

INTERSECTING IDEAS: ON TWO SPECIFIC POINTS

Shourya Sinha

§1 Introduction

The “Queue point” and “Humpty point” are two specially named points which are informally recognised by the global mathematical olympiad community. They frequently pop up in configuration-based geometry problems and are, in many ways, interlinked with one another.

This paper mainly focuses on documenting their more-or-less well-known properties (at least those which I am aware of). Keep in mind that the below configurations are probably some of the most important ones in the geometry of configurations, so be sure to be thorough with all the properties!

We also refer the reader to [math_pi_rate's blog on Queue points](#) and [Kapil Pause's handout on Humpty points](#), which were my main references. I will use Miquel's theorem, Miquel's point of a quadrilateral, Desargues's theorem, and Brocard's theorem, all of which can easily be found on Wikipedia.

§2 Frequently Used Notations

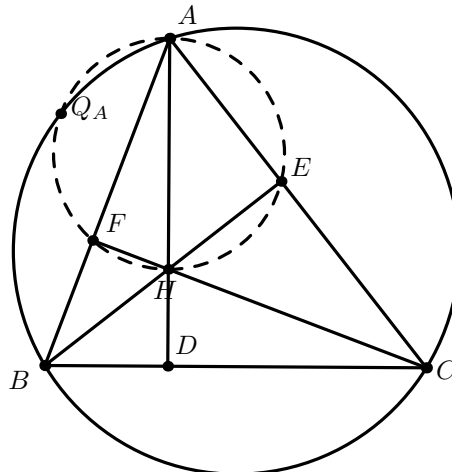
Here are a few letters that you will see often in this document:

- ω is the circumcircle of triangle ABC
- DEF is the orthic triangle of triangle ABC
- H is the orthocenter of triangle ABC
- M is the mid-point of segment BC
- R is the intersection point of lines BC and EF
- A^* is the A -antipode w.r.t (ABC)
- H' is the reflection of point H about BC

§3 The Queue Point

§3.1 Definition

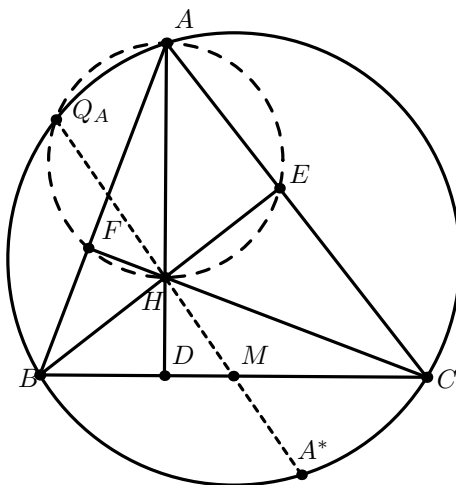
Definition 3.1. The A -Queue point, Q_A , is defined as the intersection point of ω with the circle $(AEHF)$. In other words, Q_A is the Miquel point of quadrilateral $BCEF$.



§3.2 Properties

Lemma 3.2

The points Q_A , H , M and A^* are collinear.



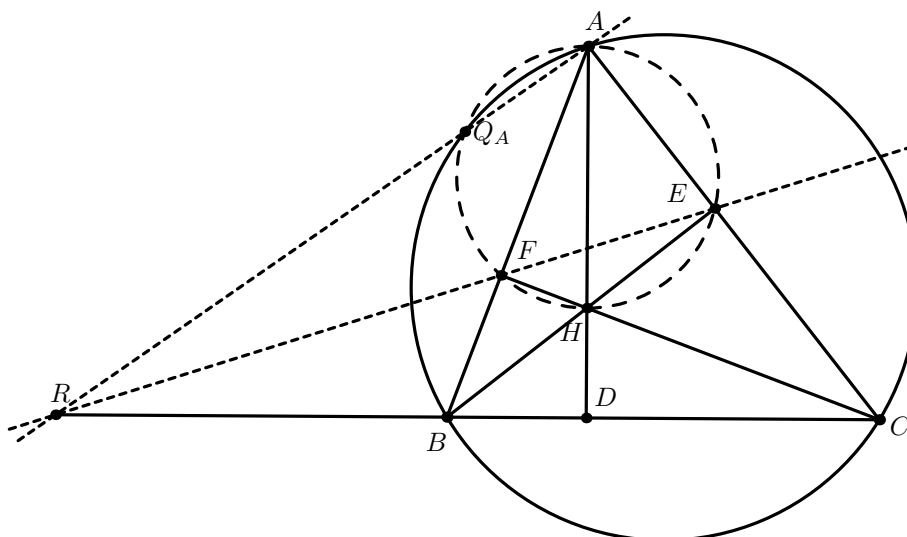
Proof. Notice that $BH \perp AC$ and $A^*C \perp AC$, which implies that $BH \parallel A^*C$. Similarly, $CH \parallel A^*B$. This implies that $BHCA^*$ is a parallelogram, or points H , M and A^* are collinear. Moreover, we have

$$\angle AQ_AH = \angle AQ_AA^* = 90^\circ$$

which finishes the proof. $\square$

Lemma 3.3

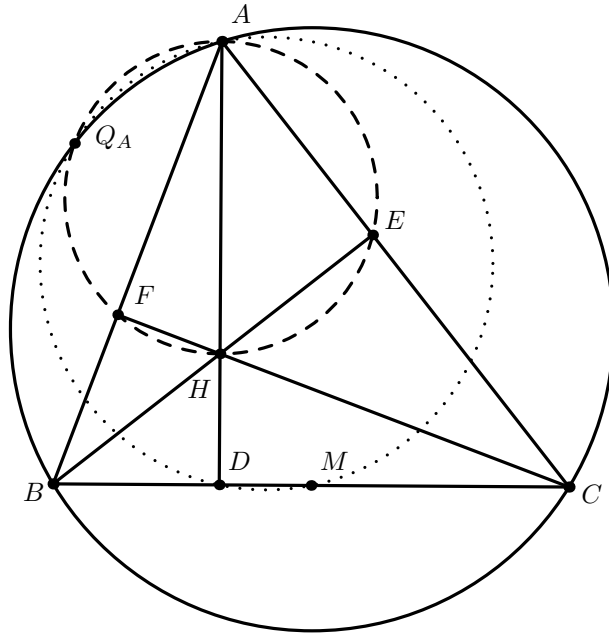
The lines AQ_A , EF and BC are concurrent.



Proof. Just notice that R is the radical center of the circles ω , $(BCEF)$ and $(AEHFQ_A)$. $\square$

Lemma 3.4

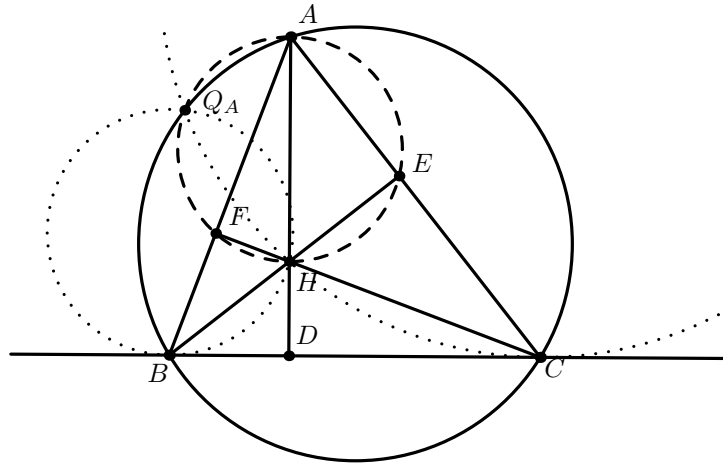
The points A , Q_A , D and M are concyclic.



Proof. Observe that $\angle AQ_A M = \angle AQ_A H = \angle AFH = 90^\circ$ and $\angle ADM = 90^\circ$. □

Lemma 3.5

The circles (BHQ_A) and (CHQ_A) are tangent to BC .



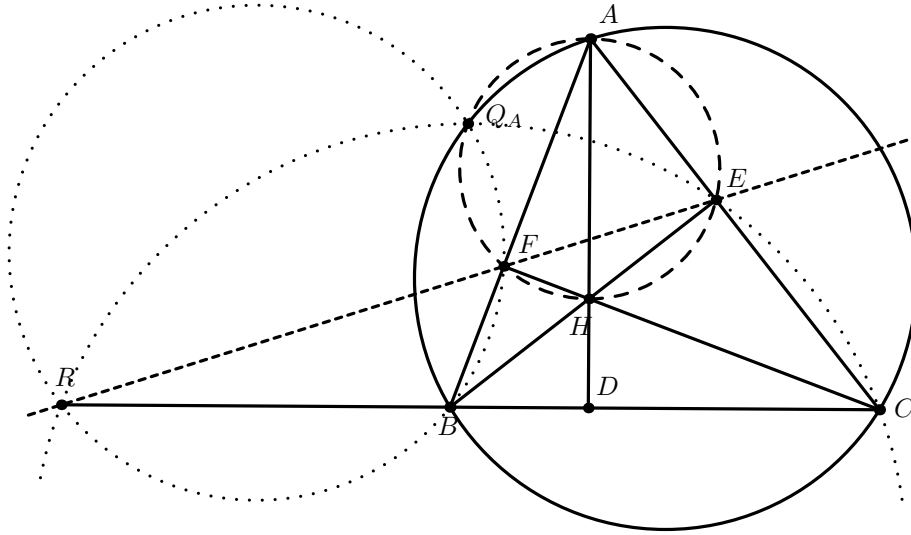
Proof. We consider the circle (BHQ_A) , since the other case would follow similarly. It is enough to show that $\angle DBH = \angle BQ_A H$. It is easy to see that $\angle DBH = 90^\circ - C$. Now, we recall [lemma 3.2](#) and extend line $Q_A H$ to meet ω at A^* . Now we have,

$$\angle BQ_A H = \angle BQ_A A^* = \angle BCA^* = 90^\circ - C$$

□

Lemma 3.6

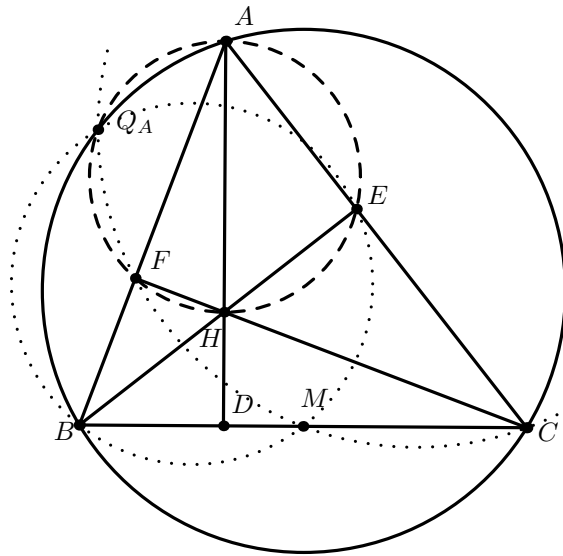
The quadrilaterals $Q_A F B R$ and $Q_A E C R$ are cyclic.



Proof. It's simple, just notice that Q_A is the Miquel point of quadrilateral $BCEF$. □

Lemma 3.7

The quadrilaterals $Q_A EMB$ and $Q_A FMC$ are cyclic.



Proof. It follows again from the properties of Miquel points of cyclic quadrilaterals. In particular, notice that Q_A is the image of $H = BE \cap CF$ w.r.t inversion about $(BCEF)$. □

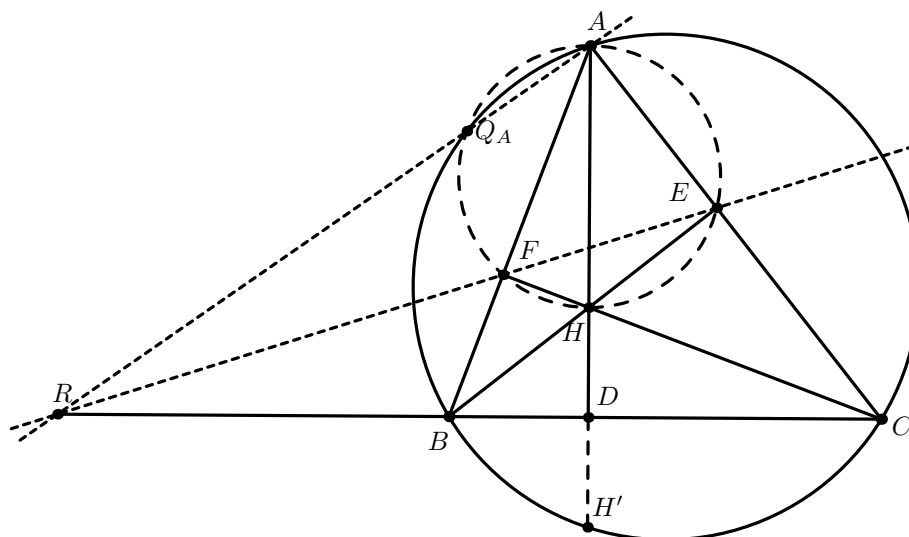
Lemma 3.8

The quadrilateral $Q_A BH'C$ is harmonic.

Proof. By lemma 3.3, we know that line AQ_A meets BC at R . Projecting from point A , we have

$$(Q_A, H'; B, C) \stackrel{A}{=} (R, D; B, C)$$

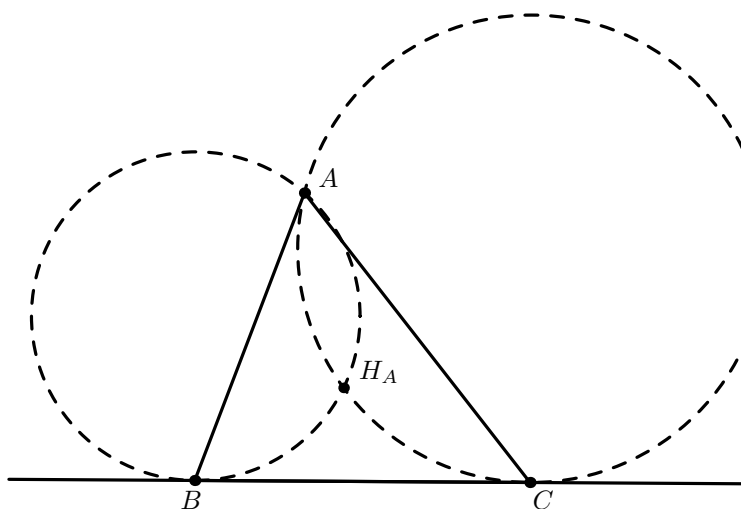
A simple application of Ceva/Menelaus yields $(R, D; B, C) = -1$. □



§4 The Humpty Point

§4.1 Definition

Definition 4.1. Define ω_B to be the circle passing through points A and B and tangent to BC at B . Define ω_C similarly. The intersection point of circles ω_B and ω_C other than the point A is known as the A -Humpty point, abbreviated H_A .



§4.2 Properties

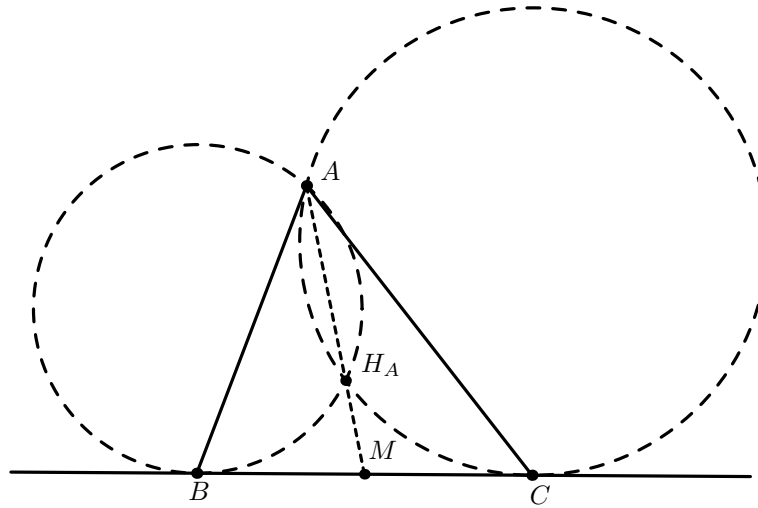
Lemma 4.2

We have that $\angle H_A AB = \angle H_A BC$ and $\angle H_A AC = \angle H_A CB$.

Proof. It follows from definition. □

Lemma 4.3

H_A lies on the A -median.



Proof. Let AH_A intersect BC at M . Then, by power-of-point,

$$MB^2 = MH_A \cdot MA = MC^2$$

which implies that M is the mid-point of BC . □

Lemma 4.4

We have that H_A lies on the A -Apollonius circle, i.e. $\frac{H_AB}{H_AC} = \frac{AB}{AC}$.

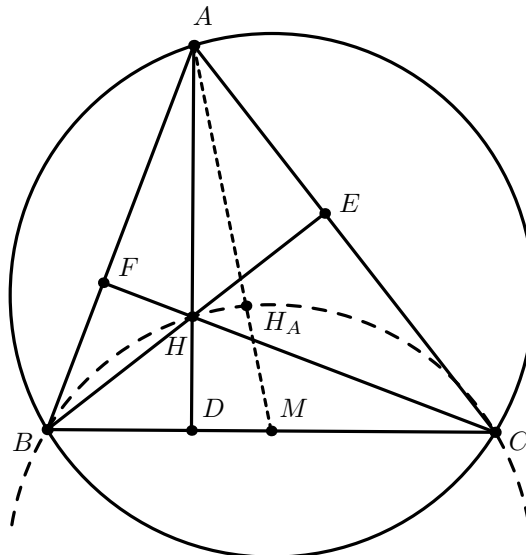
Proof. We use the sine rule. We have,

$$\begin{aligned} \frac{H_AB}{H_AC} &= \frac{\left(\frac{H_AB}{MB}\right)}{\left(\frac{H_AC}{MC}\right)} = \frac{\left(\frac{\sin \angle H_A MB}{\sin \angle BH_A M}\right)}{\left(\frac{\sin \angle H_A MC}{\sin \angle CH_A M}\right)} = \frac{\sin \angle CH_A M}{\sin \angle BH_A M} = \frac{\sin (\angle H_A CM + \angle H_A MC)}{\sin (\angle H_A BM + \angle H_A MB)} \\ &= \frac{\sin (\angle H_A AC + \angle H_A MC)}{\sin (\angle H_A AB + \angle H_A MB)} = \frac{\sin \angle ACB}{\sin \angle ABC} = \frac{AB}{AC} \end{aligned}$$

□

Lemma 4.5

We have that H_A lies on the circle (BHC) .



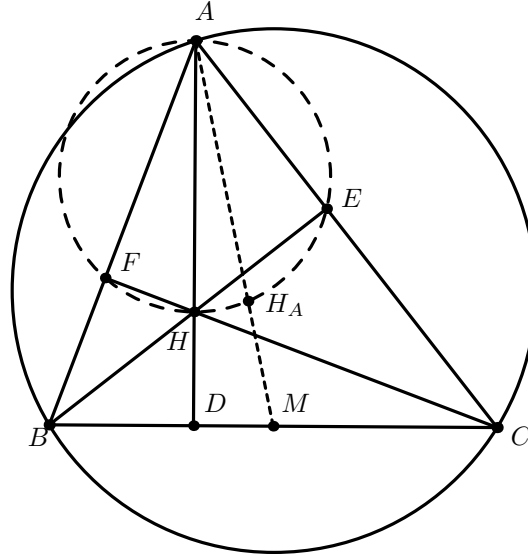
Proof. Observe that

$$\angle BH_A C = 180^\circ - \angle H_A B C - \angle H_A C B = 180^\circ - \angle H_A A B - \angle H_A A C = 180^\circ - A = \angle B H C$$

which proves that the points B , H , C , and H_A are on the same circle. $\square$

Lemma 4.6

We have that $HH_A \perp AM$, or that H_A lies on $(AEHF)$.



Proof. Consider the point X such that $ABXC$ is a parallelogram. Then, M is the mid-point of segment AX , or H_A lies on AX . Moreover, we have

$$\angle BXC = \angle BAC = 180^\circ - \angle BHC$$

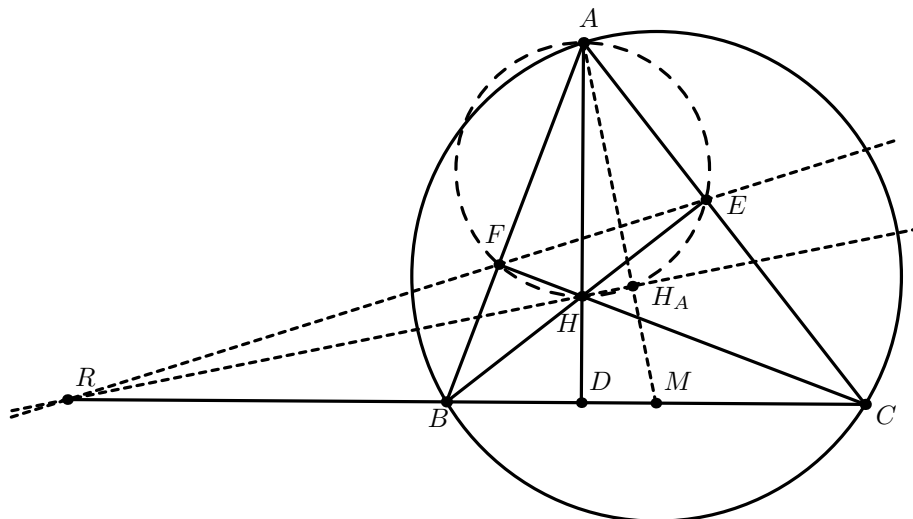
which implies that X lies on $(BHH_A C)$. Now, observe that

$$\angle HBX = \angle HBC + \angle XBC = 90^\circ - C + C = 90^\circ$$

implying HX is a diameter of $(BHH_A C X)$. Thus, $\angle HH_A X = 90^\circ$ or $HH_A \perp AM$. $\square$

Lemma 4.7

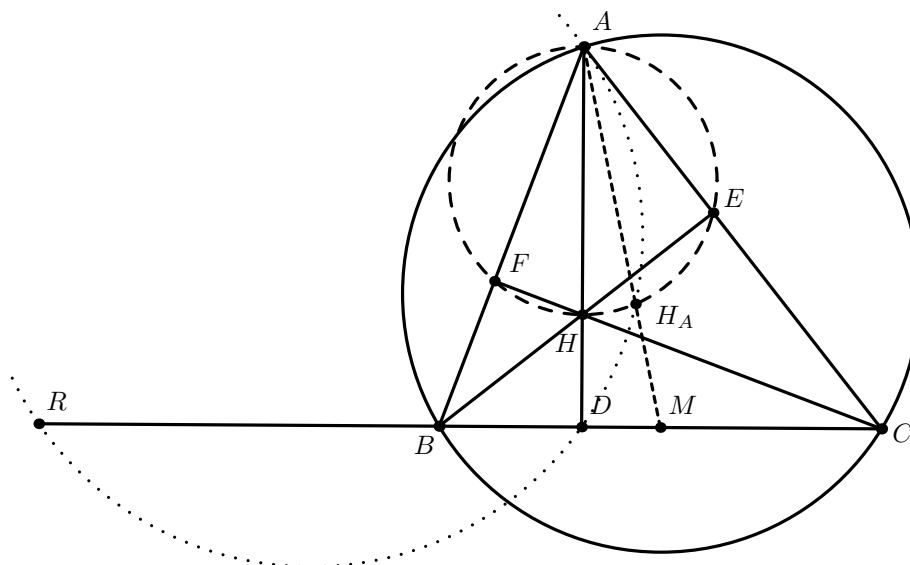
The lines HH_A , EF and BC are concurrent.



Proof. Indeed, they are concurrent at the radical center of the circles ω , (BHH_AC) and (AEH_AHF) . $\square$

Lemma 4.8

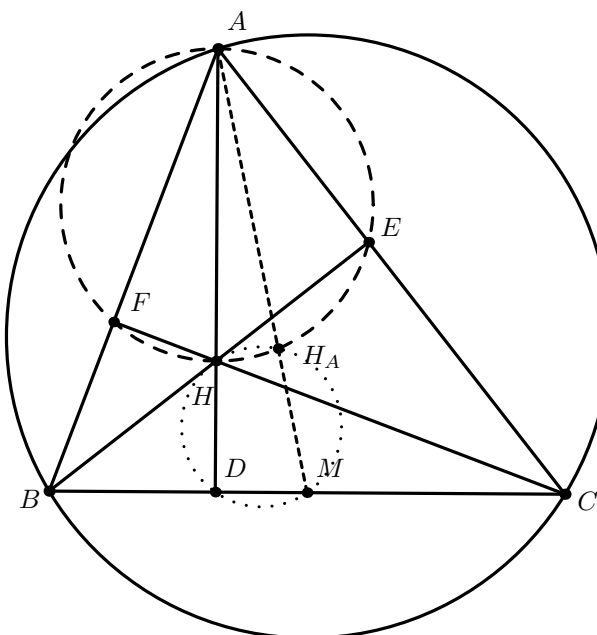
The points A , H_A , D and R are concyclic.



Proof. Brocard's theorem on quadrilateral $BCEF$ implies that M is the orthocenter of triangle AHR . This implies $AM \perp RH$ or $AH_A \perp RH_A$. Also $AD \perp RD$ which proves the concyclicity. $\square$

Lemma 4.9

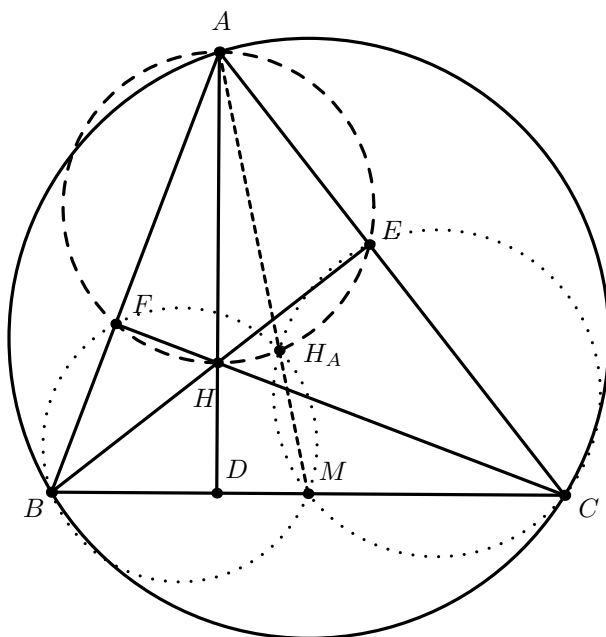
The points H , H_A , M and D are concyclic.



Proof. It follows from $\angle HH_AM = \angle HDM = 90^\circ$. $\square$

Lemma 4.10

The quadrilaterals H_AFBM and H_AECM are cyclic.



Proof. Consider the circles $(BFHD)$ and (HH_ADM) . By power-of-point

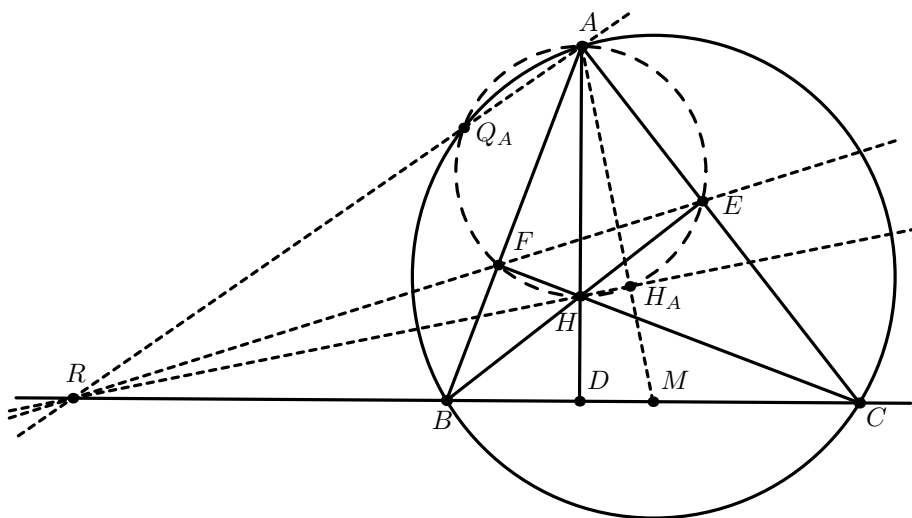
$$AF \cdot AB = AH \cdot AD = AH_A \cdot AM$$

which proves that H_AFBM is cyclic. The other case follows similarly. $\square$

§5 A few linked properties

Lemma 5.1

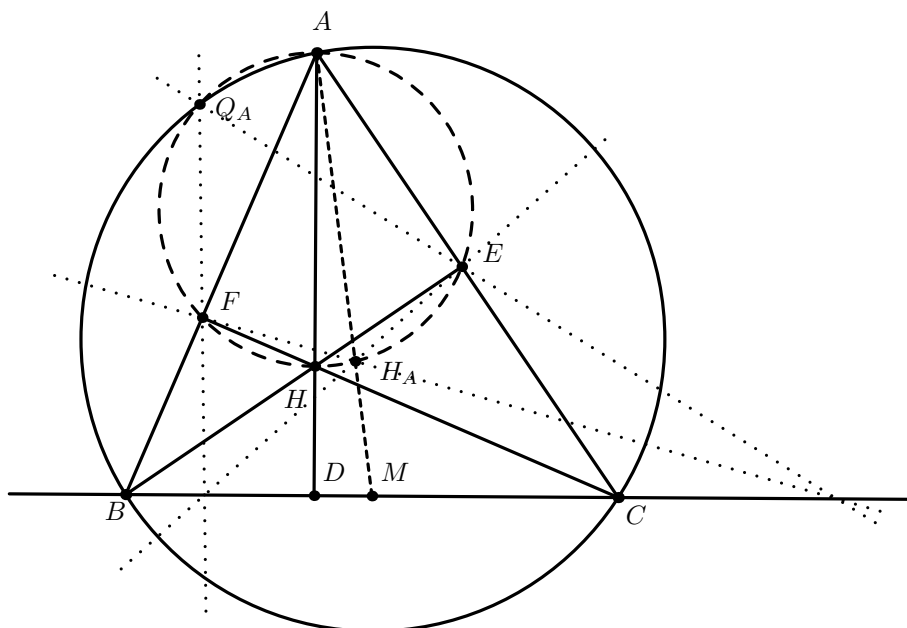
The lines AQ_A , HH_A , EF and BC are concurrent.



Proof. Combine lemma 3.3 and lemma 4.7. $\square$

Lemma 5.2

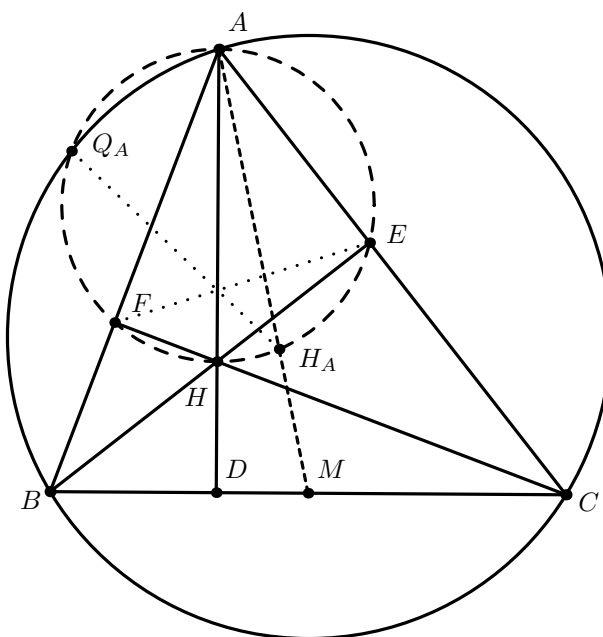
The points $Q_A E \cap H_A F$ and $Q_A F \cap H_A E$ both lie on line BC .



Proof. Use the radical center of the circles $(Q_A E M B)$, $(H_A F B M)$ and $(A E H_A H F Q_A)$ to get $Q_A E \cap H_A F \in BC$. The other case follows similarly. $\square$

Lemma 5.3

Segments AH , EF and $Q_A H_A$ are concurrent.



Proof. Apply Desargues' theorem on triangles $AQ_A E$ and $HH_A F$ to convert it to the previous lemma. $\square$

§6 Solved Problems

Problem 6.1. (USA TSTST 2017 P1) Let ABC be a triangle with circumcircle Γ , circumcenter O , and orthocenter H . Assume that $AB \neq AC$ and that $\angle A \neq 90^\circ$. Let M and N be the midpoints of sides AB and AC , respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A . Let Q be the intersection point, other than A , of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF . Prove that $PR \perp OH$.

Immediately, we recognise (at least we should be able to) Q to be the A -Queue point and R to be our much familiar radical center.

In these configurations, if you have kept a close eye, then you might spot that radical axes is a much recurring theme. The lines PR and the Euler line, OH do not appear to be much related. Moreover, one might recall that the radical axis of two circles is perpendicular to the line joining their centers. Usage of the radical axis is thus motivated.

The key step is to link the side mid-points with the feet of the altitudes using the nine-point circle of ABC . It is easy to observe that homothety at A with factor 2 yields $PA^2 = PM \cdot PN$, which implies that P lies on the radical axis of Γ and the nine-point circle, by power-of-point. Moreover, we have $RQ \cdot RA = RF \cdot RE$ which implies that R lies on the radical axis of Γ and the nine-point circle. Therefore, the radical axis of the said circles is precisely the line PR ! Also, recall that the nine-point center lies on the Euler line, and therefore we are done!

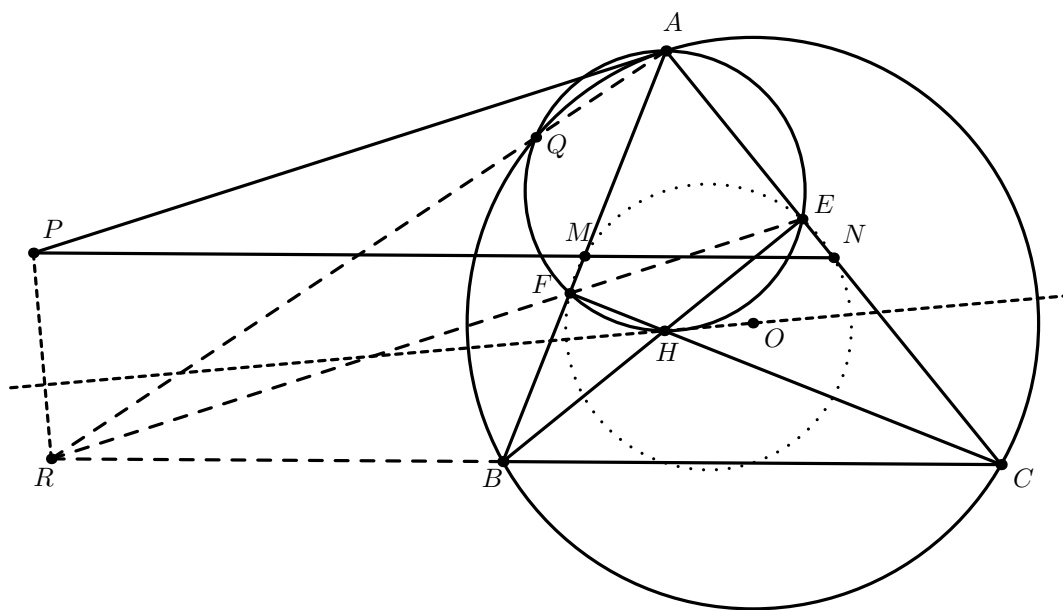
Solution. Let Ω be the nine-point circle of ABC . Observe that

$$PA^2 = PM \cdot PN$$

which implies that P lies on the radical axis of Γ and Ω . Moreover,

$$RQ \cdot RA = RF \cdot RE$$

which implies that R lies on the radical axis of Γ and Ω . Thus, PR is the required radical axis, or $PR \perp ON$, where N is the nine-point center. But N lies on OH , hence done. □

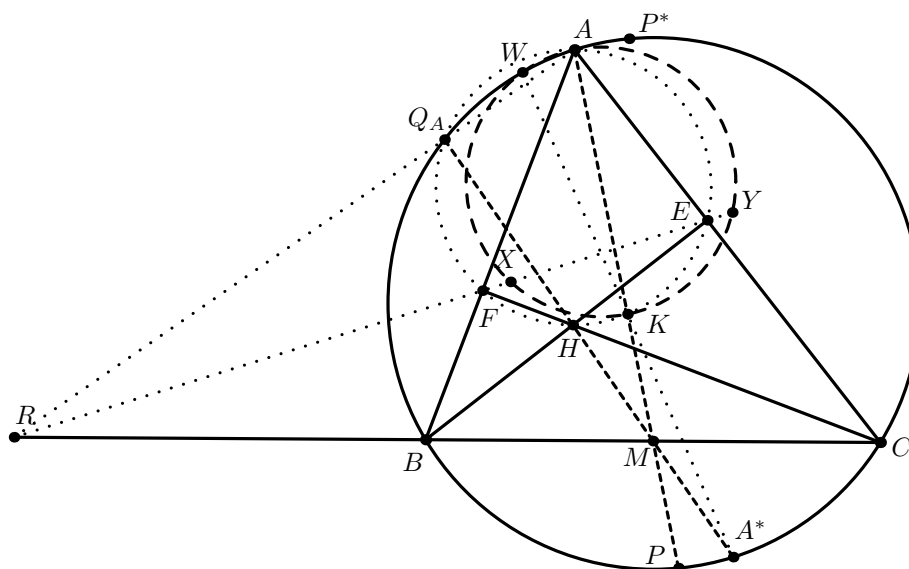


Problem 6.2. (DJGO 2023 Team P5) Let $\triangle ABC$ be a triangle with altitudes BE, CF intersecting at H . The circumcircle of $\triangle BHC$ intersects the circle with diameter AH again at point K . The circle with diameter AK intersects the circumcircle of $\triangle ABC$ at W and suppose it intersects EF at two distinct points X, Y . Prove that: $WXKY$ is a harmonic quadrilateral.

Seeing the presence of the line EF , one might be motivated to consider our familiar $R = BC \cap EF \cap KH$. Moreover, it is not hard, now, to see that $\angle RKA = \angle HKA = 90^\circ$, so the line RHK is actually tangent to $(AWXKY)$! This is exactly the sign of a harmonic quadrilateral, and we are only left to prove that RW is also tangent to $(AWXKY)$.

Perspectivity at A gives $(W, K; L, A) \stackrel{A}{=} (W, P; Q_A, P^*)$, where P^* is the antipode of P w.r.t ω (why?) and Q_A is the you-know-what. The key thing to notice now is that if A^* is the antipode of A w.r.t ω , then $P^*A^* \parallel PA$ and that K lies on WA^* , since $\angle KWA = \angle A^*WA = 90^\circ$.

Solution. We define the following points: $R = BC \cap EF$, $P = AK \cap (ABC)$, $P^* = P$ -antipode w.r.t (ABC) , $A^* = A$ -antipode w.r.t (ABC) , $L = AR \cap (AWK)$, $Q_A = AR \cap (ABC)$ and $M = AK \cap BC$.

$$(W, K; L, A) \stackrel{A}{=} (W, P; Q_A, P^*) \stackrel{A^*}{=} (K, P; M, P_\infty)$$
☐

Having brought the Humpty point into action, we are motivated to take the mid-point of BC , M . Also, taking the point R again is a good idea, as we now have five concurrent lines: BC , EF , AQ_A , HH_A and PQ (why?). We now notice that H is the orthocenter of triangle ARM , and Q_A and H_A are the feet of the M and R altitudes respectively. This motivates us to think in terms of triangle ARM .

We work backwards, assuming that the collinearity holds true. Observe carefully that the line joining the feet of the R and M altitudes meets line RM at T , which implies that T is actually the equivalent of the point R in triangle ARM . Since H is its orthocenter, it implies that $Q = TH \cap (AQ_AH_A)$ is actually the A -Humpty point of triangle ARM ! This statement is quite easy to prove with the help of a simple angle chase.

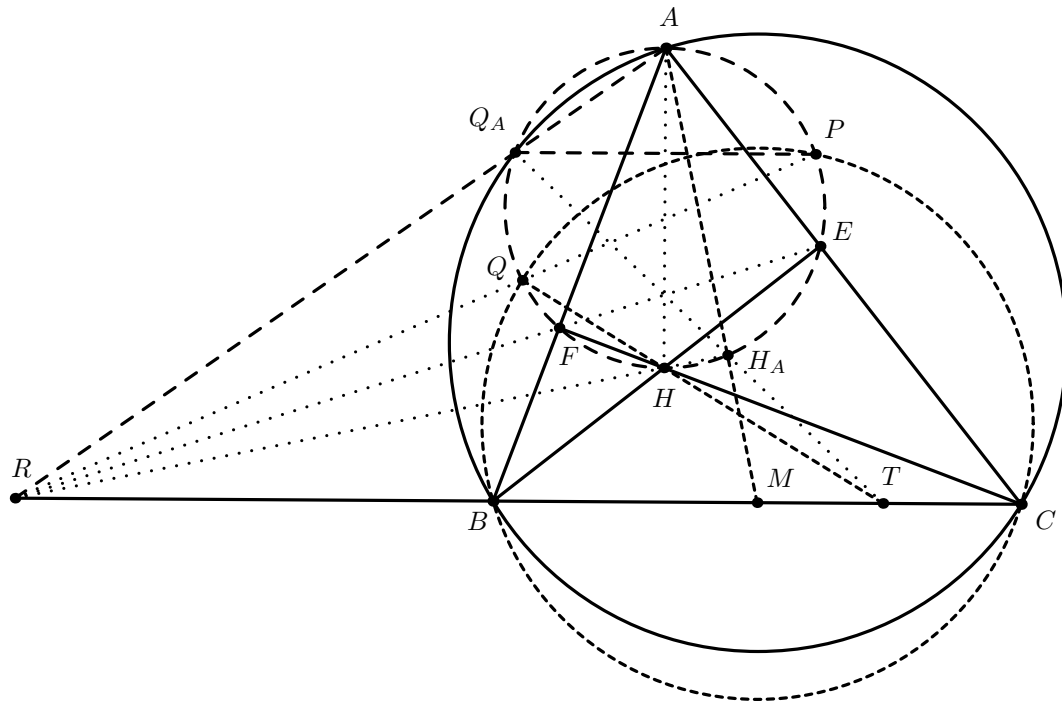
Solution. Let M be the mid-point of BC and $R = BC \cap EF$. Observe that

$$\angle QRM = \angle QPQ_A = \angle QAR$$

which implies that Q is the A -Humpty point of triangle ARM . Therefore, lines BC , QH and Q_AH_A concur at T , or Q_A , H_A and T are collinear.

We know that quadrilaterals Q_AEMB and H_AFBM are cyclic. By radical axis on circles (Q_AEMB) , (H_AFBM) and (AEH_AHFQ_A) , we get that $Q_AE \cap H_AF$ lies on BC . Desargues' theorem on triangles AQ_AE and HH_AF implies that AH , EF and Q_AH_A are concurrent, and we are done.

□



§7 More Problems

Here are a few more problems for your practice on these configurations. Remember, these problems are, by no means, easy, so don't give up trying. Enjoy solving!

Problem 7.1. (India EGMO TST 2025 P5) Let acute scalene $\triangle ABC$ have circumcircle ω . Let M be the midpoint of BC , define X as the other intersection of AM with ω . Let E, F be the feet of altitudes from B, C to AC, AB respectively. Let Q be the second intersection of the circumcircle of $\triangle AEF$ and ω . Let $Y \neq X$ be a point such that $MX = MY$ and QMY is cyclic. Finally, let S be a point on BC such that $\angle BAS = \angle MAC$. Prove that the quadrilaterals $BFYS$ and $CEYS$ are cyclic.

Problem 7.2. (APMO 2012 P4) Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC , by M the midpoint of BC , and by H the orthocenter of ABC . Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH , and F be the point of intersection (other than E) of the line ED and the circle Γ .

Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

Problem 7.3. (AQGO 2020 P1) Let ABC be a triangle with altitude AD . Let V be a variable point on (ABC) and VD intersect (ABC) at U . Let UM intersect (ABC) at X where M is the midpoint of BC . Let $AX \cap BC = Y$. Let (YVM) intersect (ABC) at a point Z . Let Z' be the reflection of Z over OH . Finally let $(Z'OH)$ intersect (ABC) at $T \neq Z'$. Then prove that TZ passes through M .

Problem 7.4. (USA TST 2005 P6) Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

Problem 7.5. (Iran MO 3rd Round 2024 Geometry P2) Let M be the midpoint of the side BC of the $\triangle ABC$. The perpendicular at A to AM meets (ABC) at K . The altitudes BE, CF of the triangle ABC meet AK at P, Q . Show that the radical axis of the circumcircles of the triangles PKE, QKF is perpendicular to BC .

Problem 7.6. (USAMO 2017 P3) Let ABC be a scalene triangle with circumcircle Ω and incenter I . Ray AI meets BC at D and meets Ω again at M ; the circle with diameter DM cuts Ω again at K . Lines MK and BC meet at S , and N is the midpoint of IS . The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2 . Prove that Ω passes through the midpoint of either IL_1 or IL_2 .

Problem 7.7. (Sharygin 2023 P21) Let $ABCD$ be a cyclic quadrilateral; M_{ac} be the midpoint of AC ; H_d, H_b be the orthocenters of $\triangle ABC, \triangle ADC$ respectively; P_d, P_b be the projections of H_d and H_b to BM_{ac} and DM_{ac} respectively. Define similarly P_a, P_c for the diagonal BD . Prove that P_a, P_b, P_c, P_d are concyclic.

Problem 7.8. (Polish MO 2018 P5) An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C , respectively. The line tangent in point A to the circle escribed on ABC crosses BC at P . The line parallel to BC that goes through point A crosses EF at Q . Prove PQ is perpendicular to the median from A of triangle ABC .

Problem 7.9. (USA TSTST 2015 P2) Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A . The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A . Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC .

Problem 7.10. (DGO 2023 Individual P1) Let ABC be an acute triangle with point D lying on the plane such that $ABDC$ is a parallelogram. H is the orthocenter of $\triangle ABC$. BH intersects CD at Y and CH intersects BD at X . The circle with diameter AH intersects the circumcircle $\triangle ABC$ again at Q . Prove that: The circumcircle of $\triangle XQY$ passes through the reflection of D over BC .