

# The Best Constant in Some Inequalities – an Elementary Approach

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In this paper we show some inequalities depending on one parameter  $k$ , and we try to find "the best" such  $k$  in each of these inequalities. We deal with two situations, namely:

1. If we know that the inequalities  $A \geq p$  and  $B \leq q$  hold, then the best  $k$  in the inequality  $A + kB \geq p + kq$  is the greatest possible such  $k$ ; indeed (except for the cases when  $B = q$ ), the inequality reads

$$\frac{A - p}{q - B} \geq k,$$

so, basically, the best  $k$  would be the minimum of the expression from the left-hand side.

2. Clearly, in an inequality of the form  $f(a, b, c) \leq k$  the best  $k$  is the smallest possible, and, ideally, this would be the maximum of the expression  $f(a, b, c)$  considering the variables  $a, b$ , and  $c$  from appropriate domains (sometimes with restrictions). Usually,  $f(a, b, c)$  (such as  $A, B, p, q$  from above) is a symmetric and homogeneous function of  $a, b, c$ .

In general there are two stages in solving such a problem: we first determine the best  $k$  (usually by considering some particular cases), then we prove that the inequality holds for the value of  $k$  that we guessed. The second phase (the proof) is usually the hard part. Let us see now some examples, all of which are solved in [1], but with a much greater amount of work. Our calculations are more simple, but only for finding the constant. Unfortunately, we only have complete proofs for the first two of the examples; for the other three we only prove the inequalities for some (not necessarily best) constants.

**Problem 1.** Determine the best (meaning the least)  $k$  such that the inequality

$$(a - b)^4 + (b - c)^4 + (c - a)^4 \leq k(a^4 + b^4 + c^4 - abc(a + b + c))$$

holds for any positive real numbers  $a, b, c$ .

**Solution.** We clearly have

$$E(a, b, c) = (a - b)^4 + (b - c)^4 + (c - a)^4 \geq 0$$

and

$$\begin{aligned} F(a, b, c) &= a^4 + b^4 + c^4 - abc(a + b + c) = \frac{1}{2}((a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 + \\ &\quad + a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2) \geq 0 \end{aligned}$$

for all  $a, b, c > 0$ , and each of these inequalities becomes an equality if and only if  $a = b = c$ . Since the required inequality can be written

$$\frac{E(a, b, c)}{F(a, b, c)} \leq k$$

for any positive  $a, b, c$  which are not all equal, it is clear that here "the best"  $k$  means "the least"  $k$ . We immediately see that

$$k \geq \frac{E(a, 0, 0)}{F(a, 0, 0)} = 2$$

and we try to prove the inequality for this value of  $k$ , that is, we need show that

$$(a-b)^4 + (b-c)^4 + (c-a)^4 \leq 2(a^4 + b^4 + c^4 - abc(a+b+c))$$

for any  $a, b, c > 0$ . By expanding and rearranging the terms that do not cancel, we see that the inequality is equivalent to

$$3(a^2b^2 + b^2c^2 + c^2a^2) + abc(a+b+c) \leq 2(a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3), \quad a, b, c > 0,$$

and that it follows by adding side by side the inequalities

$$4(a^2b^2 + b^2c^2 + c^2a^2) \leq 2(a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3)$$

and

$$abc(a+b+c) \leq a^2b^2 + b^2c^2 + c^2a^2.$$

Indeed, the second of these inequalities is  $F(a, b, c) \geq 0$  from above, while the first one is easily seen to be equivalent to

$$0 \leq ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2,$$

We conclude that  $k = 2$  is the best constant for which the inequality from problem 1 holds for all positive  $a, b, c$ .

**Problem 2.** Find the largest  $k$  for which the inequality

$$((a-b)^2 + (b-c)^2 + (c-a)^2)^3 \geq k(a-b)^2(b-c)^2(c-a)^2$$

is true for all real numbers  $a, b, c$ .

**Solution.** The inequality is symmetric, so we can assume, without loss of generality, that  $a \geq b \geq c$ . So, with  $x = a - b \geq 0$  and  $y = b - c \geq 0$  we transform the inequality into

$$(x^2 + y^2 + (x+y)^2)^3 \geq kx^2y^2(x+y)^2$$

for  $x, y \geq 0$ . Clearly, for  $x = y = 1$ , we get  $54 \geq k$ , so we try to prove that

$$(x^2 + y^2 + (x+y)^2)^3 \geq 54x^2y^2(x+y)^2, \quad \forall x, y \geq 0$$

(thus showing that the required best  $k$  is 54). This follows easily by using the well-known inequalities between the quadratic mean, arithmetic mean, and geometric mean of two nonnegative real numbers. We thus have

$$\begin{aligned} (x^2 + y^2 + (x+y)^2)^3 &\geq \left( \frac{(x+y)^2}{2} + (x+y)^2 \right)^3 = \frac{27}{8}(x+y)^4(x+y)^2 \geq \\ &\geq \frac{27}{8}(2\sqrt{xy})^4(x+y)^2 = 54x^2y^2(x+y)^2, \end{aligned}$$

as we intended to prove.

Another approach is to multiply side by side

$$4(x^2 + xy + y^2) \geq 3(x+y)^2 \quad \text{and} \quad (x^2 + xy + y^2)^2 \geq 9x^2y^2$$

in order to obtain the desired inequality in the equivalent form

$$4(x^2 + xy + y^2)^3 \geq 27x^2y^2(x+y)^2.$$

Note that equality holds when  $x = y \Leftrightarrow a + c = 2b$ , and, by symmetry, whenever the arithmetic mean of two of the numbers  $a, b, c$  equals the third.

**Problem 3.** Determine the largest value of  $k$  for which the inequality

$$(4 - 3k) \cdot \frac{(a+b)(b+c)(c+a)}{32abc} + k \cdot \left( \frac{a}{2a+b+c} + \frac{b}{a+2b+c} + \frac{c}{a+b+2c} \right) \geq 1$$

is true for any positive real numbers  $a, b, c$ .

**Partial solution.** Note that we can rewrite the inequality as

$$\begin{aligned} & \frac{(a+b)(b+c)(c+a)}{8abc} - 1 \geq \\ & \geq k \left( \frac{3(a+b)(b+c)(c+a)}{32abc} - \frac{a}{2a+b+c} - \frac{b}{a+2b+c} - \frac{c}{a+b+2c} \right), \end{aligned}$$

where both expressions from the left-hand side, and between the parentheses in the right-hand side are nonnegative (strictly positive if  $a, b, c$  are not all equal). This shows that the best  $k$  in this inequality is, indeed, the largest possible.

The fact that the left-hand side is nonnegative (for positive  $a, b, c$ ) is well-known. For the expression from the right-hand side, we can use Nesbitt's inequality

$$\sum \frac{u}{v+w} \geq \frac{3}{2}, \quad u, v, w > 0$$

(here and further all the sums are cyclic), in order to get

$$\sum \frac{a}{2a+b+c} = \frac{1}{2} \left( 3 - \sum \frac{b+c}{(a+b)+(a+c)} \right) \leq \frac{1}{2} \left( 3 - \frac{3}{2} \right) = \frac{3}{4},$$

whence

$$\frac{3(a+b)(b+c)(c+a)}{32abc} \geq \frac{3}{4} \geq \sum \frac{a}{2a+b+c},$$

meaning that the difference in the parentheses is nonnegative (and 0 if and only if  $a = b = c$ ).

Now we can write the inequality in the form

$$(4 - 3k) \left( \frac{(a+b)(b+c)(c+a)}{32abc} - \frac{1}{4} \right) \geq k \left( \frac{3}{4} - \sum \frac{a}{2a+b+c} \right),$$

and notice that we have

$$\frac{(a+b)(b+c)(c+a)}{32abc} - \frac{1}{4} = \frac{\sum c(a-b)^2}{32abc} = \frac{1}{32} \sum \frac{(a-b)^2}{ab}.$$

Also, we can easily check that

$$\sum \frac{u}{v+w} - \frac{3}{2} = \frac{1}{2} \sum \frac{(u-v)^2}{(u+w)(v+w)}$$

(which, among others, proves Nesbitt's inequality), therefore

$$\begin{aligned} \frac{3}{4} - \sum \frac{a}{2a+b+c} &= \frac{1}{2} \left( \sum \frac{b+c}{(a+b)+(a+c)} - \frac{3}{2} \right) = \\ &= \frac{1}{4} \sum \frac{(a-b)^2}{(2a+b+c)(a+2b+c)}. \end{aligned}$$

According to these identities, our inequality becomes

$$\frac{4-3k}{8} \sum \frac{(a-b)^2}{ab} \geq k \sum \frac{(a-b)^2}{(2a+b+c)(a+2b+c)}.$$

By replacing in the inequality thus obtained  $b = c = 1$  (with an arbitrary  $a$ ), we see that we must have

$$\frac{4-3k}{8} \cdot \frac{2(a-1)^2}{a} \geq k \cdot \frac{2(a-1)^2}{(2a+2)(a+3)},$$

that is,

$$\frac{4-3k}{4k} \geq \frac{a}{(a+1)(a+3)}$$

for all  $a > 0$ . (Although we can simplify by  $(a-1)^2$  only for  $a \neq 1$ , the final inequality holds for all  $a$ , by continuity.)

Now we see that the function  $h$  given by

$$h(a) = \frac{a}{(a+1)(a+3)}$$

has the derivative

$$h'(a) = \frac{3-a^2}{(a+1)^2(a+3)^2},$$

hence it has (for  $a > 0$ ) a maximum attained at  $a = \sqrt{3}$ . Consequently, this maximum is

$$\frac{\sqrt{3}}{(\sqrt{3}+1)(\sqrt{3}+3)} = \frac{4-2\sqrt{3}}{4}.$$

So, hoping for the best  $k$ , we impose the condition

$$\frac{4-3k}{4k} \geq \frac{4-2\sqrt{3}}{4} \Leftrightarrow \frac{4}{k} - 3 \geq 4-2\sqrt{3} \Leftrightarrow k \leq \frac{4}{7-2\sqrt{3}} = \frac{4(7+2\sqrt{3})}{37}.$$

Thus the best (the greatest)  $k$  could be  $\frac{4(7+2\sqrt{3})}{37} = 1.13125\dots$

Although we cannot prove the inequality for this maximal  $k$  (see [1]), and it seems that this is not easy by direct computations, we can prove it for  $k = \frac{36}{35} = 1.02857\dots$ , for which we have  $\frac{4-3k}{8k} = \frac{1}{9}$ . Thus the inequality to prove is

$$\frac{1}{9} \sum \frac{(a-b)^2}{ab} \geq \sum \frac{(a-b)^2}{(2a+b+c)(a+2b+c)},$$

and it holds because we have

$$(2a+b+c)(a+2b+c) > (2a+b)(a+2b) \geq 9ab$$

(and two more similar inequalities obtained by cyclic permutations of  $a, b, c$ ). Finally note that, in the original form, the inequality that we proved reads

$$\frac{(a+b)(b+c)(c+a)}{abc} + 36 \left( \frac{a}{2a+b+c} + \frac{b}{a+2b+c} + \frac{c}{a+b+2c} \right) \geq 35,$$

for  $a, b, c > 0$ .

**Problem 4.** Determine the smallest real number  $k$  such that the inequality

$$k \cdot \frac{a^3+b^3+c^3}{abc} + (8-24k) \cdot \frac{abc}{(a+b)(b+c)(c+a)} \geq 1$$

holds for all positive real numbers  $a, b, c$ .

**Partial solution.** We let the interested reader to see why the best  $k$  in this inequality is the smallest one (and that it must be positive), and we write the inequality in the form

$$k \left( \frac{a^3 + b^3 + c^3}{abc} - 3 \right) \geq (1 - 3k) \left( 1 - \frac{8abc}{(a+b)(b+c)(c+a)} \right),$$

or, equivalently,

$$\frac{k(a+b+c)}{2abc} \sum (a-b)^2 \geq \frac{1-3k}{(a+b)(b+c)(c+a)} \sum c(a-b)^2.$$

For  $b = c = 1$  and arbitrary  $a$  we get

$$\frac{k(a+2)}{2a} \cdot 2(a-1)^2 \geq \frac{1-3k}{2(a+1)^2} \cdot 2(a-1)^2,$$

therefore we need have

$$\frac{(a+2)(a+1)^2}{a} \geq \frac{1}{k} - 3$$

for any  $a > 0$ .

Now, the function  $h$  defined by

$$h(a) = \frac{(a+2)(a+1)^2}{a}$$

is differentiable and has the derivative

$$h'(a) = \frac{2}{a^2}(a+1) \left( a + \frac{1+\sqrt{5}}{2} \right) \left( a + \frac{1-\sqrt{5}}{2} \right),$$

thus it attains (for  $a > 0$ ) a minimum at  $a = \frac{\sqrt{5}-1}{2}$ . The minimum value is

$$h\left(\frac{\sqrt{5}-1}{2}\right) = \frac{(\sqrt{5}+3)(\sqrt{5}+1)^2}{4(\sqrt{5}-1)} = \frac{11+5\sqrt{5}}{2}$$

so  $k$  must satisfy

$$\frac{11+5\sqrt{5}}{2} \geq \frac{1}{k} - 3 \Leftrightarrow \frac{1}{k} \leq \frac{17+5\sqrt{5}}{2} \Leftrightarrow k \geq \frac{17-5\sqrt{5}}{82}.$$

We conclude that the best  $k$  could be

$$\frac{17-5\sqrt{5}}{82} = 0.07097\dots,$$

but, again, we cannot prove that the inequality is true for this  $k$ . Instead, we can prove it for  $k = \frac{1}{7} = 0.142857$ , which is pretty far from the (possibly) best  $k$ . Indeed, for this  $k$ , the inequality to prove is

$$\frac{(a+b)(b+c)(c+a)}{8abc} \sum (a+b+c)(a-b)^2 \geq \sum c(a-b)^2,$$

and it holds true simply because for positive  $a, b, c$  we have

$$\frac{(a+b)(b+c)(c+a)}{8abc} \geq 1$$

and

$$a + b + c > \max\{a, b, c\}.$$

In its initial form, the inequality that we proved (for  $k = \frac{1}{7}$ ) is

$$\frac{a^3 + b^3 + c^3}{abc} + \frac{32abc}{(a+b)(b+c)(c+a)} \geq 7, \quad a, b, c > 0.$$

**Problem 5.** Determine the smallest  $k$  such that the inequality

$$(3 - 24k) \cdot \frac{abc}{a^3 + b^3 + c^3} + k \cdot \frac{(a+b)(b+c)(c+a)}{abc} \geq 1$$

is true for all positive real numbers  $a, b, c$ .

**Partial solution.** We can rewrite the inequality as

$$k \cdot \left( \frac{(a+b)(b+c)(c+a)}{abc} - 8 \right) \geq (1 - 8k) \cdot \left( 1 - \frac{3abc}{a^3 + b^3 + c^3} \right),$$

or

$$\frac{k}{abc} \sum c(a-b)^2 \geq \frac{(1-8k)(a+b+c)}{2(a^3 + b^3 + c^3)} \sum (a-b)^2.$$

Letting  $b = c = 1$  (and noticing that  $k$  has to be positive), we see that we need have

$$\frac{2(a^3 + 2)}{a(a+2)} \geq \frac{1}{k} - 8 \Leftrightarrow k \geq \frac{a^2 + 2a}{2(a^3 + 4a^2 + 8a + 2)}$$

for all  $a > 0$ . With the help of the computer one gets that the best  $k$  (that is, the maximum of the function from the right-hand side for  $a > 0$ ) could be 0.10058... attained at 1.19071... (actually at the positive solution of the equation  $a^4 + 4a^3 - 4a - 4 = 0$ ). Of course, it would be pretty hard to prove the inequality for this value of  $k$  with elementary calculations. Nevertheless, what we can do is to show that the inequality holds for  $k = \frac{1}{9} = 0.\bar{1}$  and for  $a, b, c$  being the lengths of the sides of a (possibly degenerate) triangle—that is, for positive  $a, b, c$  satisfying  $a + b \geq c$ ,  $b + c \geq a$ , and  $c + a \geq b$ . Indeed, for  $k = \frac{1}{9}$  the inequality can be expressed in the form

$$\sum \frac{2(a^3 + b^3 + c^3)}{ab} (a-b)^2 \geq \sum (a+b+c)(a-b)^2$$

and it follows (by using  $x^3 + y^3 \geq xy(x+y)$  for positive real numbers  $x, y$ ) because we have

$$\frac{2(a^3 + b^3 + c^3)}{ab} > \frac{2(a^3 + b^3)}{ab} \geq 2(a+b) \geq a+b+c$$

and two more similar inequalities. Notice that, in the initial form, the inequality that we proved reads

$$\frac{3abc}{a^3 + b^3 + c^3} + \frac{(a+b)(b+c)(c+a)}{abc} \geq 9$$

for  $a, b, c$  being the lengths of the sides of a (possibly degenerate) triangle. It would be nice to see if this inequality holds for all positive real numbers  $a, b, c$ . Notice that this happens if two of the variables are equal, but a proof for all  $a, b, c$  (if there is one) would certainly require more complicated computations.

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## References

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