Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania*

*Solution 1 by the author*

Without loss of generality, assume that

$$a \geq b \geq 1 \geq c.$$

We have

$$b + c \leq \frac{2(a + b + c)}{3} = 2,$$

hence

$$bc \leq \frac{1}{4}(b + c)^2 \leq 1.$$

For fixed  $a$ , we may consider that  $c$  is a function of  $b$ , namely  $c(b) = 3 - a - b$ . Since  $c' = -1$ , writing the desired inequality as  $F(b) \geq 21$ , we have

$$\begin{aligned} \frac{1}{2}F'(b) &= \frac{-1}{b^3} + 3a + 3c + \left(\frac{-1}{c^3} + 3a + 3b\right)c' \\ &= \frac{1}{c^3} - \frac{1}{b^3} - 3(b - c) = (b - c) \left(\frac{b^2 + bc + c^2}{b^3c^3} - 3\right) \\ &\geq (b - c) \left(\frac{b^2 + bc + c^2}{bc} - 3\right) = \frac{(b - c)^3}{bc} \geq 0. \end{aligned}$$

Therefore,  $F(b)$  is increasing and has the minimum value when  $b$  is minimum, hence when  $b = 1$ . So, it suffices to prove the desired inequality for  $b = 1$ . We need to show that  $a + c = 2$  involves

$$\frac{1}{a^2} + \frac{1}{c^2} + 6(a + c + ac) \geq 20.$$

Denoting  $p = ac$ , the desired inequality becomes as follows:

$$\frac{2 - p}{p^2} + 3p \geq 4, \quad 3p^3 - 4p^2 - p + 2 \geq 0,$$

$$(p - 1)^2(3p + 2) \geq 0.$$

The equality occurs for  $a = b = c = 1$ .

*Solution 2 by Huseyin Yigit Emekci, Izmir, Turkey*

Without loss of generality, let  $a \geq b \geq 1 \geq c$ . Then

$$(a - 1)(b - 1)(1 - c) \geq 0.$$

Since

$$ab + bc + ca - (abc + 2) = (a - 1)(b - 1)(1 - c),$$

then  $ab + bc + ca \geq abc + 2$ . Note that

$$\begin{aligned} \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 6(ab + bc + ca) &= \frac{(ab + bc + ca)^2 - 2abc(a + b + c)}{a^2b^2c^2} + 6(ab + bc + ca) \\ &\geq \frac{(abc + 2)^2}{a^2b^2c^2} - \frac{6}{abc} + 6abc + 12. \end{aligned}$$

Hence, it suffices to show

$$\frac{(abc + 2)^2}{a^2b^2c^2} - \frac{6}{abc} + 6abc \geq 9,$$

that is,

$$\frac{6a^3b^3c^3 + a^2b^2c^2 - 2abc + 4}{a^2b^2c^2} \geq 9$$

which is equivalent to  $2(abc - 1)^2(3abc + 2) \geq 0$  which is true for  $0 < abc \leq 1$ . But by the given condition and the AM-GM Inequality we have  $abc \leq 1$ . The proof is completed. Equality occurs for  $a = b = c = 1$ .

*Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA.*

## Undergraduate problems

U703. Let  $S_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+1}$ ,  $n \geq 1$ . Evaluate  $\sum_{k=1}^{\infty} \frac{S_k}{k}$ .

*Proposed by Alessandro Ventullo, Milan, Italy*

*Solution 1 by Daniel Pascuas, Barcelona, Spain*

The binomial theorem shows that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} t^k = (1-t)^n \quad (t \in \mathbb{R}),$$

and the integration of this identity between 0 and 1 gives that

$$S_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^1 t^k dt = \int_0^1 (1-t)^n dt = \left[ -\frac{(1-t)^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}.$$

Therefore  $\sum_{k=1}^{\infty} \frac{S_k}{k} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$  and hence

$$\sum_{k=1}^{\infty} \frac{S_k}{k} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) = 1.$$

*Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

First we prove that  $S_n = \frac{1}{n+1}$ , or equivalently  $(n+1)S_n = 1$ . We have

$$(n+1)S_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{n+1}{k+1} = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} = - \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} = 1 - (1-1)^{n+1} = 1.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{S_k}{k} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) \\ &= \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) \\ &= 1. \end{aligned}$$

and we are done.

*Editor's Comments.* Kousik Sett observed that the series given by  $S_n$  is a particular case of a more general form given below.

$$\frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \cdots + (-1)^n \frac{\binom{n}{n}}{x+n} = \frac{n!}{x(x+1)(x+2) \cdots (x+n)}.$$

For  $x = 1$ , we get  $S_n$ .

*Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Danko Rusan Ramirez Shareyeva, Drexel University, Philadelphia, PA, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Gjimnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Henry Ricardo, Westchester Area Math Circle, New York, USA; Joshua Pit , Cambridge Rindge and Latin School, MA, USA; Kousik Sett, India; Nicu or Zlota, “Traian Vuia” Technical College, Foc ani, Romania; Se n M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Ayaan Alam, IIT Delhi, India; Diar Gashi, University of Primorska, Slovenia; Madani Harouach, AMMOC; Hoang-Bao Le, Dublin, Ireland; Keisha Kwok, AMMOC; Sergei Makarevich, AMMOC; Taes Padhiary, IIT Guwahati, India.*

U704. Let  $\alpha > 0$ . Evaluate

$$\lim_{n \rightarrow \infty} \int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx.$$

*Proposed Alessandro Ventullo, Milan, Italy*

*Solution 1 by the author*

Let  $f(x) = x^{2n}$  and  $g(x) = \frac{1}{(n+1)^x + 1}$ . Clearly,  $f$  is continuous and even and  $g(x) + g(-x) = 1$ . Let

$$g_1(x) = \frac{g(x) + g(-x)}{2}, \quad g_2(x) = \frac{g(x) - g(-x)}{2}.$$

It is easy to see that  $g_1$  is even and  $g_2$  is odd and  $g(x) = g_1(x) + g_2(x)$ . So,

$$\int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx = \int_{-\alpha}^{\alpha} f(x)g(x) dx = \int_{-\alpha}^{\alpha} f(x)g_1(x) dx + \int_{-\alpha}^{\alpha} f(x)g_2(x) dx.$$

Since  $f g_2$  is odd and we are integrating in  $[-\alpha, \alpha]$ , it follows that the second integral is zero. So,

$$\int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx = \int_{-\alpha}^{\alpha} f(x)g_1(x) dx = \int_{-\alpha}^{\alpha} \frac{1}{2} x^{2n} dx = \int_0^{\alpha} x^{2n} dx = \frac{\alpha^{2n+1}}{2n+1}.$$

Thus,

$$\lim_{n \rightarrow \infty} \int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx = \begin{cases} 0 & \text{if } 0 < \alpha \leq 1 \\ +\infty & \text{if } \alpha > 1. \end{cases}$$

*Solution 2 by Henry Ricardo, Westchester Area Math Circle, New York, USA*

We have

$$\begin{aligned} \int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx &= \int_{-\alpha}^0 \frac{x^{2n}}{(n+1)^x + 1} dx + \int_0^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx \\ &\stackrel{x \rightarrow -u}{=} \int_0^{\alpha} \frac{(n+1)^u u^{2n}}{(n+1)^u + 1} du + \int_0^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx \\ &= \int_0^{\alpha} \frac{[(n+1)^x + 1] x^{2n}}{(n+1)^x + 1} dx \\ &= \int_0^{\alpha} x^{2n} dx = \frac{\alpha^{2n+1}}{2n+1}. \end{aligned}$$

Now we see that

$$\lim_{n \rightarrow \infty} \int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx = \lim_{n \rightarrow \infty} \frac{\alpha^{2n+1}}{2n+1} = \begin{cases} 0 & \text{if } 0 < \alpha \leq 1 \\ \infty & \text{if } \alpha > 1. \end{cases}$$

*Solution 3 by Joshua Pit  , Cambridge Rindge and Latin School, MA, USA*

$$\text{Let } I_n(\alpha) := \int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx$$

Assume  $0 < \alpha \leq 1$ . For  $x \in [-\alpha, \alpha]$  we have  $0 \leq x^{2n} \leq 1$  and  $(n+1)^x + 1 \geq 1$ , hence

$$\forall x \in [-\alpha, \alpha], \forall n \in \mathbb{N}, \quad 0 \leq \frac{x^{2n}}{(n+1)^x + 1} \leq 1.$$

For every  $x \in (-\alpha, \alpha)$ ,  $|x| < 1$  so  $x^{2n} \rightarrow 0$ , and at  $x = 1$  (only present when  $\alpha = 1$ ) we have  $\frac{1}{(n+1)+1} \rightarrow 0$ . Thus the integrand converges to 0 almost everywhere on  $[-\alpha, \alpha]$ , and it is dominated by the integrable function 1. By the Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} I_n(\alpha) = \int_{-\alpha}^{\alpha} 0 \, dx = 0.$$

Assume  $\alpha > 1$ . Set  $\varepsilon := \frac{\alpha-1}{2} > 0$  and consider the subinterval  $J = [1 + \varepsilon, 1 + 2\varepsilon] \subset (1, \alpha]$ . For  $x \in J$  we have  $x^{2n} \geq (1 + \varepsilon)^{2n}$  and  $(n+1)^x + 1 \leq (n+1)^\alpha + 1 \leq 2(n+1)^\alpha$  (for  $n \geq 1$ ). Hence

$$I_n(\alpha) \geq \int_J \frac{x^{2n}}{(n+1)^x + 1} \, dx \geq |J| \cdot \frac{(1 + \varepsilon)^{2n}}{2(n+1)^\alpha} = \varepsilon \frac{(1 + \varepsilon)^{2n}}{2(n+1)^\alpha}.$$

Therefore  $\lim_{n \rightarrow \infty} I_n(\alpha) = +\infty$ .

We proved that

$$\lim_{n \rightarrow \infty} I_n(\alpha) = \begin{cases} 0, & 0 < \alpha \leq 1, \\ +\infty, & \alpha > 1. \end{cases}$$

*Solution 4 by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia*

Denote the limit to be found by  $L$ . We start by observing that

$$\frac{1}{(n+1)^x + 1} = \frac{1}{e^{\log(n+1)x} + 1}.$$

Now the Euler polynomials  $E_n(x)$ , where  $n$  is a non-negative integer, are defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Setting  $x = 0$  and  $t = \log(n+1)x$  in this generating function, we find

$$\frac{2}{e^{\log(n+1)x} + 1} = \sum_{k=0}^{\infty} E_k(0) \frac{\log^k(n+1)}{k!} x^k,$$

or

$$\frac{1}{(n+1)^x + 1} = \frac{1}{2} E_0(0) + \frac{1}{2} \sum_{k=1}^{\infty} E_k(0) \frac{\log^k(n+1)}{k!} x^k. \quad (1)$$

From properties of the Euler polynomials at zero argument [see, for example, Section 20.7 on page 184 of: K. Oldham, J. Myland, and J. Spanier, *An Atlas of Functions*, 2nd ed. (Springer, New York, 2009)], one has

$$E_0(0) = 1 \quad \text{and} \quad E_k(0) = -\frac{2(2^{k+1} - 1)B_{k+1}}{k+1}.$$

Here  $k$  is a positive integer while  $B_k$  are the Bernoulli numbers. Eq. (1) may therefore be rewritten as

$$\frac{1}{(n+1)^x + 1} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{(2^{k+1} - 1)B_{k+1} \log^k(n+1)}{(k+1)k!} x^k.$$

Since  $B_k = 0$  for all odd  $k > 1$ , terms in the sum will be zero when  $k$  is even. Reindexing the sum by  $k \mapsto 2k - 1$ , we have

$$\frac{1}{(n+1)^x + 1} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)B_{2k} \log^{2k-1}(n+1)}{2k(2k)!} x^{2k-1}.$$

For the integral that appears in the limit  $L$ , this may be written as

$$\int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx = \frac{1}{2} \int_{-\alpha}^{\alpha} x^{2n} dx - \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) B_{2k} \log^{2k-1}(n+1)}{2k(2k)!} \int_{-\alpha}^{\alpha} x^{2n+2k-1} dx.$$

As the integrand in the right most integral is odd between symmetric limits, it evaluates to zero. The remaining integral is elementary. Thus

$$\int_{-\alpha}^{\alpha} \frac{x^{2n}}{(n+1)^x + 1} dx = \frac{\alpha^{2n+1}}{2n+1}.$$

So for the limit  $L$ , as  $\alpha > 0$ , we immediately see that

$$L = \lim_{n \rightarrow \infty} \frac{\alpha^{2n+1}}{2n+1} = \begin{cases} 0, & 0 < \alpha \leq 1, \\ \infty, & \alpha > 1. \end{cases}$$

*Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Ayaan Alam, IIT Delhi, India; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Taes Padhiary, IIT Guwahati, India.*

U705. Let  $a, b, c$  be positive real numbers such that at most one of them is less than 1 and  $a^5 + b^5 + c^5 = 3$ .  
Prove that

$$(a^2 + b^2 + c^2) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

*Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania*

*Solution by the author*

Assume that

$$a \geq b \geq 1 \geq c,$$

and denote by  $F$  the left hand side of the inequality. For fixed  $a$ , we may assume that  $c$  and  $F$  are functions of  $b$ . By differentiating the equality constraint, we get

$$b^4 + c^4 c' = 0, \quad c' = -b^4 c^{-4},$$

therefore

$$\begin{aligned} F'(b) &= 2(b + cc') \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + (a^2 + b^2 + c^2) \left( \frac{-1}{b^2} + \frac{-c'}{c^2} \right) \\ &= 2(b - b^4 c^{-3}) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + (a^2 + b^2 + c^2) \left( \frac{-1}{b^2} + b^4 c^{-6} \right). \end{aligned}$$

We will show that  $F'(b) \geq 0$ . Denoting  $x = \frac{b}{c}$ , we get

$$F'(b) = -2c(x^4 - x) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + b^{-2}(a^2 + b^2 + c^2)(x^6 - 1).$$

Since  $x \geq 1$  and  $a \geq b$ , we have

$$\begin{aligned} F'(b) &\geq -2c(x^4 - x) \left( \frac{2}{b} + \frac{1}{c} \right) + b^{-2}(2b^2 + c^2)(x^6 - 1) \\ &= -2(x^4 - x) \left( \frac{2}{x} + 1 \right) + \left( 2 + \frac{1}{x^2} \right) (x^6 - 1) \\ &= \frac{(x^3 - 1)(2x^5 - x^3 - 2x^2 + 1)}{x^2} = \frac{(x^3 - 1)(x - 1)(2x^4 + 2x^3 + x^2 - x - 1)}{x^2} \geq 0. \end{aligned}$$

From  $F'(b) \geq 0$ , it follows that  $F(b)$  is increasing and has the minimum value when  $b$  is minimum, hence when  $b = 1$ . Thus, it suffices to prove the desired inequality for  $b = 1$ . We need to show that

$$(a^2 + c^2 + 1) \left( \frac{1}{a} + \frac{1}{c} + 1 \right) \geq 9$$

for  $a \geq 1 \geq c > 0$  such that  $a^5 + c^5 = 2$ . Let  $S = \frac{a+c}{2} > \frac{1}{2}$  and  $p = ac$ . From

$$2 = a^5 + c^5 = (a^2 + c^2)(a^3 + c^3) - a^2 c^2 (a + c) = (4S^2 - 2p)(8S^3 - 6Sp) - 2Sp^2,$$

we obtain

$$5Sp^2 = 20S^3p - 16S^5 + 1, \quad p = 2S^2 - \sqrt{\frac{4S^5 + 1}{5S}}.$$

We claim that

$$p \leq \frac{5S^2 - 1}{4},$$

that is

$$\sqrt{\frac{4S^5 + 1}{5S}} \geq \frac{3S^2 + 1}{4}.$$

By squaring, the inequality becomes

$$16(4S^5 + 1) \geq 5S(3S^2 + 1)^2, \quad 19S^5 - 30S^3 - 5S + 16 \geq 0,$$

$$(S - 1)^2(19S^3 + 38S^2 + 27S + 16) \geq 0.$$

Write now the desired inequality as follows:

$$(4S^2 - 2p + 1) \left( \frac{2S}{p} + 1 \right) \geq 9,$$

$$p^2 + 2(2 + S - S^2)p - S(4S^2 + 1) \leq 0, \quad 5Sp^2 + 10S(2 + S - S^2)p - 5S^2(4S^2 + 1) \leq 0,$$

$$(20S^3p - 16S^5 + 1) + 10S(2 + S - S^2)p - 5S^2(4S^2 + 1) \leq 0,$$

$$16S^5 + 20S^4 + 5S^2 - 1 \geq 10S(S^2 + S + 2)p.$$

Since  $p \leq (5S^2 - 1)/4$ , we have

$$16S^5 + 20S^4 + 5S^2 - 1 - 10S(S^2 + S + 2)p \geq 16S^5 + 20S^4 + 5S^2 - 1 - \frac{5S(S^2 + S + 2)(5S^2 - 1)}{2}$$

$$= \frac{7S^5 + 15S^4 - 45S^3 + 15S^2 + 10S - 2}{2} = \frac{(S - 1)^2(7S^3 + 29S^2 + 6S - 2)}{2} \geq 0.$$

So, the proof is completed. The equality occurs when  $a = b = c = 1$ .

*Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA; Keisha Kwok, AMMOC; Madani Harouach, AMMOC Parth Andhare, AMMOC.*

U706. Let  $(F_n)_{n \geq 1}$  be the Fibonacci sequence defined by  $F_1 = 1$ ,  $F_2 = 1$ , and  $F_{n+2} = F_{n+1} + F_n$ , for  $n \geq 1$ . Evaluate

$$\int \sqrt{x^{F_1} \sqrt{x^{F_2} \sqrt{x^{F_3} \sqrt{x^{F_4} \dots}}} dx.$$

*Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

*Solution 1 by Joshua Pité, Cambridge Rindge and Latin School, MA, USA*

$$\text{Set } f(x) = \sqrt{x^{F_1} \sqrt{x^{F_2} \sqrt{x^{F_3} \sqrt{x^{F_4} \dots}}}} \quad (x \geq 0).$$

Let  $\varphi = \frac{1 + \sqrt{5}}{2}$ . Since  $\varphi < 2$ , the series  $\sum_{k=1}^{\infty} \frac{F_k}{2^k}$  converges. Let  $S = \sum_{k=1}^{\infty} \frac{F_k}{2^k}$ . The factor  $x^{F_k}$  sits under  $k$

successive square roots, contributing exponent  $F_k/2^k$ , hence  $f(x) = x^S$ .

Using  $F_{k+2} = F_{k+1} + F_k$  and shifting indices,

$$\begin{aligned} S - \frac{F_1}{2} - \frac{F_2}{4} &= \sum_{k=3}^{\infty} \frac{F_k}{2^k} = \sum_{k=1}^{\infty} \frac{F_{k+2}}{2^{k+2}} \\ &= \sum_{k=1}^{\infty} \frac{F_{k+1}}{2^{k+2}} + \sum_{k=1}^{\infty} \frac{F_k}{2^{k+2}} = \frac{1}{2} \left( \sum_{k=1}^{\infty} \frac{F_{k+1}}{2^{k+1}} \right) + \frac{1}{4} S \\ &= \frac{1}{2} \left( S - \frac{F_1}{2} \right) + \frac{1}{4} S. \end{aligned}$$

With  $F_1 = F_2 = 1$  this gives  $S - \frac{3}{4} = \frac{3}{4}S - \frac{1}{4}$ , hence  $\frac{1}{4}S = \frac{1}{2}$  and  $S = 2$ . Therefore  $f(x) = x^2$  and

$$\int \sqrt{x^{F_1} \sqrt{x^{F_2} \sqrt{x^{F_3} \sqrt{x^{F_4} \dots}}} dx = \int x^2 dx = \frac{x^3}{3} + C.$$

*Solution 2 by Diar Gashi, University of Primorska, Slovenia*

Notice that  $\forall x \in \mathbb{R}$ ,

$$\sqrt{x^{F_1} \sqrt{x^{F_2} \sqrt{x^{F_3} \sqrt{x^{F_4} \dots}}}} = x^{\sum_{k=1}^{\infty} \frac{F_k}{2^k}}.$$

And on the other hand, if the generating function for  $F_k$  is

$$G(x) = \sum_{k=1}^{\infty} F_k \cdot x^k,$$

then, by the properties of Fibonacci sequence,

$$\begin{aligned} G(x) - xG(x) - x^2G(x) &= \sum_{k=1}^{\infty} F_k \cdot x^k - \sum_{k=2}^{\infty} F_{k-1} \cdot x^k - \sum_{k=3}^{\infty} F_{k-2} \cdot x^k \\ &= F_1x + F_2x^2 - F_1x^2 + \sum_{k=3}^{\infty} (F_k - F_{k-1} - F_{k-2}) \cdot x^k \\ &= x. \end{aligned}$$

Hence,

$$\sum_{k=1}^{\infty} F_k \cdot x^k = G(x) = \frac{x}{1 - x - x^2},$$

and plugging in  $x = \frac{1}{2}$ , we get

$$\sum_{k=1}^{\infty} \frac{F_k}{2^k} = \frac{\frac{1}{2}}{1 - \frac{1}{2} - (\frac{1}{2})^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2.$$

Hence the value of the integral is

$$\int x^2 dx = \frac{x^3}{3} + C.$$

*Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Devis Alvarado, UNAH y UP-NFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Kousik Sett, India; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Ayaan Alam, IIT Delhi, India; Sergei Makarevich, AMMOC; Taes Padhihary, IIT Guwahati, India.*

U707. Let  $p$  be a prime. Find all  $p$ -groups of order less than 60 and with exactly 4 automorphisms.

*Proposed by Alessandro Ventullo, Milan, Italy*

*Solution by the author*

Let  $G$  be a finite group such that  $|\text{Aut}(G)| = 4$ , where  $\text{Aut}(G)$  is the group of automorphisms of  $G$ . First, if  $G$  is abelian, by the Fundamental Theorem of Finite Abelian Groups, then  $G \simeq \mathbb{Z}_{p^{e_1}} \times \mathbb{Z}_{p^{e_2}} \times \dots \times \mathbb{Z}_{p^{e_n}}$ , where  $1 \leq e_1 \leq e_2 \leq \dots \leq e_n$ . If  $n = 1$ , we can easily see that

$$|\text{Aut}(G)| = |\text{Aut}(\mathbb{Z}_{p^{e_1}})| = \varphi(p^{e_1}) = p^{e_1-1}(p-1) = 4,$$

so either  $p = 2$  and  $e_1 = 3$  or  $p = 5$  and  $e_1 = 1$ , which gives  $G \simeq \mathbb{Z}_5$  or  $G \simeq \mathbb{Z}_8$ .

To compute the size of the automorphism group in the more general case, we can use the Hillar-Rhea formula<sup>1</sup>

$$|\text{Aut}(G)| = \prod_{k=1}^n (p^{d_k} - p^{k-1}) \prod_{j=1}^n (p^{e_j})^{n-d_j} \prod_{i=1}^n (p^{e_i-1})^{n-c_i+1}, \quad (1)$$

where  $d_k = \max\{l : e_l = e_k\}$  and  $c_k = \min\{l : e_l = e_k\}$ . Since  $d_k \geq k$ , then if  $n \geq 2$  we have that

$\prod_{k=1}^n (p^{d_k} - p^{k-1})$  is divisible by  $p$ . So, for  $n \geq 2$  we must have  $p = 2$  and since  $|G| < 60$ , then  $n \leq 5$ . If  $n \geq 3$ ,

then  $\prod_{k=1}^n (2^{d_k} - 2^{k-1})$  is divisible by  $(2^{d_2} - 2)(2^{d_3} - 2^2) = 8(2^{d_2-1} - 1)(2^{d_3-2} - 1)$ . So,  $n = 2$  and from (1) we get

$$|\text{Aut}(G)| = \prod_{k=1}^2 (2^{d_k} - 2^{k-1}) \prod_{j=1}^2 (2^{e_j})^{2-d_j} \prod_{i=1}^2 (2^{e_i-1})^{2-c_i+1}.$$

If  $e_1 = e_2$ , then  $d_1 = 2$  and  $|\text{Aut}(G)|$  would be divisible by  $2^2 - 2^0 = 3$ , impossible. So,  $e_1 < e_2$  and we need to test  $(e_1, e_2) \in \{(1, 2), (1, 3), (1, 4), (2, 3)\}$ . We see that

$$|\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_4)| = 8, \quad |\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_8)| = 16,$$

$$|\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_{16})| = 32, \quad |\text{Aut}(\mathbb{Z}_4 \times \mathbb{Z}_8)| = 128,$$

so we have no solutions for  $n = 2$ .

Now, let  $Z(G)$  be the center of  $G$  and let  $\text{Inn}(G)$  be the group of the inner automorphisms of  $G$ . If  $G$  is not abelian,  $G/Z(G)$  is not cyclic and since  $G/Z(G) \simeq \text{Inn}(G)$ , it follows that  $|\text{Inn}(G)| \geq 4$ . Since  $\text{Inn}(G)$  is a subgroup of  $\text{Aut}(G)$ , it follows that  $\text{Inn}(G) = \text{Aut}(G)$ . Since  $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G) = \{1\}$  and since by Gaschutz's Theorem every  $p$ -group has nontrivial outer automorphism group, we have no solutions in this case.

## Remark

The problem becomes pretty hard if we are interested in finding all finite groups with exactly 4 automorphisms. We can still invoke Gaschutz's Theorem for the  $p$ -groups, but we have to use more complicated ideas for that. For example, we could use that every prime divisor  $p$  of  $|G|$  is at most 5 and we could use a boundary for the order of the derived group  $G'$  (see the article by Benjamin Sambale - *On a theorem of Ledermann and Neumann*<sup>2</sup>).

<sup>1</sup>Hillar, C. J. and Rhea, D. L. - *Automorphisms of finite abelian groups*, The American Mathematical Monthly, 114(10):917-923 (2007)

<sup>2</sup><https://arxiv.org/abs/1909.13220>

U708. Prove that

$$\frac{\sin x}{x} + \frac{(\sin x)^4}{x^4} \geq 2 \cos x + (\cos x)^2 (\sin x)^4$$

for all  $0 < x \leq \pi/2$ .

*Proposed by Paolo Perfetti, Università di "Tor Vergata", Rome, Italy*

*Solution by Theo Koupeles, Clark College, Washington, USA*

The desired inequality holds as an equality when  $x = 0$  and as a strict inequality when  $x = \pi/2$ . Let  $0 < x < \pi/2$ .

(i) Let  $1 \leq x < \pi/2$ , where  $x$  is in radians. It is well-known (see, for example, the solution to problem U666) that  $\sin x \geq x \cdot \cos^{1/3} x$ , for all  $0 \leq x \leq \pi/2$ . Setting  $t = \cos^{1/3} x$ , where  $0 < t \leq \cos^{1/3} 1 < 0.82$ , it suffices to show that  $t + t^4 \geq 2t^3 + t^6(1 - t^6)^2$ , or after expanding and dividing by  $t(1 - t) > 0$ ,

$$S(t) := \sum_{n=5}^{10} (t^{n+6} - t^n) + 1 + t(1 - t) \geq 0.$$

But

$$S(t) = (1 - t)^2(t^{14} + 3t^{13} + 6t^{12} + 10t^{11}) + (1 - t)^4(15t^8 + 51t^7 + 113t^6) + s_1(t) + s_2(t),$$

where  $s_1(t) = 205t^9 - 490t^8 + 400t^7 - 114t^6 + 0.75$ , and  $s_2(t) = 0.25 + t - t^2 - t^5$ . We get  $s'_2(t) = 1 - 2t - 5t^4$ ,  $s''_2(t) = -2 - 20t^3 < 0$ ,  $s_2(0) = 0.25 > 0$ , and  $s_2(0.82) \approx 0.027 > 0$ ; thus,  $s_2(t) > 0$  for all  $0 < t < 0.82$ . Also,  $s'_1(t) = t^5 p(t)$ , where  $p(t) = 1845t^3 - 3920t^2 + 2800t - 684$ . The discriminant of the cubic  $p(t)$  is negative, and thus it has one real root. Using Cardano's method we find that  $p(t_o) = 0$  where  $t_o \approx 0.87$ . Also,  $p(0) = -684 < 0$  and  $p(0.82) \approx -6.53 < 0$ . Thus,  $s'_1(t) < 0$  for  $0 < t < 0.82$ . We also have  $s_1(0) = 0.75 > 0$  and  $s_1(0.82) \approx 0.0065 > 0$ . Thus,  $s_1(t) > 0$ . Therefore,  $S(t) > 0$  for all  $0 < t \leq \cos^{1/3} 1 < 0.82$ .

(ii) Let  $0 < x < 1$ . Using the Taylor series expansion of  $\sin x$  and  $\cos x$  it suffices to show that

$$\begin{aligned} \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right) + \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right)^4 &\geq 2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \\ &+ \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^2 \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^4. \end{aligned}$$

Expanding and rearranging we get

$$\begin{aligned} &x^2(1 - x^2) \left[ 21609x^{24} + x^{20}(105646397 - 2225727x^2) \right. \\ &\quad + x^{16}(59213985533 - 3046155235x^2) \\ &\quad + x^{12}(8070520289597 - 814049420419x^2) \\ &\quad \left. + x^8(291538341058877 - 57685280747203x^2) \right] \\ &+ x^2 \left[ 995575768688323 \left(x^2 - \frac{1}{2}\right)^4 + 1094746191237754(1 - x^2) \left(x^2 - \frac{1}{2}\right)^2 \right. \\ &\quad \left. + \frac{1}{16} \cdot (8406084760768120x^4 - 6273981849098336x^2 + 1508013147000661) \right] \geq 0, \end{aligned}$$

which is obvious for all  $0 < x < 1$  as each quantity in parenthesis is nonnegative.

*Also solved by Joshua Pit , Cambridge Rindge and Latin School, MA, USA; Nicu or Zlota, "Traian Vuia" Technical College, Foc ani, Romania.*

## Olympiad problems

O703. Let  $ABC$  be an acute triangle. Prove that

$$\frac{\cos A}{\cos(B-C)} + \frac{\cos B}{\cos(C-A)} + \frac{\cos C}{\cos(A-B)} \geq 1 + \frac{2}{3}(\cos^2 A + \cos^2 B + \cos^2 C).$$

*Proposed by Titu Andreescu, USA and Marius Stănean, Romania*

*Solution 1 by the authors*

Denote  $a = 2 \cos A$ ,  $b = 2 \cos B$ ,  $c = 2 \cos C$ , then  $a, b, c \geq 0$  and  $a^2 + b^2 + c^2 + abc = 4$ . The inequality can be rewritten as

$$\sum_{cyc} \frac{2a}{bc + \sqrt{(4-b^2)(4-c^2)}} \geq 1 + \frac{a^2 + b^2 + c^2}{6},$$

or considering that

$$\begin{aligned} \sqrt{(4-b^2)(4-c^2)} &= \sqrt{16 - 4(b^2 + c^2) + b^2c^2} \\ &= \sqrt{4a^2 + 4abc + b^2c^2} = 2a + bc, \end{aligned}$$

it becomes

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{c}{c+ab} \geq \frac{5}{3} - \frac{abc}{6}.$$

With the substitutions

$$a = \sqrt{\frac{4yz}{(x+y)(z+x)}}, \quad b = \sqrt{\frac{4zx}{(y+z)(x+y)}}, \quad c = \sqrt{\frac{4xy}{(z+x)(y+z)}},$$

the inequality becomes

$$\frac{y+z}{2x+y+z} + \frac{z+x}{x+2y+z} + \frac{x+y}{x+y+2z} \geq \frac{5}{3} - \frac{4xyz}{3(x+y)(y+z)(z+x)}.$$

Assuming  $x+y+z=3$  (due homogeneity of the inequality) and denoting  $q = xy + yz + zx$ ,  $r = xyz$  we need to show that

$$\frac{3(36-q-r)}{54+3q+r} - \frac{5}{3} + \frac{4r}{9q-3r} \geq 0.$$

Clearing denominators and expanding, it becomes

$$3r^2 + (27-q)r + 27q - 12q^2 \geq 0.$$

Setting  $q = 3(1-t^2)$  where  $t \in [0, 1)$ , we need to show that

$$r^2 + (8+t^2)r + 9(1-t^2)(4t^2-1) \geq 0.$$

The inequality is true for  $2t > 1$ . If  $2t \leq 1$  then we notice that  $r \geq (1-2t)(1+t)^2$ , so it suffices to prove that

$$(1-2t)^2(1+t)^4 + (8+t^2)(1-2t)(1+t)^2 + 9(1-t^2)(4t^2-1) \geq 0,$$

which is

$$2t^2(1-2t)(2-t)(t+1)(t+4) \geq 0,$$

clearly true. The equality holds when  $x = y = z$  which means  $A = B = C = 60^\circ$ .

*Solution 2 by Theo Koupelis, Clark College, Washington, USA*

We have

$$\frac{\cos A}{\cos(B-C)} = -\frac{\cos(B+C)}{\cos(B-C)} = \frac{\tan B \tan C - 1}{\tan B \tan C + 1},$$

and cyclic expressions. Let  $(x, y, z) = (\tan A, \tan B, \tan C)$ , where, for an acute triangle,  $x, y, z$  are positive, real numbers satisfying the well-known expression  $x + y + z = xyz$ . The desired inequality is equivalent to

$$\frac{xy-1}{xy+1} + \frac{yz-1}{yz+1} + \frac{zx-1}{zx+1} - 1 \geq \frac{2}{3} \left( \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \right).$$

Consolidating we get

$$\frac{2x^2y^2z^2 - 2(xy + yz + zx) - 4}{x^2y^2z^2 + xyz(x + y + z) + (xy + yz + zx) + 1} \geq \frac{2}{3} \cdot \frac{x^2y^2 + y^2z^2 + z^2x^2 + 2(x^2 + y^2 + z^2) + 3}{x^2y^2z^2 + x^2y^2 + y^2z^2 + z^2x^2 + x^2 + y^2 + z^2 + 1}.$$

Let  $p = x + y + z$ ,  $q = xy + yz + zx$ , and  $r = xyz = p$ . Using AM-GM we get  $r = p \geq 3r^{1/3}$ , and thus  $r \geq 3\sqrt{3}$  and  $q \geq 3r^{2/3} \geq 9$ . Substituting into the above equation we get

$$\frac{2p^2 - 2q - 4}{2p^2 + q + 1} \geq \frac{2}{3} \cdot \frac{q^2 - 2pr + 2(p^2 - 2q) + 3}{p^2 + (q^2 - 2pr) + (p^2 - 2q) + 1} \iff \frac{p^2 - q - 2}{2p^2 + q + 1} \geq \frac{1}{3} \cdot \frac{q - 3}{q - 1}.$$

Clearing denominators we get

$$p^2(q + 3) - 4q^2 - q + 9 \geq 0.$$

From Schur's inequality we get

$$p = r \geq \frac{4pq - p^3}{9} \implies p^2 \geq 4q - 9.$$

Thus, it suffices to show that

$$(4q - 9)(q + 3) - 4q^2 - q + 9 \geq 0 \iff q \geq 9,$$

which is obvious. Equality occurs for an equilateral triangle.

*Also solved by Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Hoang-Bao Le, Dublin, Ireland.*

O704. Find all positive integers  $n$  such that  $\sqrt{n^3+1} + \sqrt{n+2}$  is an integer.

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution 1 by Joshua Pit  , Cambridge Rindge and Latin School, MA, USA*

Observe that  $n = 2$  is a solution. We show this is the only one. Set

$$k = \sqrt{n^3+1} + \sqrt{n+2}$$

and assume that  $k$  is an integer.

Squaring both sides gives,

$$k^2 = (\sqrt{n^3+1} + \sqrt{n+2})^2 = n^3 + n + 3 + 2\sqrt{(n^3+1)(n+2)}.$$

For  $k$  to be an integer, the expression under the square root must be a perfect square. Thus,  $(n^3+1)(n+2)$  must be a perfect square.

Let  $P(n) = (n^3+1)(n+2) = (n^2+3n+2)(n^2-n+1) = n^4+2n^3+n+2$ .

For  $n \geq 3$ ,

$$(n^2+n-1)^2 = n^4+2n^3-n^2-2n+1, \quad \text{and} \quad (n^2+n)^2 = n^4+2n^3+n^2.$$

Compute the differences,

$$P(n) - (n^2+n-1)^2 = (n^4+2n^3+n+2) - (n^4+2n^3-n^2-2n+1) = n^2+3n+1$$

$$P(n) - (n^2+n)^2 = (n^4+2n^3+n+2) - (n^4+2n^3+n^2) = -n^2+n+2.$$

Thus, for all  $n \geq 3$ ,

$$(n^2+n-1)^2 < P(n) < (n^2+n)^2.$$

Since  $n^2+n$  and  $n^2+n-1$  are consecutive integers, their squares are consecutive perfect squares. Therefore,  $P(n)$  lies strictly between two consecutive perfect squares and cannot itself be a perfect square for  $n \geq 3$ . This concludes the proof, thus, the only positive integer solution is  $n = 2$ .

*Solution 2 by Ioan Viorel Codreanu, Satulung, Maramures, Romania*

Assume there exists an integer  $m$  with

$$\sqrt{n^3+1} + \sqrt{n+2} = m.$$

Then

$$\sqrt{n^3+1} = m - \sqrt{n+2}.$$

Squaring gives

$$n^3+1 = m^2 + n + 2 - 2m\sqrt{n+2} \implies 2m\sqrt{n+2} = m^2 + n + 1 - n^3 \in \mathbb{Z}.$$

Hence  $\sqrt{n+2} \in \mathbb{Q}$ . As  $n+2 \in \mathbb{N}$ , it follows that  $\sqrt{n+2} \in \mathbb{N}$ . Write  $n+2 = b^2$  with  $b \in \mathbb{N}$ . Then

$$\sqrt{n^3+1} = m - b \in \mathbb{N},$$

so  $n^3+1 = a^2$  for some  $a \in \mathbb{N}$ . Thus

$$a^2 - n^3 = 1.$$

By the Catalan-Mih  ilescu theorem, the only consecutive perfect powers greater than 1 are  $3^2$  and  $2^3$ . Hence  $a = 3$  and  $n = 2$ .

Indeed, for  $n = 2$  we have  $\sqrt{2^3+1} + \sqrt{2+2} = 5$ , which is an integer. Therefore the unique positive integer  $n$  satisfying the condition is  $n = 2$ .

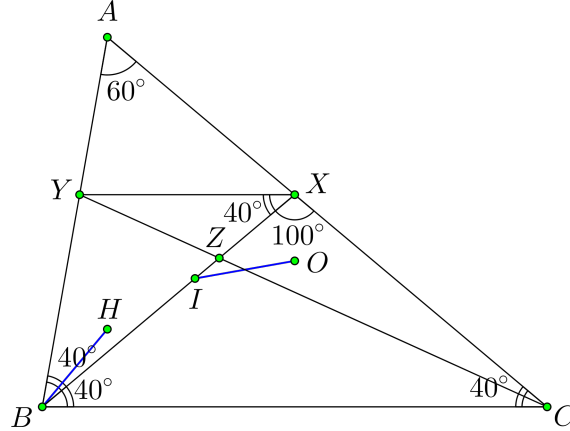
*Also solved by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; G. C. Greubel, Newport News, VA, USA; Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Sundares H. R., Shivamogga, Karnataka, India; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Hoang-Bao Le, Dublin, Ireland; Madani Harouach, AMMOC; Prasanna Pawar, AMMOC, Bengaluru, India; Taes Padhiary, IIT Guwahati, India.*

O705. Let  $ABC$  be a triangle with  $\angle A = 60^\circ$ , circumcenter  $O$ , orthocenter  $H$ , incenter  $I$  and such that  $BH = OI$ . The angle bisector  $BI$  intersects  $AC$  in  $X$  and the line through  $X$  parallel to  $BC$  intersects  $AB$  in  $Y$ . Prove that  $CY$  is a symmedian in triangle  $XBC$ .

*Proposed by Mihaela Berindeanu, Bucharest, Romania*

*Solution by Kousik Sett, India*

Let  $R$  and  $r$  be the circumradius and inradius of triangle  $ABC$  with  $\angle A = 60^\circ$ . Then  $\angle B + \angle C = 120^\circ$ .



Since  $BH = OI$ , we have  $2R \cos B = \sqrt{R^2 - 2Rr}$ . Then

$$\cos^2 B = \frac{1}{4} - \frac{r}{2R}.$$

Since  $\angle A = 60^\circ$ , the following well-known identity

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

implies

$$\frac{r}{R} = \cos B + \cos C - \frac{1}{2}.$$

Therefore,

$$\cos^2 B = \frac{1}{4} - \frac{1}{2} \left( \cos B + \cos C - \frac{1}{2} \right),$$

which implies

$$\begin{aligned} 2 \cos^2 B + \cos B &= 1 - \cos C = 1 - \cos(120^\circ - B) \\ &= 1 + \frac{1}{2} \cos B - \frac{\sqrt{3}}{2} \sin B, \end{aligned}$$

which gives

$$(4 \cos^2 B + \cos B - 2)^2 = 3(1 - \cos^2 B).$$

After simplification, we obtain

$$16 \cos^4 B + 8 \cos^3 B - 12 \cos^2 B - 4 \cos B + 1 = 0,$$

which can be written as

$$(2 \cos B + 1)(8 \cos^3 B - 6 \cos B + 1) = 0,$$

which implies

$$(2 \cos B + 1)(2 \cos 3B + 1) = 0.$$

The first factor  $2\cos B + 1 = 0$  gives  $\cos B = -\frac{1}{2}$ , i.e.,  $\angle B = 120^\circ$ , which is not possible as it implies  $\angle C = 0^\circ$ . The second factor gives  $\cos 3B = -\frac{1}{2}$ , which implies  $\angle B = \frac{2n\pi}{3} \pm \frac{2\pi}{9}$ , where  $n \in \mathbb{Z}$ . For  $n = 0$ ,  $\angle B = \frac{2\pi}{9} = 40^\circ$ . For  $n = 1$ ,  $\angle B = \frac{2\pi}{3} \pm \frac{2\pi}{9}$ , i.e.,  $\angle B = \frac{4\pi}{9} = 80^\circ$  or,  $\angle B = \frac{8\pi}{9} = 160^\circ$ . Therefore, the distinct values for  $\angle B$  within the range  $[0, 180^\circ]$  are  $40^\circ$ ,  $80^\circ$ , and  $160^\circ$ . But  $\angle B < 120^\circ$ , so  $\angle B = 40^\circ$ , or  $80^\circ$  and since  $\angle C = 120^\circ - B$ , we have  $\angle C = 80^\circ$ , or  $40^\circ$ .

Without loss of generality, we may assume that  $\angle B > \angle C$ . Therefore, we choose  $\angle B = 80^\circ$  and  $\angle C = 40^\circ$ . Since  $BX$  is the angle bisector of  $\angle B$ , we have  $\angle CBX = \angle B/2 = 40^\circ = \angle BCX$ . So,  $\angle BXC = 100^\circ$ . Since  $YX \parallel BC$ , we have  $\angle BXY = \angle CBX = 40^\circ$ . Therefore,  $\angle CXY = \angle BXC + \angle BXY = 100^\circ + 40^\circ = 140^\circ$ . Since  $BX$  is the angle bisector of  $\angle B$ ,  $\angle XBY = \angle B/2 = 40^\circ = \angle BXY$ . So,  $BY = XY$ . Let  $CY$  intersects  $BX$  at  $Z$ .

Therefore,

$$\frac{XZ}{ZB} = \frac{[CXZ]}{[CBZ]} = \frac{[YXZ]}{[YBZ]} = \frac{[CXZ] + [YXZ]}{[CBZ] + [YBZ]} = \frac{[CXY]}{[CBY]} = \frac{\frac{1}{2} \cdot CX \cdot XY \cdot \sin 140^\circ}{\frac{1}{2} \cdot BC \cdot BY \cdot \sin 80^\circ}.$$

Since  $\sin 140^\circ = \sin(180^\circ - 40^\circ) = \sin 40^\circ$  and  $XY = BY$ , we have

$$\frac{XZ}{ZB} = \frac{CX}{BC} \cdot \frac{\sin 40^\circ}{\sin 80^\circ}.$$

Since  $\angle AXB = 180^\circ - \angle BXC = 180^\circ - 100^\circ = 80^\circ$ , applying the Law of Sines to triangle  $ABX$  and applying angle bisector theorem to triangle  $ABC$ , we obtain

$$\frac{\sin 40^\circ}{\sin 80^\circ} = \frac{AX}{AB} = \frac{CX}{BC}.$$

Therefore, by combining the above two relations, we obtain

$$\frac{XZ}{ZB} = \frac{CX}{BC} \cdot \frac{CX}{BC} = \frac{CX^2}{BC^2}.$$

The above relation confirms that  $CY$  is the symmedian in triangle  $XBC$  and we are done!

*Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Keisha Kwok, AMMOC; Madani Harouach, AMMOC.*

O706. Prove that for all nonnegative real numbers  $a, b, c$ ,

$$\sqrt{a^3 + b^3 + c^3 + \frac{1}{4}} + 2abc + \frac{1}{2} \geq a + b + c.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

*Solution by Theo Koupelis, Clark College, Washington, USA*

Let  $p = a + b + c$ ,  $q = ab + bc + ca$ , and  $r = abc$ . The desired inequality is equivalent to

$$\sqrt{p^3 - 3pq + 3r + \frac{1}{4}} \geq p - \frac{1}{2} - 2r, \quad (1)$$

which is obvious if its right-hand-side is negative. Thus, we only need examine the case

$$r \leq \frac{2p-1}{4}, \quad \text{where } p \geq \frac{1}{2}. \quad (2)$$

Squaring (1) and simplifying, we get

$$f(r) := 4r^2 - r(4p+1) + p^2 - p - p^3 + 3pq \leq 0. \quad (3)$$

For given appropriate values of  $p, q$ , we have  $f'(r) = 8r - (4p+1) < 0$  because of (2), and  $f''(r) = 8 > 0$ . Also,

$$f\left(\frac{2p-1}{4}\right) = \frac{-2p^3 - 3p + 1 + 6pq}{2} < 0,$$

because  $p \geq 1/2$  and by AM-GM  $q \leq p^2/3$ . Thus,  $f(r)$  is a decreasing function for all  $0 \leq r \leq \frac{2p-1}{4}$ , and therefore it suffices to show that (3) holds when  $r = 0$ . In this case, at least one of the variables  $a, b, c$  is equal to zero. Without loss of generality, let  $c = 0$ . Then  $p^2 \geq 4q$ , and thus it suffices to show that  $q \leq (p^3 - p^2 + p)/(3p) = (p^2 - p + 1)/3$ , which is obvious, because

$$\frac{p^2}{4} \leq \frac{p^2 - p + 1}{3} \iff (p-2)^2 \geq 0.$$

Equality occurs when  $(a, b, c) = (1, 1, 0)$  and cyclic permutations.

*Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania.*

O707. Prove that  $\frac{1}{2}$  is the largest positive value of the constant  $k$  such that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} - 4 \geq k(a^2 + b^2 + c^2 + d^2 - 4)$$

for any positive real numbers  $a, b, c, d$  satisfying  $ab + bc + cd + da = 4$ .

*Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania*

*Solution by the author*

By choosing  $b = d$  and  $c = 1$ , we have  $a = \frac{2-b}{b}$ , where  $0 < b < 2$ , while the inequality becomes

$$\begin{aligned} \frac{1}{a^2} + \frac{2}{b^2} - 3 &\geq k(a^2 + 2b^2 - 3), \\ \frac{b^2}{(2-b)^2} + \frac{2}{b^2} - 3 &\geq k \left[ \frac{(2-b)^2}{b^2} + 2b^2 - 3 \right], \\ \frac{b^4}{(2-b)^2} + 2 - 3b^2 &\geq k \left[ (2-b)^2 + 2b^4 - 3b^2 \right]. \end{aligned}$$

For  $b \rightarrow 0$ , we get the necessary condition  $k \leq \frac{1}{2}$ . To show that  $\frac{1}{2}$  is the largest positive value of the constant  $k$ , we need to prove that

$$2 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq a^2 + b^2 + c^2 + d^2 + 4$$

for  $(a+c)(b+d) = 4$ . Write the inequality as

$$\frac{2(a^2 + c^2)}{a^2 c^2} + (b^2 + d^2) \left( \frac{2}{b^2 d^2} - 1 \right) \geq a^2 + c^2 + 4,$$

assume that  $b + d \leq a + c$ , hence  $b + d \leq 2$ , and denote

$$\begin{aligned} x &= \frac{a+c}{2}, & p &= ac, & x^2 &\geq p, \\ y &= \frac{b+d}{2}, & q &= bd, & xy &= 1, & q &\leq y^2 \leq 1. \end{aligned}$$

We will show first that

$$(b^2 + d^2) \left( \frac{2}{b^2 d^2} - 1 \right) \geq \frac{4}{y^2} - 2y^2.$$

Since  $b^2 + d^2 \geq 2y^2$ , this is true if

$$2y^2 \left( \frac{2}{q^2} - 1 \right) \geq \frac{4}{y^2} - 2y^2.$$

which is equivalent to the obvious inequality

$$y^4 - q^2 \geq 0.$$

To complete the proof, we need to show that

$$\frac{2(a^2 + c^2)}{a^2 c^2} + \frac{4}{y^2} - 2y^2 \geq a^2 + c^2 + 4,$$

that is

$$\frac{2(a^2 + c^2)}{p^2} + 4x^2 - \frac{2}{x^2} \geq 4x^2 - 2p + 4,$$

$$\frac{a^2 + c^2}{p^2} - \frac{1}{x^2} + p \geq 2.$$

Since  $a^2 + c^2 \geq 2p$  and  $x^2 \geq p$ , we have

$$\frac{a^2 + c^2}{p^2} - \frac{1}{x^2} + p \geq \frac{2}{p} - \frac{1}{p} + p = \frac{1}{p} + p \geq 2.$$

For  $k = \frac{1}{2}$ , the equality occurs when  $a = b = c = d = 1$ .

*Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA.*

O708. Prove that in any triangle  $ABC$ , the following inequality holds:

$$\frac{\cos \frac{A}{2} + \cos \frac{B}{2}}{\sin \frac{C}{2}} + \frac{\cos \frac{B}{2} + \cos \frac{C}{2}}{\sin \frac{A}{2}} + \frac{\cos \frac{C}{2} + \cos \frac{A}{2}}{\sin \frac{B}{2}} \geq 6\sqrt{3}.$$

*Proposed by Marius Stănean, Zalău, Romania*

*Solution 1 by the author*

With the substitutions  $A \rightarrow \pi - 2A$ ,  $B \rightarrow \pi - 2B$ ,  $C \rightarrow \pi - 2C$ , we need to show that in any acute angled triangle  $ABC$ , we have

$$\frac{\sin A + \sin B}{\cos C} + \frac{\sin B + \sin C}{\cos A} + \frac{\sin C + \sin A}{\cos B} \geq 6\sqrt{3}.$$

After squaring, this inequality becomes

$$\sum_{cyc} \frac{\sin^2 A + \sin^2 B + 2 \sin A \sin B}{\cos^2 C} + 2 \sum_{cyc} \frac{\sin^2 B + \sin A \sin B + \sin B \sin C + \sin C \sin A}{\cos C \cos A} \geq 108.$$

Denote  $x = 2 \cos A$ ,  $y = 2 \cos B$ ,  $z = 2 \cos C$ , then  $x, y, z > 0$  and  $x^2 + y^2 + z^2 + xyz = 4$ . The inequality can be rewritten as

$$\sum_{cyc} \frac{8 - x^2 - y^2 + 2\sqrt{(4 - x^2)(4 - y^2)}}{z^2} + 2 \sum_{cyc} \frac{4 - y^2 + \sqrt{(4 - x^2)(4 - y^2)} + \sqrt{(4 - y^2)(4 - z^2)} + \sqrt{(4 - z^2)(4 - x^2)}}{zx} \geq 108.$$

But

$$\sqrt{(4 - x^2)(4 - y^2)} = \sqrt{16 - 4(x^2 + y^2) + x^2 y^2} = \sqrt{4z^2 + 4xyz + x^2 y^2} = 2z + xy,$$

and similarly for any other two variables. The inequality becomes

$$\sum_{cyc} \frac{4 + z^2 + xyz + 4z + 2xy}{z^2} + 2 \sum_{cyc} \frac{x^2 + z^2 + xyz + 2(x + y + z) + xy + yz + zx}{zx} \geq 108,$$

or

$$\sum \frac{4}{z^2} + \sum \frac{xy}{z} + \sum \frac{4}{z} + \sum \frac{2xy}{z^2} + 2 \sum \left( \frac{z}{x} + \frac{x}{z} \right) + 2 \sum x + 4 \sum x \sum \frac{1}{xy} + 2 \sum xy \sum \frac{1}{xy} \geq 105,$$

or

$$\sum \frac{x^2 + y^2 + z^2 + xyz}{z^2} + \sum \frac{xy}{z} + \sum \frac{4}{z} + \sum \frac{2xy}{z^2} + 2 \sum \left( \frac{z}{x} + \frac{x}{z} \right) + 2 \sum x + 4 \sum x \sum \frac{1}{xy} + 2 \sum xy \sum \frac{1}{xy} \geq 105,$$

or

$$\sum \left( \frac{z^2}{x^2} + \frac{x^2}{z^2} \right) + 2 \sum \frac{xy}{z} + \sum \frac{4}{z} + \sum \frac{2xy}{z^2}$$

$$+2 \sum \left( \frac{z}{x} + \frac{x}{z} \right) + 2 \sum x + 4 \sum x \sum \frac{1}{xy} + 2 \sum xy \sum \frac{1}{xy} \geq 102.$$

But it is known as  $x + y + z \leq 3$  and  $xyz \leq 1$  so  $xy + yz + zx \leq \frac{(x + y + z)^2}{3} \leq x + y + z$ . Using the AM-GM Inequality, we have

$$\begin{aligned} \sum \left( \frac{z^2}{x^2} + \frac{x^2}{z^2} \right) &\geq 6, \\ \frac{4}{x} + \frac{4}{y} + \frac{4}{z} &\geq \frac{12}{\sqrt[3]{xyz}} \geq 12, \\ \frac{2xy}{z^2} + \frac{2yz}{x^2} + \frac{2zx}{y^2} &\geq 6 \sqrt[3]{\frac{xy}{z^2} \cdot \frac{yz}{x^2} \cdot \frac{zx}{y^2}} = 6, \\ 2 \sum \left( \frac{z}{x} + \frac{x}{z} \right) &\geq 12, \\ 4 \sum x \sum \frac{1}{xy} + 2 \sum xy \sum \frac{1}{xy} &\geq 6 \sum xy \sum \frac{1}{xy} \geq 54, \end{aligned}$$

and it remains to show that

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} + x + y + z \geq 6.$$

But

$$\begin{aligned} x + y + z &= \sqrt{x^2 + y^2 + z^2 + 2(xy + yz + zx)} \geq \sqrt{4 - xyz + 6 \sqrt[3]{x^2 y^2 z^2}} \\ &\geq \sqrt{4 + 5xyz} \geq 2 + xyz. \end{aligned}$$

Hence, we prove that

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} + xyz \geq 4.$$

Let

$$x = \sqrt{\frac{4vw}{(u+v)(w+u)}}, \quad y = \sqrt{\frac{4wu}{(u+v)(v+w)}}, \quad z = \sqrt{\frac{4uv}{(v+w)(w+u)}}, \quad u, v, w > 0$$

and the inequality becomes

$$\frac{2w}{u+v} + \frac{2u}{v+w} + \frac{2v}{w+u} + \frac{8uvw}{(u+v)(v+w)(w+u)} \geq 4,$$

or

$$u^3 + v^3 + w^3 + 3uvw \geq uv(u+v) + vw(v+w) + wu(w+u),$$

which is Schur's Inequality.

**Observation.** The inequality is stronger than the inequality J570 from Mathematical Reflections 5-2021:

Let  $ABC$  be an acute angled triangle. Prove that

$$\left( \frac{\sin A + \sin B}{\cos C} \right)^2 + \left( \frac{\sin B + \sin C}{\cos A} \right)^2 + \left( \frac{\sin C + \sin A}{\cos B} \right)^2 \geq 36.$$

*Solution 2 by Arkady Alt, San Jose, California, USA*

By the AM-GM Inequality,

$$\frac{\cos \frac{A}{2} + \cos \frac{B}{2}}{\sin \frac{C}{2}} \geq \frac{2 \sqrt{\cos \frac{A}{2} \cos \frac{B}{2}}}{\sin \frac{C}{2}}.$$

Summing and applying the AM-GM Inequality again to the three nonnegative terms gives

$$\begin{aligned} \sum_{cyc} \frac{\cos \frac{A}{2} + \cos \frac{B}{2}}{\sin \frac{C}{2}} &\geq \sum_{cyc} \frac{2\sqrt{\cos \frac{A}{2} \cos \frac{B}{2}}}{\sin \frac{C}{2}} \\ &\geq 6 \left( \prod_{cyc} \frac{\sqrt{\cos \frac{A}{2} \cos \frac{B}{2}}}{\sin \frac{C}{2}} \right)^{1/3} = 6 \left( \prod_{cyc} \cot \frac{A}{2} \right)^{1/3}. \end{aligned}$$

Recall that  $\cot \frac{A}{2} = \frac{s-a}{r}$ , where  $s$  is the semiperimeter and  $r$  is the inradius. Hence

$$\prod_{cyc} \cot \frac{A}{2} = \frac{(s-a)(s-b)(s-c)}{r^3}.$$

By Heron's formula and  $K = rs$ ,

$$(s-a)(s-b)(s-c) = \frac{K^2}{s} = r^2 s,$$

so  $\prod_{cyc} \cot \frac{A}{2} = \frac{s}{r}$  and therefore

$$\sum_{cyc} \frac{\cos \frac{A}{2} + \cos \frac{B}{2}}{\sin \frac{C}{2}} \geq 6 \left( \frac{s}{r} \right)^{1/3}.$$

It remains to show  $\frac{s}{r} \geq 3\sqrt{3}$ . By the AM-GM Inequality,

$$(s-a)(s-b)(s-c) \leq \left( \frac{(s-a) + (s-b) + (s-c)}{3} \right)^3 = \left( \frac{s}{3} \right)^3.$$

Since  $(s-a)(s-b)(s-c) = r^2 s$ , we get  $r^2 s \leq \frac{s^3}{27}$ , i.e.  $s \geq 3\sqrt{3}r$ . Hence  $\left( \frac{s}{r} \right)^{1/3} \geq \sqrt{3}$ , and thus

$$\sum_{cyc} \frac{\cos \frac{A}{2} + \cos \frac{B}{2}}{\sin \frac{C}{2}} \geq 6\sqrt{3}.$$

Equality holds for the equilateral triangle.

*Solution 3 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam*  
Let

$$x = \tan \frac{A}{2}, \quad y = \tan \frac{B}{2}, \quad z = \tan \frac{C}{2}.$$

Then

$$\sin \frac{A}{2} = \frac{x}{\sqrt{1+x^2}}, \quad \cos \frac{A}{2} = \frac{1}{\sqrt{1+x^2}},$$

and cyclically for  $y, z$ . Since  $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ , the tangent identity gives  $xy + yz + zx = 1$ .

With  $f(t) = \frac{\sqrt{1+t^2}}{t}$  and  $g(t) = \frac{1}{\sqrt{1+t^2}}$ , we have

$$\frac{\cos \frac{A}{2} + \cos \frac{B}{2}}{\sin \frac{C}{2}} = \frac{\sqrt{1+z^2}}{z} \left( \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \right) = f(z)g(x) + f(z)g(y).$$

Summing cyclically,

$$S = \sum_{cyc} [f(z)g(x) + f(x)g(y)].$$

Group symmetric pairs and apply the AM-GM Inequality:

$$f(z)g(x) + f(x)g(z) \geq 2\sqrt{f(z)g(x)f(x)g(z)} = 2\sqrt{\frac{1}{xz}} = \frac{2}{\sqrt{xz}},$$

$$f(x)g(y) + f(y)g(x) \geq \frac{2}{\sqrt{xy}}, \quad f(y)g(z) + f(z)g(y) \geq \frac{2}{\sqrt{yz}}.$$

Therefore

$$S \geq 2 \left( \frac{1}{\sqrt{xy}} + \frac{1}{\sqrt{yz}} + \frac{1}{\sqrt{zx}} \right). \quad (1)$$

By Titu's lemma,

$$\sum_{cyc} \frac{1}{\sqrt{xy}} = \sum_{cyc} \frac{1^2}{\sqrt{xy}} \geq \frac{(1+1+1)^2}{\sqrt{xy} + \sqrt{yz} + \sqrt{zx}}$$

and by the Cauchy-Schwarz Inequality,

$$(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})^2 \leq 3(xy + yz + zx) = 3.$$

Hence  $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \leq \sqrt{3}$  and

$$\sum_{cyc} \frac{1}{\sqrt{xy}} \geq \frac{9}{\sqrt{3}} = 3\sqrt{3}.$$

Combining with (1) yields  $S \geq 2 \cdot 3\sqrt{3} = 6\sqrt{3}$ . Equality holds if and only if  $x = y = z = \frac{1}{\sqrt{3}}$ , which corresponds to  $A = B = C = \frac{\pi}{3}$ .

*Also solved by Gjmnazi i Specializuar Matematikor Group, Prishtine, Kosovo; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA; Hoang-Bao Le, Dublin, Ireland; Keisha Kwok, AMMOC; Madani Harouach, AMMOC; Medha Shrivastava, AMMOC, India.*