

Junior Problems

J709. Let a, b, c be distinct nonzero real numbers such that

$$\frac{a}{b^2} + ab + \frac{b}{a^2} = 3 \quad \text{and} \quad \frac{b}{c^2} - bc + \frac{c}{b^2} = -3.$$

Evaluate

$$\frac{c}{a^2} + 8ca - \frac{a}{c^2}.$$

Proposed by Adrian Andreescu, Dallas, USA

J710. Prove that for all positive real numbers a, b, c

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(ab+bc+ca)}{2(a+b+c)^2} \geq 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J711. Prove that

$$\left| \frac{x}{x^2+1} + \frac{6x}{x^2+36} \right| \leq \frac{7}{10}$$

for all $x \in \mathbb{R}$.

Proposed by An Zhenping, Xianyang Normal University, China

J712. Let ABC be a triangle with circumcenter O and let X be the midpoint of side BC . The circumcircle Ω of triangle BOC intersects AB and AC in D and E respectively. The line OX intersects again Ω in Y and the lines AY and DE intersect in Z . Prove that

$$2\overrightarrow{ZX} = \overrightarrow{DB} + \overrightarrow{EC}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

J713. Let a, b, c be positive real numbers such that $ab + bc + ca = 3abc$. Prove that

$$1 \leq abc \leq \frac{(a+b+c)^3}{12(a+b+c)-9}.$$

Proposed by Titu Andreescu, Dallas, USA

J714. Let x, y, z be nonnegative real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$xy + yz + zx + \frac{3}{2}\sqrt{x^2 + y^2 + z^2 - xy - yz - zx} \geq 3.$$

Proposed by Marius Stănean, Zalău, Romania

Senior Problems

S709. Find all pairs (m, n) of positive integers such that

$$m + 4\sqrt{n-1} \quad \text{and} \quad n + 4\sqrt{m-1}$$

are both perfect squares.

Proposed by Titu Andreescu, Dallas, USA

S710. Let a, b, c be nonnegative real numbers such that $a + b + c = ab + bc + ca > 0$. Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 2\sqrt{2} + (3 - 2\sqrt{2})\sqrt{abc}.$$

Proposed by Marius Stănean, Zalău, Romania

S711. Show that in any triangle ABC the following inequality holds:

$$\sum \frac{\tan \frac{A}{2}}{\cos^2 \frac{B-C}{2}} \geq \sum \frac{1}{\sin A + \sin B}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

S712. Let x, y, z be positive real numbers such that $xy + yz + zx + xyz = 4$. Prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 2 \left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} \right).$$

Proposed by Mircea Lascu, Zalău, România

S713. Let $k > 2$ be an integer. Prove that for each $j \leq k$ there are distinct prime numbers $p_1, \dots, p_j$ such that

$$p_1^{a_1} + \dots + p_j^{a_j} \equiv j - k \pmod{p_1 \dots p_j},$$

for some positive integers $a_1, \dots, a_j$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

S714. Prove that there is no positive integer n such that $2^n + 3^n + 5^n$ is a perfect square.

Proposed by Mohammadreza Ghelichkhani, Shahed University, Tehran, Iran.

Undergraduate Problems

U709. Let n be a positive integer and let U_n be the set of the n -th roots of unity. Let $P(x) \in \mathbb{C}[x]$ with $\deg P \leq n-1$ and let $H(P)$ be the height of P , that is, the maximum of the moduli of the coefficients of P . If there is $M > 0$ such that $|P(x)| \leq M$ for all $x \in U_n$, prove that $H(P) \leq M$.

Proposed by Alessandro Ventullo, Milan, Italy

U710. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function such that

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x^n) dx = \sin 1.$$

Prove that there exists $c \in (0, 1)$ such that $f(c) = \cos c$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

U711. Evaluate

$$\sum_{n=1}^{\infty} \left[n^2 \sum_{k=n+1}^{\infty} \frac{1}{k^2} - n + \frac{1}{2} - \frac{1}{6n} \right]^2.$$

Proposed by Paolo Perfetti, Universita di "Tor Vergata", Rome, Italy

U712. What is the smallest positive value of the exponent k such that the inequality

$$(ab)^k + (ac)^k + (ad)^k + (bc)^k + (bd)^k + (cd)^k \geq 6$$

holds for any nonnegative real number a, b, c, d with $ab + bc + cd + da = 4$?

Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania

U713. Let $f(x) = x^5 - 3x^3 + 1 \in \mathbb{Q}[x]$. Determine the Galois group of $f(x)$ over $\mathbb{Q}$ and then find whether the equation $f(x) = 0$ is solvable by radicals.

Proposed by Alessandro Ventullo, Milan, Italy

U714. Evaluate

$$\int_0^{\infty} \frac{\arctan x}{\sqrt[4]{x^8 + 1}} dx.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania

Olympiad Problems

- O709.** Let $ABCDEF$ be a hexagon in the plane, and let M, N, P, Q, R, S be the centers of the equilateral triangles constructed on its sides AB, BC, CD, DE, EF , and FA respectively such that either all these triangles are built outside the hexagon, or they are all towards its interior. Let X, Y, Z be the midpoints of the line segments MQ, NR , and PS respectively. Prove that the triangle XYZ is equilateral.

Proposed by Hayk Sedrakian and Maximilian Wolfensberger

- O710.** Let $n \geq 3$ be a positive integer. A finite set S of positive integers is called *balanced* if there exist two disjoint, nonempty subsets A and B of S such that

$$\sum_{a \in A} a = \sum_{b \in B} b.$$

Let S be a set of 2^n positive integers, each of which is at most $2^{2^n - n - 1}$. Prove that S is balanced.

Proposed by Alessandro Ventullo, Milan, Italy

- O711.** Let $x, y, z > 0$ be real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$x^2 + y^2 + \frac{2(x + y + z - 2)}{2 - z} \geq 4.$$

Proposed by Marius Stănean, Zalău, Romania

- O712.** Let a, b, c, d be nonnegative real numbers such that at most one is less than 1 and $ab + bc + cd + da = 4$. Prove that

$$\frac{1}{ab + 3} + \frac{1}{ac + 3} + \frac{1}{ad + 3} + \frac{1}{bc + 3} + \frac{1}{bd + 3} + \frac{1}{cd + 3} \geq \frac{3}{2}.$$

Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania

- O713.** Prove that the equation $x^2 + x^2y + 2y^2 - yz^2 + 3 = 0$ has no integer solutions.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

- O714.** Consider a triangle ABC and its circumcircle Ω . Let there be a point D on the line AB and let E be the intersection of the circumcircle of $\triangle BCD$, denoted ω , with line AC . Consider P the intersection of the circumcircle of $\triangle ADE$ with Ω , $P \neq A$, and Q and R the intersections of lines PD and PE with circle ω . Let M be the midpoint of PQ . Prove that, as D is moving along the line AB , M is moving along a circle.

Proposed by Iancu-Ioan Sandea, Cluj-Napoca, Romania