

Junior problems

J709. Let a, b, c be distinct nonzero real numbers such that

$$\frac{a}{b^2} + ab + \frac{b}{a^2} = 3 \quad \text{and} \quad \frac{b}{c^2} - bc + \frac{c}{b^2} = -3.$$

Evaluate

$$\frac{c}{a^2} + 8ca - \frac{a}{c^2}.$$

Proposed by Adrian Andreescu, Dallas, USA

Solution 1 by the author

We have

$$a^3 + (ab)^3 + b^3 - 3a(ab)b = 0$$

and

$$b^3 + (-bc)^3 + c^3 - 3b(-bc)c = 0.$$

We use the fact that if u, v, w are real numbers, not all equal, then $u^3 + v^3 + w^3 - 3uvw = 0$ if and only if $u + v + w = 0$.

Hence $a + ab + b = 0$ and $b - bc + c = 0$, yielding

$$\frac{1}{a} + \frac{1}{b} = -1 \quad \text{and} \quad \frac{1}{b} + \frac{1}{c} = 1.$$

It follows that $\frac{1}{a} + 2 - \frac{1}{c} = 0$, which is equivalent to

$$c + 2ca + (-a) = 0.$$

Then

$$c^3 + (2ca)^3 + (-a)^3 - 3c(2ca)(-a) = 0,$$

implying

$$\frac{c}{a^2} + 8ca - \frac{a}{c^2} = -6.$$

Solution 2 by Diar Gashi, University of Primorska, Slovenia

Using the well-known identity

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z) \frac{1}{2} [(x - y)^2 + (y - z)^2 + (z - x)^2],$$

the first equation can be rewritten as:

$$\frac{1}{a^3} + \frac{1}{b^3} + 1^3 - 3 \frac{1}{ab} = 0 \iff \left(\frac{1}{a} + \frac{1}{b} + 1 \right) \left[\left(\frac{1}{a} - 1 \right)^2 + \left(\frac{1}{b} - 1 \right)^2 + \left(\frac{1}{a} - \frac{1}{b} \right)^2 \right] = 0,$$

and the second equation as:

$$\frac{1}{b^3} + \frac{1}{c^3} - 1^3 + 3 \frac{1}{bc} = 0 \iff \left(\frac{1}{b} + \frac{1}{c} - 1 \right) \left[\left(\frac{1}{b} + 1 \right)^2 + \left(\frac{1}{c} + 1 \right)^2 + \left(\frac{1}{b} - \frac{1}{c} \right)^2 \right] = 0.$$

Notice that in the first equation, the second expression is zero if and only if $\frac{1}{a} - 1 = \frac{1}{b} - 1 = \frac{1}{a} - \frac{1}{b} = 0$, impossible because a, b and c must be distinct.

Similarly, in the second equation, the second expression is zero if and only if $\frac{1}{b} + 1 = \frac{1}{c} + 1 = \frac{1}{b} - \frac{1}{c} = 0$, which violates once again the condition that a , b and c are distinct.

So,

$$\frac{1}{a} + \frac{1}{b} = -1 \quad \text{and} \quad \frac{1}{b} + \frac{1}{c} = 1.$$

Hence, $2 = (\frac{1}{b} + \frac{1}{c}) - (\frac{1}{a} + \frac{1}{b}) = \frac{1}{c} - \frac{1}{a}$. Using this, we get that

$$\frac{1}{c^3} - \frac{1}{a^3} = \left(\frac{1}{c} - \frac{1}{a}\right) \left(\left(\frac{1}{c} - \frac{1}{a}\right)^2 + 3\frac{1}{ac}\right) = 2\left(2^2 + 3\frac{1}{ac}\right) = 8 + 6\frac{1}{ac},$$

so

$$\frac{c}{a^2} + 8ca - \frac{a}{c^2} = -ac\left(\frac{1}{c^3} - \frac{1}{a^3}\right) + 8ac = -8ac - 6 + 8ac = -6.$$

Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; G. C. Greubel, Newport News, VA, USA; Hannah Yoon, Gwinnett School of Mathematics, Science, and Technology, GA, USA; Specialized Mathematical Gymnasium, Prishtine, Kosovo; Sundaresh H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Aarnav Sivachandran, AMMOC, USA, CT; Dow Lee, St. George's School, Vancouver, Canada; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Nguyen Tuan Minh; Polyhedra, Polk State College, FL, USA.

J710. Prove that for all positive real numbers a, b, c

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(ab+bc+ca)}{2(a+b+c)^2} \geq 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by The Ring Lords, Baden-Württemberg, Germany

By Titu's Lemma, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{a(b+c)} + \frac{b^2}{b(c+a)} + \frac{c^2}{c(a+b)} \geq \frac{(a+b+c)^2}{a(b+c) + b(c+a) + c(a+b)}.$$

Simplifying the denominator gives

$$a(b+c) + b(c+a) + c(a+b) = 2(ab+bc+ca),$$

so that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

It remains to prove the inequality

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} + \frac{9(ab+bc+ca)}{2(a+b+c)^2} \geq 3.$$

Finally, by the AM-GM inequality,

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} + \frac{9(ab+bc+ca)}{2(a+b+c)^2} \geq 2\sqrt{\frac{(a+b+c)^2}{2(ab+bc+ca)} \cdot \frac{9(ab+bc+ca)}{2(a+b+c)^2}} = 3,$$

as desired.

The equality holds if and only if $a = b = c$.

Solution 2 by Kousik Sett, India

We have

$$\begin{aligned} & \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(ab+bc+ca)}{2(a+b+c)^2} \\ &= \frac{a^2}{ab+ca} + \frac{9(ab+ca)}{4(a+b+c)^2} + \frac{b^2}{bc+ab} + \frac{9(ab+bc)}{4(a+b+c)^2} + \frac{c^2}{ca+bc} + \frac{9(bc+ca)}{4(a+b+c)^2} \end{aligned}$$

Applying the AM-GM Inequality on successive pair of terms, we obtain

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(ab+bc+ca)}{2(a+b+c)^2} \geq \frac{3a}{a+b+c} + \frac{3b}{a+b+c} + \frac{3c}{a+b+c} = 3,$$

and the equality holds when

$$2(a+b+c) = 3(b+c) = 3(c+a) = 3(a+b),$$

that is when $a = b = c$.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Specialized Mathematical Gymnasium, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Yiji Guangxi, China; Batyrkhan Beiganov, Almaty, Kazakhstan; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Nguyen Tuan Minh; Polyhedra, Polk State College, FL, USA; Thomas McCormick, AMMOC.

J711. Prove that

$$\left| \frac{x}{x^2 + 1} + \frac{6x}{x^2 + 36} \right| \leq \frac{7}{10}$$

for all $x \in \mathbb{R}$.

Proposed by An Zhenping, Xianyang Normal University, China

Solution 1 by Theo Koupelis, Clark College, Washington, USA

Clearing denominators, the desired inequality is equivalent to

$$10|x|(x^2 + 6) \leq x^4 + 37x^2 + 36 \iff (|x| - 3)^2(|x| - 2)^2 \geq 0,$$

which is obvious. Equality occurs when $x \in \{-3, -2, 2, 3\}$.

Solution 2 by Kousik Sett, India

For $x = 0$, the inequality is trivial. Assume $x \neq 0$ and let $xy = 6$. Then either $x, y > 0$ or $x, y < 0$.

If $x, y > 0$ then we have

$$\begin{aligned} \frac{x}{x^2 + 1} + \frac{6x}{x^2 + 36} &= \frac{x}{x^2 + 1} + \frac{y}{y^2 + 1} = \frac{(x + y)(xy + 1)}{x^2y^2 + x^2 + y^2 + 1} \\ &= \frac{7(x + y)}{37 + (x + y)^2 - 2xy} = \frac{7(x + y)}{25 + (x + y)^2} = \frac{7}{\frac{25}{(x + y)} + (x + y)}. \end{aligned}$$

From the AM-GM Inequality, we have

$$\frac{25}{(x + y)} + (x + y) \geq 2 \cdot 5 = 10$$

and consequently,

$$\frac{x}{x^2 + 1} + \frac{6x}{x^2 + 36} \leq \frac{7}{10},$$

and the equality holds when $x + y = 5$ and $xy = 6$, that is when $x = 2$ or 3 .

If $x, y < 0$ then assume $x' = -x$ and $y' = -y$ and consequently, from the above, we have

$$\frac{x'}{x'^2 + 1} + \frac{6x'}{x'^2 + 36} \leq \frac{7}{10} \quad \text{or} \quad \frac{x}{x^2 + 1} + \frac{6x}{x^2 + 36} \geq -\frac{7}{10},$$

and the equality holds for $x = -2$ or -3 . Therefore, the result follows.

Also solved by Alex Habin Song, Northview High School in Johns Creek, GA, USA; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; G. C. Greubel, Newport News, VA, USA; Hannah Yoon, Gwinnett School of Mathematics, Science, and Technology, GA, USA; Henry Ricardo, Westchester Area Math Circle, New York, USA; Jonathan Lee, Buford High School Buford, GA, USA; Kalkin Gandhi, AMMOC Bengaluru, India; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Specialized Mathematical Gymnasium, Prishtine, Kosovo; Sundares H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Batyrkhan Beiganov, Almaty, Kazakhstan; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Dow Lee, St. George's School, Vancouver, Canada; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Nguyen Tuan Minh; Polyhedra, Polk State College, FL, USA; Thomas McCormick, AMMOC.

J712. Let ABC be a triangle with circumcenter O and let X be the midpoint of side BC . The circumcircle Ω of triangle BOC intersects AB and AC in D and E respectively. The line OX intersects again Ω in Y and the lines AY and DE intersect in Z . Prove that

$$2\overrightarrow{ZX} = \overrightarrow{DB} + \overrightarrow{EC}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

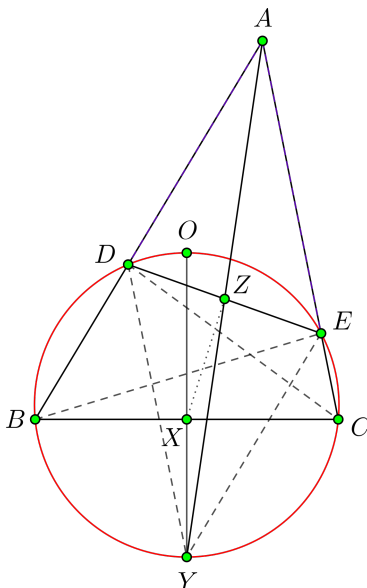
Solution 1 by Kousik Sett, India

Since O is the circumcenter of triangle ABC and X is the midpoint of BC , it follows that line OX is the perpendicular bisector of BC . Moreover, since O , X , and Y are collinear, then we have Y is the midpoint of arc BC not containing O . Therefore, DY and EY are the angle bisectors of angles BDC and BEC , respectively.

It is well-known that $\angle BOC = 2\angle BAC$.

Thus,

$$\angle BDY = \frac{1}{2}\angle BDC = \frac{1}{2}\angle BOC = \angle BAC \quad \text{and} \quad \angle YEC = \frac{1}{2}\angle BEC = \frac{1}{2}\angle BOC = \angle BAC.$$



Hence it follows that $DY \parallel AE$ and $EY \parallel AD$. Therefore, $ADYE$ is a parallelogram. Further, since $Z = AY \cap DE$, we conclude that Z is the midpoint of DE . Let the position vectors of B , C , E , and D be $\vec{b}$, $\vec{c}$, $\vec{e}$, and $\vec{d}$, respectively. Then the position vectors of X and Z are

$$\frac{\vec{b} + \vec{c}}{2} \quad \text{and} \quad \frac{\vec{d} + \vec{e}}{2}.$$

Therefore,

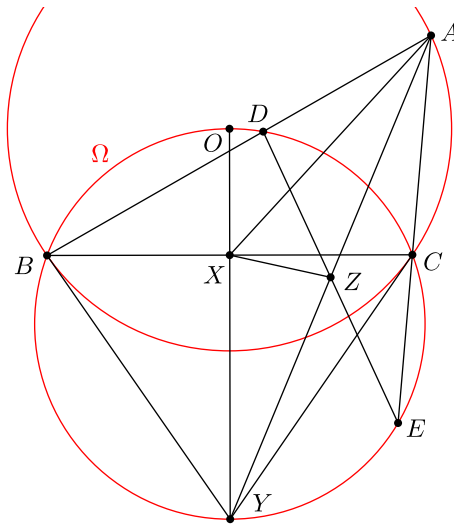
$$2\overrightarrow{ZX} = 2\left(\frac{\vec{b} + \vec{c}}{2} - \frac{\vec{d} + \vec{e}}{2}\right) = \vec{b} - \vec{d} + \vec{c} - \vec{e} = \overrightarrow{DB} + \overrightarrow{EC}.$$

Editor's Comment. The solution implicitly assumes that the triangle is acute, something which is not explicitly stated in the hypothesis (we thank Theo Koupelis for noticing that). However, the conclusion holds in general, even if the triangle is obtuse. As a matter of fact, O and Y are the midpoints of the two arcs BC of Ω . Therefore, for any point $D \in \Omega$, the lines DO and DY are the internal and external angle bisectors of $\angle BDC$ (in some order) and similarly EO and EY are the internal and external angle bisectors

of $\angle BEC$. If θ is the (smaller) angle $\angle BOC$, since $D \in \Omega$, the inscribed angle $\angle BDC$ equals either θ or $180^\circ - \theta$. Because DY is an internal or external bisector of $\angle BDC$, we have $\angle(BD, DY) = \frac{\theta}{2}$. Similarly, since $E \in \Omega$ and EY is an internal or external bisector of $\angle BEC$, we have $\angle(EY, EC) = \frac{\theta}{2}$. On the other hand, the (smaller) angle between the lines AB and AC is also $\frac{\theta}{2}$. Therefore $DY \parallel AC$ and $EY \parallel AB$, i.e. $DY \parallel AE$ and $EY \parallel AD$. From now on everything proceeds exactly as in the first solution.

Solution 2 by Polyhedra, Polk State College, FL, USA

By the power of a point, $AD \cdot AB = AC \cdot AE$, so $\triangle ADE \sim \triangle ACB$. Since YB and YC are tangent to the circumcircle of $\triangle ABC$, AY is the A -symmedian of $\triangle ABC$, so Z is the midpoint of DE , from which the claim follows immediately.



Also solved by Specialized Mathematical Gymnasium, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Batyrkhan Beiganov, Almaty, Kazakhstan; Diar Gashi, University of Primorska, Slovenia; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Nguyen Tuan Minh.

J713. Let a, b, c be positive real numbers such that $ab + bc + ca = 3abc$. Prove that

$$1 \leq abc \leq \frac{(a+b+c)^3}{12(a+b+c)-9}.$$

Proposed by Titu Andreescu, Dallas, USA

Solution 1 by Diar Gashi, University of Primorska, Slovenia

By the AM-GM Inequality,

$$3abc = ab + bc + ca \geq 3\sqrt[3]{(abc)^2} \implies \sqrt[3]{abc} \geq 1 \implies abc \geq 1.$$

Notice that the RHS is equivalent to the following

$$4(a+b+c) \cdot 3abc - 9abc \leq (a+b+c)^3,$$

that is,

$$\begin{aligned} & 4(a+b+c)(ab+bc+ca) - 9abc \leq (a+b+c)^3 \\ \iff & 12abc + 4 \sum_{cyc} (a^2b + ab^2) - 9abc \leq \sum_{cyc} (a^3) + 3 \sum_{cyc} (a^2b) + 3 \sum_{cyc} (ab^2) + 6abc \\ \iff & \sum_{cyc} (a^2b + ab^2) \leq \sum_{cyc} (a^3) + 3abc, \end{aligned}$$

which is well-known to be Schur's inequality.

The equality holds if and only if $a = b = c = 1$.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Without loss of generality we may assume that $a \geq b \geq c > 0$. Let $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$, $p = x + y + z$, $q = xy + yz + zx$, and $r = xyz$. Then $0 < x \leq y \leq z$, condition $ab + bc + ca = 3abc$ reads as $p = 3$, and we have to prove that

$$r \leq \min\left(1, \frac{q^3}{12q - 9r}\right).$$

Since $p = 3$, the AM-GM inequality directly gives that $r \leq 1$, so it only remains to show that

$$r \leq \frac{q^3}{12q - 9r}. \quad (*)$$

But $q \geq 3r^{\frac{2}{3}}$ (by the AM-GM inequality) and $r^{\frac{2}{3}} \geq r$ (since $0 < r \leq 1$), so $q \geq 3r$. In particular, $12q \geq 36r > 9r$, and therefore we may multiply $(*)$ by $12q - 9r > 0$ and arrange terms to obtain the equivalent inequality

$$0 \leq 9r^2 - 12qr + q^3. \quad (**)$$

Now the function $f_q(t) = 9t^2 - 12qt + q^3$ satisfies that $f'_q(t) = 18t - 12q = 18(t - \frac{2q}{3}) < 0$, for $t < \frac{2q}{3}$, so f_q is decreasing on the interval $(-\infty, \frac{2q}{3}]$. Since r belongs to that interval (because $r \leq \frac{q}{3}$), it follows that we only have to prove $(**)$ for the maximal value of r , for fixed q . We know that this maximal value is assumed when $x = y$. (See Corollary 1 on page 57 from the book *118 Inequalities for Mathematics Competitions* by Titu Andreescu and Marius Stălean, published by XYZ Press in 2019). But for $x = y$ and $z = 3 - 2x$, $0 < x \leq 1$, we have that

$$9r^2 - 12qr + q^3 = 9x^4(3 - 2x)^2 - 12x^2(3 - 2x)(x^2 + 2x(3 - 2x)) + (x^2 + 2x(3 - 2x))^3 = 9x^4(x - 1)^2$$

so $(**)$ holds, and we are done.

Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Kousik Sett, India; Kristine Lee, Mount Juliet High School, TN, USA; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Specialized Mathematical Gymnasium, Prishtine, Kosovo; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Batyrkhan Beiganov, Almaty, Kazakhstan; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Nguyen Tuan Minh; Polyhedra, Polk State College, FL, USA; Thomas McCormick, AMMOC.

J714. Let x, y, z be nonnegative real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$xy + yz + zx + \frac{3}{2}\sqrt{x^2 + y^2 + z^2 - xy - yz - zx} \geq 3.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

Since $x + y + z \leq 3$, we have

$$9 \geq (x + y + z)^2 \geq 3(xy + yz + zx) \implies 3 \geq xy + yz + zx$$

hence

$$\sqrt{3(3 - xy - yz - zx)} \geq 3 - xy - yz - zx.$$

Therefore it suffices to prove that

$$\frac{3}{2}\sqrt{x^2 + y^2 + z^2 - xy - yz - zx} \geq \sqrt{3(3 - xy - yz - zx)},$$

or

$$3(x^2 + y^2 + z^2) + xy + yz + zx \geq 12,$$

or

$$3(x^2 + y^2 + z^2) + xy + yz + zx \geq 3(x^2 + y^2 + z^2 + xyz),$$

or

$$xy + yz + zx \geq 3xyz.$$

But this follows by AM-GM Inequality, i.e.

$$xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2} \geq 3xyz$$

since it is clear that $xyz \leq 1$.

The equality holds when $x = y = z = 1$ or $x = 2, y = z = 0$ and its cyclic permutations.

Solution 2 by Daniel Pascuas, Barcelona, Spain

Without loss of generality we may assume that $0 \leq x \leq y \leq z$. Since $x^2 + y^2 + z^2 + xyz = 4$, we may write the inequality to be proved as

$$\frac{3}{2}\sqrt{4 - xyz - xy - yz - zx} \geq 3 - (xy + yz + zx)$$

If we let $p = x + y + z$, $q = xy + yz + zx$, and $r = xyz$, then the above inequality reads as

$$\frac{3}{2}\sqrt{4 - r - q} \geq 3 - q. \quad (*)$$

Note that $3q \leq p^2$ (since $p^2 - 3q = x^2 + y^2 + z^2 - (xy + yz + zx) = \frac{1}{2}((x - y)^2 + (y - z)^2 + (z - x)^2) \geq 0$). Moreover, it is known that the condition $x^2 + y^2 + z^2 + xyz = 4$ implies that $p \leq 3$ (see page 77 on the book *New, Newer, and Newest Inequalities* by Titu Andreescu and Marius Stănean, published by XYZ Press in 2021). It follows that $q \leq 3$, that is, $3 - q \geq 0$, and so, by squaring and arranging terms, $(*)$ is equivalent to

$$15q - 4q^2 \geq 9r \quad (*)$$

But we know that, for fixed p and q , r is maximal when $x = y$ (see Corollary 1 on page 57 from the book *118 Inequalities for Mathematics Competitions* by Titu Andreescu and Marius Stănean, published by XYZ Press in 2019). Thus we only have to prove $(*)$ when $0 \leq x = y \leq z$, that is, when $z \geq 0$ and $x = y = tz$, for some $0 \leq t \leq 1$. Then $q = x^2 + 2xz = (t^2 + 2t)z^2$, $r = x^2z = t^2z^3$, and condition $x^2 + y^2 + z^2 + xyz = 4$

reads as $t^2 z^2(2+z) + z^2 = 4$, that is, $t = \frac{\sqrt{2-z}}{z}$ and $0 < z \leq 2$. Now $0 \leq t \leq 1$ means that $0 \leq z^2 + z - 2$, that is, $0 < z \leq 1$ (since $z^2 + z - 2 = (z-1)(z+2)$). Therefore

$$\begin{aligned} 15q - 4q^2 - 9r &= 15(t^2 + 2t)z^2 - 4t^2(t+2)^2 z^4 - 9t^2 z^3 \\ &= 15(2-z) - 4(2-z)^2 - 2z(1-8z)\sqrt{2-z} - 16z^2(2-z) - 9z(2-z). \end{aligned}$$

In order to prove that this quantity is nonnegative let us write it in terms of $\sqrt{2-z}$ by taking into account that $z = 2 - (\sqrt{2-z})^2$. We have that $15q - 4q^2 - 9r = P(\sqrt{2-z})$, where

$$\begin{aligned} P(X) &= -16X^6 + 16X^5 + 69X^4 - 62X^3 - 67X^2 + 60X = P_1(X)P_2(X), \\ P_1(X) &= X(4X+5)(X-1)^2, \quad \text{and} \quad P_2(X) = 12 + X - 4X^2. \end{aligned}$$

Note that $\{\sqrt{2-z} : 0 < z \leq 1\} = [1, \sqrt{2})$, $P_1(1) = 0$, and $P_1(X) > 0$, for any $X > 1$. Moreover, P_2 is decreasing on $[1, \infty)$ (since $P_2'(X) = 1 - 8X < 0$, for any $X \geq 1$), so $P_2(X) \geq P_2(\sqrt{2}) > 0$, for any $X \in [1, \sqrt{2})$. Hence it turns out that $15q - 4q^2 - 9r = P(\sqrt{2-z}) = P_1(\sqrt{2-z})P_2(\sqrt{2-z}) \geq 0$, for any $0 < z \leq 1$, and that completes the proof of the inequality at the statement. Note that we have also shown that the inequality becomes an equality if and only if $x = y = z = 1$.

Also solved by Arkady Alt, San Jose, California, USA; Specialized Mathematical Gymnasium, Prishtine, Kosovo; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Polyhedra, Polk State College, FL, USA.

Senior problems

S709. Find all pairs (m, n) of positive integers such that

$$m + 4\sqrt{n-1} \quad \text{and} \quad n + 4\sqrt{m-1}$$

are both perfect squares.

Proposed by Titu Andreescu, Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Let $m + 4\sqrt{n-1} = s^2$ and $n + 4\sqrt{m-1} = t^2$, where $s, t \in \mathbb{N}$. Since $4\sqrt{n-1} = s^2 - m$ and $4\sqrt{m-1} = t^2 - n$ are integers, it follows that $m-1 = a^2$ and $n-1 = b^2$ for some $a, b \in \mathbb{N}$. Then, the problem is equivalent to find the natural numbers a and b such that $a^2 + 4b + 1$ and $b^2 + 4a + 1$ are both perfect squares. By symmetry, we can assume that $a \leq b$. Since

$$b^2 < b^2 + 4a + 1 \leq b^2 + 4b + 1 < b^2 + 4b + 4 = (b+2)^2,$$

it follows that $b^2 + 4a + 1 = (b+1)^2$, i.e. $b = 2a$. Hence, $a^2 + 4b + 1 = a^2 + 8a + 1$ and

$$(a+1)^2 \leq a^2 + 8a + 1 < (a+4)^2.$$

We have three cases.

- (i) If $a^2 + 8a + 1 = (a+1)^2$, then $a = 0$ and so $b = 0$, which yields $(m, n) = (1, 1)$.
- (ii) If $a^2 + 8a + 1 = (a+2)^2$, then $4a = 3$, which gives no integer solution.
- (iii) If $a^2 + 8a + 1 = (a+3)^2$, then $2a = 8$, i.e. $a = 4$ and so $b = 8$. It follows that $(m, n) = (17, 65)$.

In conclusion, $(m, n) \in \{(1, 1), (17, 65), (65, 17)\}$.

Also solved by Anita Tufisi, Romania; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; G. C. Greubel, Newport News, VA, USA; Specialized Mathematical Gymnasium, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Aarnav Sivachandran, AMMOC, USA, CT; Batyrkhan Beiganov, Almaty, Kazakhstan; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam; Nguyen Tuan Minh.

S710. Let a, b, c be nonnegative real numbers such that $a + b + c = ab + bc + ca > 0$. Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 2\sqrt{2} + (3 - 2\sqrt{2})\sqrt{abc}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Without loss of generality, we may assume that $a \geq b \geq c$. We have

$$a + b + c = ab + bc + ca \geq c^2 + bc + ca \implies c \leq 1$$

and

$$(a + b + c)^2 \geq 3(ab + bc + ca) = 3(a + b + c) \implies a + b + c \geq 3,$$

so $a + b \geq 2$. Also, we have

$$3ab \geq ab + bc + ca = \frac{(a + b + c)^2}{ab + bc + ca} \geq 3 \implies ab \geq 1.$$

The inequality can be rewritten as

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \left(1 - (3 - 2\sqrt{2})\sqrt{ab} \right) \geq 2\sqrt{2},$$

or

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \left(1 - \frac{\sqrt{ab}}{3 + 2\sqrt{2}} \right) \geq 2\sqrt{2},$$

We have three cases:

Case 1. $\sqrt{ab} \geq 3 + 2\sqrt{2}$, since $c \leq 1$ it suffices to prove that

$$\sqrt{a} + \sqrt{b} + \left(1 - \frac{\sqrt{ab}}{3 + 2\sqrt{2}} \right) \geq 2\sqrt{2},$$

or

$$\sqrt{a} + \sqrt{b} \geq 2\sqrt{2} - 1 + (3 - 2\sqrt{2})\sqrt{ab},$$

or, after squaring both sides

$$a + b + 2\sqrt{ab} \geq 9 - 4\sqrt{2} + 2(8\sqrt{2} - 11)\sqrt{ab} + (17 - 12\sqrt{2})ab,$$

or

$$a + b + 8(3 - 2\sqrt{2})\sqrt{ab} \geq 9 - 4\sqrt{2} + (17 - 12\sqrt{2})ab,$$

which is true because

$$a + b \geq ab \geq (17 - 12\sqrt{2})ab,$$

$$8(3 - 2\sqrt{2})\sqrt{ab} \geq 8 > 9 - 4\sqrt{2}.$$

Case 2. $2 \leq \sqrt{ab} < 3 + 2\sqrt{2}$, the inequality is obviously true because

$$LHS \geq \sqrt{a} + \sqrt{b} \geq 2\sqrt[4]{ab} \geq 2\sqrt{2}.$$

The equality holds when $a = b = 2, c = 0$

Case 3. If $1 \leq \sqrt{ab} < 2$ then we have

$$(\sqrt{ab} - 1)(\sqrt{ab} - 2) \leq 0 \iff ab - 3\sqrt{ab} + 2 \leq 0 \iff \frac{ab + \sqrt{ab} + 2}{2} \leq 2\sqrt{ab} \leq a + b,$$

by the AM-GM Inequality. We get

$$2(a + b - 1) \geq \sqrt{ab}(\sqrt{ab} + 1).$$

It follows that

$$c = \frac{a + b - ab}{a + b - 1} = 1 - \frac{ab - 1}{a + b - 1} \geq 1 - \frac{2(ab - 1)}{\sqrt{ab}(\sqrt{ab} + 1)} = -1 + \frac{2}{\sqrt{ab}}.$$

Returning to our problem, it is sufficient to show

$$2\sqrt[4]{ab} + \sqrt{-1 + \frac{2}{\sqrt{ab}}} \left(1 - \frac{\sqrt{ab}}{3 + 2\sqrt{2}} \right) \geq 2\sqrt{2},$$

or if setting $t = \sqrt[4]{ab} \in [1, \sqrt{2})$ it becomes

$$\sqrt{2 - t^2}(3 + 2\sqrt{2} - t^2) \geq 2(3 + 2\sqrt{2})t(\sqrt{2} - t),$$

or, after squaring both sides and factoring, this becomes

$$(\sqrt{2} - t)(t - 1)^2 \left(t^3 + (2 + \sqrt{2})t^2 + (65 + 46\sqrt{2})t + 17\sqrt{2} + 24 \right) \geq 0$$

which is clearly true.

The equality holds when $t = 1$ which means $a = b = c = 1$.

Also solved by Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA.

S711. Show that in any triangle ABC the following inequality holds:

$$\sum \frac{\tan \frac{A}{2}}{\cos^2 \frac{B-C}{2}} \geq \sum \frac{1}{\sin A + \sin B}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam

We have the identity

$$\frac{\tan \frac{A}{2}}{\cos^2 \frac{B-C}{2}} = \frac{2 \sin A}{(\sin B + \sin C)^2},$$

since

$$\sin B + \sin C = 2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} = 2 \cos \frac{A}{2} \cos \frac{B-C}{2},$$

and

$$\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}}.$$

Therefore the given inequality can be rewritten in the form

$$\sum_{\text{cyc}} \frac{2 \sin A}{(\sin B + \sin C)^2} \geq \sum_{\text{cyc}} \frac{1}{\sin A + \sin B}.$$

By the sine law,

$$\sin A = \frac{a}{2R}, \quad \sin B = \frac{b}{2R}, \quad \sin C = \frac{c}{2R},$$

where a, b, c are the side lengths and R is the circumradius. Hence

$$\frac{2 \sin A}{(\sin B + \sin C)^2} = \frac{2 \cdot \frac{a}{2R}}{\left(\frac{b}{2R} + \frac{c}{2R}\right)^2} = \frac{4Ra}{(b+c)^2}, \quad \frac{1}{\sin A + \sin B} = \frac{1}{\frac{a}{2R} + \frac{b}{2R}} = \frac{2R}{a+b}.$$

Thus the inequality is equivalent to

$$\sum_{\text{cyc}} \frac{2a}{(b+c)^2} \geq \sum_{\text{cyc}} \frac{1}{a+b}.$$

By Chebyshev's inequality applied to the similarly ordered sequences

$$\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \right) \quad \text{and} \quad \left(\frac{2a}{b+c}, \frac{2b}{c+a}, \frac{2c}{a+b} \right),$$

we obtain

$$\sum_{\text{cyc}} \frac{2a}{(b+c)^2} = \sum_{\text{cyc}} \frac{2a}{b+c} \cdot \frac{1}{b+c} \geq \left(\sum_{\text{cyc}} \frac{1}{b+c} \right) \left(\sum_{\text{cyc}} \frac{2a}{3(b+c)} \right).$$

By Nesbitt's inequality,

$$\sum_{\text{cyc}} \frac{a}{b+c} \geq \frac{3}{2},$$

so

$$\sum_{\text{cyc}} \frac{2a}{(b+c)^2} \geq \left(\sum_{\text{cyc}} \frac{1}{b+c} \right) \cdot 1 = \sum_{\text{cyc}} \frac{1}{a+b}.$$

This proves the desired inequality. Equality holds if and only if $a = b = c$, i.e. when the triangle is equilateral.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $BC = a, AC = b, AB = c$, and let s, R, E be the semiperimeter, circumradius, and area of the triangle. Using the law of cosines we get that in any triangle

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \text{and} \quad \tan \frac{A}{2} = \frac{(s-b)(s-c)}{E},$$

and cyclic expressions, where we used Heron's formula for the area. Also

$$\frac{1}{\sin A + \sin B} = \frac{2R}{a + b},$$

and cyclic expressions, because $(a, b, c) = (2R \sin A, 2R \sin B, 2R \sin C)$. Finally,

$$\begin{aligned} 2 \cos^2 \frac{B-C}{2} &= 1 + \cos(B-C) = 1 + \cos B \cos C + \sin B \sin C \\ &= 1 + \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{4a^2bc} + \frac{bc}{4R^2} \\ &= \frac{2(b+c)^2(s-b)(s-c)}{a^2bc} = \frac{(b+c)^2}{2Ra} \cdot \tan \frac{A}{2}, \end{aligned}$$

and cyclic expressions, where we used $E = abc/(4R)$. Substituting, we get that the desired inequality is equivalent to

$$\frac{2a}{(b+c)^2} + \frac{2b}{(c+a)^2} + \frac{2c}{(a+b)^2} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

Clearing denominators we get

$$2 \sum_c a^5 + 3 \sum_c a^4(b+c) - 2 \sum_c a^3(b^2+c^2) - 4abc \sum_c ab \geq 0,$$

which is obvious by Muirhead's inequality because $(5, 0, 0) \succ (4, 1, 0) \succ (3, 2, 0) \succ (2, 2, 1)$. Equality occurs for an equilateral triangle.

Also solved by Arkady Alt, San Jose, California, USA; Marin Chirciu, Colegiul National Zinca Golescu Pitesti, Romania; Specialized Mathematical Gymnasium, Prishtine, Kosovo; The Ring Lords, Baden-Württemberg, Germany; Yun Zhang, Shaan xi province, China and Kaiqing Gao, Shaan xi province, China.

S712. Let x, y, z be positive real numbers such that $xy + yz + zx + xyz = 4$. Prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 2 \left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} \right).$$

Proposed by Mircea Lascu, Zalău, Romania

Solution 1 by the author

After clearing denominators and expanding this becomes

$$x^2y^2 + y^2z^2 + z^2x^2 + xyz(xy + yz + zx) \geq 2xyz(x + y + z),$$

or

$$(xy + yz + zx)^2 + xyz(xy + yz + zx) \geq 4xyz(x + y + z),$$

or

$$(xy + yz + zx)(xy + yz + zx + xyz) \geq 4xyz(x + y + z),$$

that is

$$xy + yz + zx \geq xyz(x + y + z),$$

or

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z.$$

Homogenizing the inequality with the substitutions

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b},$$

the inequality becomes

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 4 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right).$$

The inequality can be rewritten as

$$\begin{aligned} & \frac{a+b+c-c}{c} + \frac{b+c+a-a}{a} + \frac{c+a+b-b}{b} \\ & \geq 4 \left(\frac{a+b+c-b-c}{b+c} + \frac{b+c+a-c-a}{c+a} + \frac{c+a+b-a-b}{a+b} \right), \end{aligned}$$

or

$$(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 3 \geq 4(a+b+c) \left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) - 12.$$

Dividing both sides by $a+b+c$, this is just

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{9}{a+b+c} \geq \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{c+a},$$

which is Popoviciu's Inequality applied to the convex function $f(x) = \frac{1}{x}$.

Solution 2 by Kousik Sett, India

Using the substitution

$$x = \frac{bc}{a}, \quad y = \frac{ca}{b}, \quad z = \frac{ab}{c},$$

the equality

$$xy + yz + zx + xyz = 4$$

can be written as

$$a^2 + b^2 + c^2 + abc = 4.$$

Notice that $a, b, c \in (0, 2)$ and hence we'll make the substitution for $a = 2 \cos A$, $b = 2 \cos B$, $c = 2 \cos C$. Therefore, the constraint becomes

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1 \Rightarrow A + B + C = \pi.$$

Thus, there is a triangle of angles A, B, C , for which the inequalities

$$\cos A + \cos B + \cos C \leq \frac{3}{2} \quad \text{and} \quad \cos^2 A + \cos^2 B + \cos^2 C \geq \frac{3}{4}$$

are well known (See Andreescu, A., Andreescu, T., Mushkarov, O., *113 Geometric Inequalities from the AwesomeMath Summer Program*, XYZ press, p.58).

Combining the above two inequalities, we obtain

$$2(\cos^2 A + \cos^2 B + \cos^2 C) \geq \cos A + \cos B + \cos C,$$

which is equivalent to

$$a^2 + b^2 + c^2 \geq a + b + c.$$

On other hand using the substitution

$$x = \frac{bc}{a}, \quad y = \frac{ca}{b}, \quad z = \frac{ab}{c},$$

the given inequality can be written as

$$a^4 + b^4 + c^4 + abc(a^2 + b^2 + c^2) \geq 2(a^2b^2 + b^2c^2 + c^2a^2).$$

Since

$$a^2 + b^2 + c^2 \geq a + b + c,$$

it suffices to prove

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq 2(a^2b^2 + b^2c^2 + c^2a^2).$$

Recall the Schur's inequality: For any nonnegative real numbers a, b, c and any positive number k , the inequality holds

$$a^k(a - b)(a - c) + b^k(b - c)(b - a) + c^k(c - a)(c - b) \geq 0.$$

For $k = 2$, the above inequality can be rewritten as

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 2(a^2b^2 + b^2c^2 + c^2a^2)$$

by the AM-GM inequality and we are done. The equality holds for $a = b = c$ that is for $x = y = z$.

Solution 3 by Daniel Pascuas, Barcelona, Spain

Without loss of generality we may assume that $x \geq y \geq z > 0$. Let $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$. Then $0 < a \leq b \leq c$, the condition $xy + yz + zx + xyz = 4$ reads as $a + b + c + 1 = 4abc$, and we want to prove that

$$a^2 + b^2 + c^2 + a + b + c \geq 2(ab + bc + ca). \quad (*)$$

Let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$. Then we know that $p + 1 = 4r$ and we want to prove that

$$p^2 - 2q + p \geq 2q, \quad \text{that is,} \quad p^2 + p \geq 4q.$$

But, given p and r , the value of q is maximal when $b = c$ (see, for instance, Corollary 1 on page 57 from the book *118 Inequalities for Mathematics Competitions* by Titu Andreescu and Marius Stănean, published by XYZ Press in 2019), so we only have to prove $(*)$ in this case, that is,

$$a^2 + a + 2b \geq 4ab, \quad \text{when } 0 < a \leq b \text{ and } a + 2b + 1 = 4ab^2,$$

or equivalently,

$$a^2 + 4ab(b-1) - 1 \geq 0, \quad \text{when } 0 < a \leq b \text{ and } a + 2b + 1 = 4ab^2,$$

Let $b > 0$. Then $0 < a \leq b$ means that $a = tb$, where $0 < t \leq 1$. The condition $a + 2b + 1 = 4ab^2$ says that $(t+2)b + 1 = 4tb^3$, that is, $t = \frac{1}{b(2b-1)}$, i.e. $a = \frac{1}{2b-1}$, where $b \neq \frac{1}{2}$. Moreover,

$$0 < t \leq 1 \iff 1 \leq b(2b-1) \iff 0 \leq 2b^2 - b - 1 = 2(b + \frac{1}{2})(b-1) \iff b \geq 1.$$

Now for $b \geq 1$ and $a = \frac{1}{2b-1}$, we have that

$$a^2 + 4ab(b-1) - 1 = \frac{8b^3 - 16b^2 + 8b}{(2b-1)^2} = \frac{8b(b-1)^2}{(2b-1)^2} \geq 0,$$

which completes the proof of the inequality at the statement. Note that we have also proved that the inequality becomes an equality if and only if $x = y = z = 1$.

Also solved by Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Specialized Mathematical Gymnasium, Prishtine, Kosovo; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Xiang Jiang, Shanghai, China; Yiji Guangxi, China; Nguyen Tuan Minh; Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam.

S713. Let $k > 2$ be an integer. Prove that for each $j \leq k$ there are distinct prime numbers $p_1, \dots, p_j$ such that

$$p_1^{a_1} + \dots + p_j^{a_j} \equiv j - k \pmod{p_1 \dots p_j},$$

for some positive integers $a_1, \dots, a_j$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Specialized Mathematical Gymnasium, Prishtine, Kosovo

We proceed by induction on j .

Set

$$P_j := \prod_{i=1}^j p_i \quad \text{and} \quad S_j := \sum_{i=1}^j p_i^{a_i}.$$

We wish to achieve

$$S_j \equiv j - k \pmod{P_j}.$$

Base Case ($j = 1$). We need a prime p_1 and integer $a_1 > 0$ such that

$$p_1^{a_1} \equiv 1 - k \pmod{p_1}.$$

Since $p_1^{a_1} \equiv 0 \pmod{p_1}$ for any $a_1 \geq 1$, this reduces to

$$0 \equiv 1 - k \pmod{p_1},$$

i.e. $p_1 \mid (k - 1)$. Choose any prime divisor p_1 of $k - 1$ and take $a_1 = 1$. Then

$$S_1 = p_1^{a_1} \equiv 1 - k \pmod{p_1},$$

so the base case holds.

Inductive Step. Assume that for some $1 \leq j < k$ there exist distinct primes $p_1, \dots, p_j$ and positive integers $a_1, \dots, a_j$ satisfying

$$S_j \equiv j - k \pmod{P_j}.$$

We will construct p_{j+1} and a_{j+1} such that

$$S_{j+1} := S_j + p_{j+1}^{a_{j+1}} \equiv (j + 1) - k \pmod{P_{j+1}},$$

where $P_{j+1} = P_j \cdot p_{j+1}$.

Consider the integer

$$T := S_j + k - (j + 1) = (S_j + k - j) - 1.$$

By the inductive hypothesis, $S_j + k - j \equiv 0 \pmod{P_j}$, hence

$$T \equiv -1 \pmod{P_j}.$$

Therefore $\gcd(T, P_j) = 1$, so T has a prime divisor not among $p_1, \dots, p_j$. Let p_{j+1} be any prime divisor of T . By definition,

$$T \equiv 0 \pmod{p_{j+1}} \implies S_j \equiv j + 1 - k \pmod{p_{j+1}}. \quad (1)$$

Since p_{j+1} is coprime to P_j , there exists an integer $m \geq 1$ such that

$$p_{j+1}^m \equiv 1 \pmod{P_j}.$$

Choose $a_{j+1} = m$. Then

$$p_{j+1}^{a_{j+1}} \equiv 1 \pmod{P_j}.$$

For every $i = 1, \dots, j$, we have

$$S_{j+1} = S_j + p_{j+1}^{a_{j+1}} \equiv (j - k) + 1 = j + 1 - k \pmod{p_i}.$$

Together with (1), which ensures the same congruence modulo p_{j+1} , the Chinese Remainder Theorem yields

$$S_{j+1} \equiv j + 1 - k \pmod{P_{j+1}}.$$

Thus the construction works for $j + 1$, and since all primes remain distinct, the statement holds for all $1 \leq j \leq k$.

S714. Prove that there is no positive integer n such that $2^n + 3^n + 5^n$ is a perfect square.

Proposed by Mohammadreza Ghelichkhani, Shahed University, Tehran, Iran

Solution 1 by the author

If n is even, then $2^n + 3^n + 5^n \equiv 2 \pmod{4}$, which is a contradiction since a perfect square is congruent to 0 or 1 $\pmod{4}$.

If n is odd, we have $v_2(3^n + 5^n) = 3$. Thus, if $n \geq 4$, it follows that $v_2(2^n + 3^n + 5^n) = 3$, leading to another contradiction.

Checking the cases $n = 1$ and $n = 3$, we find that neither is a solution. Therefore, there are no solutions.

Solution 2 by Anita Tufisi, Romania

Let

$$S_n = 2^n + 3^n + 5^n.$$

We have two cases.

(i) If n is odd, we have $2^n \equiv 2 \pmod{8}$, $3^n \equiv 3 \pmod{8}$, and $5^n \equiv 5 \pmod{8}$. Thus

$$S_n \equiv 2 + 3 + 5 = 10 \equiv 2 \pmod{8}.$$

But squares modulo 8 can only be 0, 1, 4, so S_n cannot be a perfect square.

(ii) If n is even, then $2^n \equiv 1 \pmod{3}$, $3^n \equiv 0 \pmod{3}$, $5^n \equiv 1 \pmod{3}$. Hence

$$S_n \equiv 1 + 0 + 1 = 2 \pmod{3},$$

but squares modulo 3 are only 0 or 1. Thus S_n cannot be a perfect square in this case either.

Since S_n fails to be a quadratic residue modulo 8 when n is odd, and fails to be a quadratic residue modulo 3 when n is even, no positive integer n makes it a perfect square.

Solution 3 by Huynh Quoc Thang Ngo, Le Quy Don high school, Vietnam

If n is even, we have

$$2^n + 3^n + 5^n \equiv (-1)^n + 3^n + (-1)^n \equiv 1 + 0 + 1 \equiv 2 \pmod{3}.$$

Since for any integer a we have $a^2 \equiv 0, 1 \pmod{3}$, this is a contradiction. Thus n is odd, and then

$$2^n + 3^n + 5^n \equiv 2^n + 3^n + (-2)^n \equiv 3^n \pmod{7}.$$

As n is odd, then $n \equiv 1, 3, 5 \pmod{6}$. Note that $\gcd(3, 7) = 1$, so by Fermat's Little Theorem $3^6 \equiv 1 \pmod{7}$. Hence

$$3^n = 3^{6k+d} = (3^6)^k 3^d \equiv 3^d \pmod{7}$$

for some $d \in \{1, 3, 5\}$, and therefore

$$3^n \equiv 3, 5, 6 \pmod{7}.$$

But $b^2 \equiv 0, 1, 2, 4 \pmod{7}$ for any integer b , so 3^n cannot be a quadratic residue modulo 7. Thus $2^n + 3^n + 5^n$ cannot be a perfect square.

In short, there is no positive integer n such that $2^n + 3^n + 5^n$ is a perfect square.

Also solved by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jamal Dine Chergui, Abdelmalek Essaadi University, Morocco; Specialized Mathematical Gymnasium, Prishtine, Kosovo; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Batyrkhan Beiganov, Almaty, Kazakhstan; Daniel Pascuas, Barcelona, Spain.

Undergraduate problems

U709. Let n be a positive integer and let U_n be the set of the n -th roots of unity. Let $P(x) \in \mathbb{C}[x]$ with $\deg P \leq n-1$ and let $H(P)$ be the height of P , that is, the maximum of the moduli of the coefficients of P . If there is $M > 0$ such that $|P(x)| \leq M$ for all $x \in U_n$, prove that $H(P) \leq M$.

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Let m be a positive integer. If m is not divisible by n , then $e^{\frac{2im\pi}{n}} \neq 1$ and

$$\sum_{x \in U_n} x^m = \sum_{k=0}^{n-1} \left(e^{\frac{2ik\pi}{n}} \right)^m = \sum_{k=0}^{n-1} e^{\frac{2ikm\pi}{n}} = \sum_{k=0}^{n-1} \left(e^{\frac{2im\pi}{n}} \right)^k = \frac{1 - e^{2im\pi}}{1 - e^{\frac{2im\pi}{n}}} = 0.$$

If m is divisible by n , then

$$\sum_{x \in U_n} x^m = \sum_{k=0}^{n-1} \left(e^{\frac{2ik\pi}{n}} \right)^m = \sum_{k=0}^{n-1} e^{\frac{2ikm\pi}{n}} = \sum_{k=0}^{n-1} 1 = n.$$

Now, let $P(x) = \sum_{k=0}^{n-1} a_k x^k \in \mathbb{C}[x]$ such that $|P(x)| \leq M$ for some $M > 0$. Then, $H(P) = \max_{0 \leq k \leq n-1} |a_k|$. We will prove that $|a_k| \leq M$ for all $k = 0, 1, \dots, n-1$. Observe that for $k = 0, 1, \dots, n-1$ we have

$$\sum_{x \in U_n} \frac{P(x)}{x^k} = \sum_{x \in U_n} \sum_{j=0}^{n-1} a_j x^{j-k} = \sum_{j=0}^{n-1} a_j \sum_{x \in U_n} x^{j-k} = na_k$$

because $|j-k| < n$ and $j-k$ is divisible by n if and only if $j=k$. Therefore,

$$|a_k| = \frac{1}{n} \left| \sum_{x \in U_n} \frac{P(x)}{x^k} \right| \leq \frac{1}{n} \sum_{x \in U_n} \left| \frac{P(x)}{x^k} \right| \leq \frac{1}{n} \sum_{x \in U_n} M = M,$$

as we wanted to prove.

Solution 2 by Diar Gashi, University of Primorska, Slovenia

Let $\omega = e^{2\pi i/n}$ be a nontrivial root of unity, so that $U_n = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$.

Because $\deg(P) \leq n-1$, the coefficients $a_0, a_1, \dots, a_{n-1}$ are uniquely determined by the values of P at these n distinct points. This correspondence can be expressed using the discrete Fourier transform (DFT):

$$P(\omega^j) = \sum_{k=0}^{n-1} a_k \omega^{jk}, \quad j = 0, 1, \dots, n-1.$$

Since the matrix $(\omega^{jk})_{0 \leq j, k \leq n-1}$ is unitary up to a factor $\sqrt{n}$, we can invert this system to recover the coefficients a_k as

$$a_k = \frac{1}{n} \sum_{j=0}^{n-1} P(\omega^j) \omega^{-jk}, \quad k = 0, 1, \dots, n-1.$$

Now take absolute values:

$$|a_k| = \left| \frac{1}{n} \sum_{j=0}^{n-1} P(\omega^j) \omega^{-jk} \right| \leq \frac{1}{n} \sum_{j=0}^{n-1} |P(\omega^j)|,$$

where we used the triangle inequality and the fact that $|\omega^{-jk}| = 1$ for all j, k . By hypothesis, we have $|P(\omega^j)| \leq M$ for every j , so

$$|a_k| \leq \frac{1}{n} \sum_{j=0}^{n-1} M = M.$$

Since this inequality holds for every $k = 0, 1, \dots, n-1$, it follows that

$$H(P) = \max_{0 \leq k \leq n-1} |a_k| \leq M.$$

This completes the proof.

Also solved by The Ring Lords, Baden-Württemberg, Germany; Ayaan Alam, IIT Delhi; Daniel Pascuas, Barcelona, Spain.

U710. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function such that

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x^n) dx = \sin 1.$$

Prove that there exists $c \in (0, 1)$ such that $f(c) = \cos c$.

Proposed Mihaela Berindeanu, Bucharest, Romania

Solution 1 by The Ring Lords, Baden-Württemberg, Germany

Perform the change of variables $y = x^n$. Then $x = y^{\frac{1}{n}}$ and $dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$. Hence

$$n \int_0^1 x^n f(x^n) dx = \int_0^1 y^{\frac{1}{n}} f(y) dy.$$

We claim the right-hand side converges to $\int_0^1 f(y) dy$ as $n \rightarrow \infty$. Indeed, for $0 \leq y \leq 1$ we have $0 \leq y^{\frac{1}{n}} \leq 1$, so

$$\left| \int_0^1 y^{\frac{1}{n}} f(y) dy - \int_0^1 f(y) dy \right| \leq \max_{0 \leq y \leq 1} |f(y)| \int_0^1 \left| 1 - y^{\frac{1}{n}} \right| dy.$$

But the last integral can be computed exactly:

$$\int_0^1 \left| 1 - y^{\frac{1}{n}} \right| dy = \int_0^1 \left(1 - y^{\frac{1}{n}} \right) dy = 1 - \int_0^1 y^{\frac{1}{n}} dy = 1 - \frac{1}{1 + \frac{1}{n}} = \frac{1}{n+1}.$$

Therefore

$$\left| \int_0^1 y^{\frac{1}{n}} f(y) dy - \int_0^1 f(y) dy \right| \leq \frac{1}{n+1} \max_{0 \leq y \leq 1} |f(y)| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

so $\int_0^1 y^{\frac{1}{n}} f(y) dy \rightarrow \int_0^1 f(y) dy$.

Now assume $\int_0^1 f(y) dy = \sin 1$. Define

$$G(x) = \int_0^x f(y) dy - \sin x, \quad 0 \leq x \leq 1.$$

Because f is continuous, G is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Moreover

$$G(0) = \int_0^0 f(y) dy - \sin 0 = 0, \quad G(1) = \int_0^1 f(y) dy - \sin 1 = 0.$$

Since $G(0) = G(1)$, Rolle's theorem gives a point $c \in (0, 1)$ with $G'(c) = 0$. But

$$G'(x) = \left(\frac{d}{dx} \int_0^x f(y) dy \right) - \cos x = f(x) - \cos x,$$

so at $x = c$ we have $f(c) - \cos c = 0$, i.e. $f(c) = \cos c$.

Solution 2 by Joshua Pit  , Cambridge Rindge and Latin School, MA, USA

Set $t = x^n$, then

$$n \int_0^1 x^n f(x^n) dx = n \int_0^1 t f(t) \frac{1}{n} t^{\frac{1}{n}-1} dt = \int_0^1 t^{\frac{1}{n}} f(t) dt.$$

Hence,

$$\lim_{n \rightarrow \infty} \int_0^1 t^{\frac{1}{n}} f(t) dt = \sin 1.$$

Since f is continuous on $[0, 1]$, it is bounded, and $0 \leq t^{\frac{1}{n}} \leq 1$ for all $t \in [0, 1]$. Moreover $t^{\frac{1}{n}} \rightarrow 1$ for all $t \in (0, 1]$ and $t^{\frac{1}{n}} \rightarrow 0$ at $t = 0$, so $t^{\frac{1}{n}} \rightarrow 1$ for all $t \in (0, 1]$. By the Dominated Convergence Theorem,

$$\int_0^1 f(t) dt = \sin 1 = \int_0^1 \cos t dt.$$

Let g be the continuous function on $[0, 1]$ defined by $g(t) = f(t) - \cos t$.

Then

$$\int_0^1 g(t) dt = \int_0^1 f(t) dt - \int_0^1 \cos t dt = 0.$$

If $g \geq 0$ on $[0, 1]$, then $g = 0$. Similarly if $g \leq 0$ then $g = 0$. Otherwise g takes both positive and negative values, and by the IVT there exists $c \in (0, 1)$ with $g(c) = 0$.

We proved that there exists $c \in (0, 1)$ such that $f(c) = \cos c$.

Also solved by Anita Tufisi, Romania; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Florin Rotaru, Focșani, Romania; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Sunghoon Kim, Seoul National University, Korea; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia.

U711. Evaluate

$$\sum_{n=1}^{\infty} \left[n^2 \sum_{k=n+1}^{\infty} \frac{1}{k^2} - n + \frac{1}{2} - \frac{1}{6n} \right]^2.$$

Proposed by Paolo Perfetti, Università di "Tor Vergata", Rome, Italy

Solution by the author

The series is

$$\sum_{n=1}^{\infty} \left[n^4 \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^2 + \left(n - \frac{1}{2} + \frac{1}{6n} \right)^2 + 2 \left(-n^3 + \frac{n^2}{2} - \frac{n}{6} \right) \sum_{k=n+1}^{\infty} \frac{1}{k^2} \right] \doteq A + B + C$$

We need the asymptotic expansion of $\sum_{k=n}^{\infty} \frac{1}{k^2}$ as a function of n . We use Cesàro–Stolz.

$$\lim_{n \rightarrow \infty} n \sum_{k=n}^{\infty} \frac{1}{k^2} = \lim_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} 1/k^2}{1/n} = \lim_{n \rightarrow \infty} \frac{\sum_{k=n+1}^{\infty} 1/k^2 - \sum_{k=n}^{\infty} 1/k^2}{1/(n+1) - 1/n} = \lim_{n \rightarrow \infty} \frac{n(n+1)}{n^2} = 1$$

$$\lim_{n \rightarrow \infty} n^2 \sum_{k=n}^{\infty} \frac{1}{k^2} - n = \lim_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} \frac{1}{k^2} - \frac{1}{n}}{1/n^2} = \lim_{n \rightarrow \infty} \frac{-1/n^2 - 1/(n+1) + 1/n}{1/(n+1)^2 - 1/n^2} = \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \frac{1}{2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n^3 \sum_{k=n}^{\infty} \frac{1}{k^2} - n^2 - \frac{n}{2} &= \lim_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} \frac{1}{k^2} - \frac{1}{n} - \frac{1}{2n^2}}{1/n^3} = \\ &= \lim_{n \rightarrow \infty} \frac{-1/n^2 - 1/(n+1) - 1/(2(n+1)^2) + 1/n + 1/(2n^2)}{1/(n+1)^3 - 1/n^3} = \lim_{n \rightarrow \infty} \frac{n^2 + n}{2(3n^2 + 3n + 1)} = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n^4 \sum_{k=n}^{\infty} \frac{1}{k^2} - n^3 - \frac{n^2}{2} - \frac{n}{6} &= \lim_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} \frac{1}{k^2} - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3}}{1/n^4} = \\ &= \lim_{n \rightarrow \infty} \frac{-1/n^2 - 1/(n+1) - 1/(2(n+1)^2) - 1/(6(n+1)^3) + 1/n + 1/(2n^2) + 1/(6n^3)}{1/(n+1)^4 - 1/n^4} = \\ &= \lim_{n \rightarrow \infty} \frac{-(n^2 + n)}{6(4n^3 + 6n^2 + 4n + 1)} = 0 \end{aligned}$$

This means that $\sum_{k=n}^{\infty} \frac{1}{k^2} = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{6n^3} + o(\frac{1}{n^4})$ or $\sum_{k=n+1}^{\infty} \frac{1}{k^2} = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} + o(\frac{1}{n^4})$

We use Abel's "Summation by parts" several times.

$$\sum_{k=1}^n k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \doteq A_n$$

A.

$$\begin{aligned} \sum_{n=1}^N n^4 \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^2 &= A_N \left(\sum_{k=N+1}^{\infty} \frac{1}{k^2} \right)^2 + \sum_{n=2}^N A_{n-1} \left(\left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^2 - \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^2 \right) = \\ &= A_N \left(\sum_{k=N+1}^{\infty} \frac{1}{k^2} \right)^2 + \sum_{n=2}^N A_{n-1} \left[\frac{1}{n^4} + \frac{2}{n^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \right] \end{aligned}$$

$$\begin{aligned}
A_N \left(\sum_{k=N+1}^{\infty} \frac{1}{k^2} \right)^2 &= \frac{N^3}{5} + \frac{3N^2}{10} - \frac{N}{20} - \frac{3}{40} + O\left(\frac{1}{N}\right) \\
\sum_{n=2}^N \frac{A_{n-1}}{n^4} &= \frac{N^2}{10} - \frac{2N}{5} + \frac{H_N}{3} - \frac{\zeta(3)}{30} + o(1) \\
B_n = \frac{2A_{n-1}}{n^2} &= \frac{2n^3}{5} - n^2 + \frac{2n}{3} - \frac{1}{15n}, \quad \sum_{k=1}^n B_k = \frac{n^4}{10} - \frac{2n^3}{15} - \frac{n^2}{15} + \frac{n}{6} - \frac{H_n}{15} \doteq C_n \\
\sum_{k=1}^{n-1} B_k &= \frac{n^4}{10} - \frac{8n^3}{15} + \frac{14n^2}{15} - \frac{n}{2} - \frac{H_{n-1}}{15} \doteq C_{n-1} \\
\sum_{n=2}^N B_n \sum_{k=n+1}^{\infty} \frac{1}{k^2} &= C_N \sum_{k=N+1}^{\infty} \frac{1}{k^2} - C_1 \sum_{k=2}^{\infty} \frac{1}{k^2} + \sum_{n=2}^N C_{n-1} \frac{1}{n^2} = \\
&= \left(\frac{N^3}{10} - \frac{11N^2}{60} + \frac{N}{60} + \frac{8}{45} \right) + o(1) + \sum_{n=2}^N \left(\frac{n^2}{10} - \frac{8n}{15} + \frac{14}{15} - \frac{1}{2n} - \frac{H_{n-1}}{15n^2} \right) = \\
&= \left(\frac{N^3}{10} - \frac{11N^2}{60} + \frac{N}{60} + \frac{8}{45} \right) + \frac{N^3}{30} - \frac{13N^2}{60} + \frac{41N}{60} - \frac{1}{2} - \frac{H_N}{2} + \frac{1}{2} - \frac{\zeta(3)}{15} + o(1) = \\
&= \frac{2N^3}{15} - \frac{2N^2}{5} + \frac{7N}{10} - \frac{H_N}{2} + \frac{8}{45} - \frac{\zeta(3)}{15} + o(1)
\end{aligned}$$

because

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 -x^{n-1} \ln(1-x) dx = \int_0^1 \frac{\ln^2(1-x) dx}{x} = \\
\int_0^1 \frac{\ln^2 x dx}{1-x} &= \sum_{k=0}^{\infty} \int_0^1 x^k \ln^2 x dx = \sum_{k=0}^{\infty} \frac{2}{(k+1)^3} = 2\zeta(3) =
\end{aligned}$$

We have obtained

$$\begin{aligned}
\sum_{n=1}^N n^4 \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^2 &= \frac{N^3}{5} + \frac{3N^2}{10} - \frac{N}{20} - \frac{3}{40} + O\left(\frac{1}{N}\right) + \\
&+ \frac{N^2}{10} - \frac{2N}{5} + \frac{H_N}{3} - \frac{\zeta(3)}{30} + o(1) + \\
&+ \frac{2N^3}{15} - \frac{2N^2}{5} + \frac{7N}{10} - \frac{H_N}{2} + \frac{8}{45} - \frac{\zeta(3)}{15} + o(1) = \frac{N^3}{3} + \frac{N}{4} + \frac{37}{360} - \frac{H_N}{6} - \frac{\zeta(3)}{10} + o(1)
\end{aligned}$$

B.

$$\sum_{n=1}^N \left(n - \frac{1}{2} + \frac{1}{6n} \right)^2 = \frac{N^3}{3} + \frac{N}{4} + \frac{\pi^2}{216} - \frac{H_N}{6} + o(1)$$

C.

$$\begin{aligned}
\sum_{n=1}^N 2\left(-n^3 + \frac{n^2}{2} - \frac{n}{6}\right) &= \frac{-N^4}{2} - \frac{2N^3}{3} - \frac{N^2}{6} \doteq R_N \\
\sum_{n=1}^N 2\left(-n^3 + \frac{n^2}{2} - \frac{n}{6}\right) \sum_{k=n+1}^{\infty} \frac{1}{k^2} &= R_N \sum_{k=N+1}^{\infty} \frac{1}{k^2} + \sum_{n=2}^{\infty} R_{n-1} \frac{1}{n^2} = \\
&= \frac{-N^3}{2} - \frac{5N^2}{12} + \frac{N}{12} - \frac{1}{36} - \frac{N^3}{6} + \frac{5N^2}{12} - \frac{7N}{12} + \frac{H_N}{3} = \frac{-2N^3}{3} - \frac{N}{2} - \frac{1}{36} + \frac{H_N}{3} + o(1)
\end{aligned}$$

Finally the sum of the quantities A,B,C is

$$\begin{aligned} & \left(\frac{N^3}{3} + \frac{N}{4} + \frac{37}{360} - \frac{H_N}{6} - \frac{\zeta(3)}{10} \right) + \left(\frac{N^3}{3} + \frac{N}{4} + \frac{\pi^2}{216} - \frac{H_N}{6} \right) + \\ & + \left(\frac{-2N^3}{3} - \frac{N}{2} - \frac{1}{36} + \frac{H_N}{3} \right) + o(1) = \frac{3}{40} + \frac{\pi^2}{216} - \frac{\zeta(3)}{10} + o(1) \end{aligned}$$

Also solved by Anita Tufisi, Romania; Xiao Jiang, Shanghai, China; Madani Harouach, AMMOC.

U712. What is the smallest positive value of the exponent k such that the inequality

$$(ab)^k + (ac)^k + (ad)^k + (bc)^k + (bd)^k + (cd)^k \geq 6$$

holds for any nonnegative real number a, b, c, d with $ab + bc + cd + da = 4$?

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania

Solution by the author

By choosing $d = b$ and $c = 0$, the constraint $(a + c)(b + d) = 4$ becomes $ab = 2$, while the inequality can be written as follows:

$$2(ab)^k + b^{2k} \geq 6, \quad 2^{k+1} + b^{2k} \geq 6.$$

For $b \rightarrow 0$, the inequality involves $2^k \geq 3$, that is $k \geq k_0 = \frac{\ln 3}{\ln 2} \approx 1.585$. So, we only need to prove the desired inequality for $k = k_0$, when $2^k = 3$. Without loss of generality, assume that

$$a^k + c^k \geq b^k + d^k, \quad a \geq c, \quad b \geq d.$$

By Lemma below, for fixed a and c , the left side of the required inequality is minimum when $d = b$. Thus, it suffices to consider this case, when we need to show that $b(a + c) = 2$ involves

$$2(a^k + c^k)b^k + a^k c^k + b^{2k} \geq 6.$$

Since $a^k c^k + b^{2k} \geq 2b^k \sqrt{a^k c^k}$, it suffices to show that

$$a^k + c^k + \sqrt{a^k c^k} \geq \frac{3}{b^k},$$

that is

$$a^k + c^k + \sqrt{a^k c^k} \geq 3 \left(\frac{a + c}{2} \right)^k.$$

For $c = 0$, the inequality is an equality. For $c > 0$, due to homogeneity, we can set $c = 1$ and $a \geq 1$. So, we need to show that $f_1(a) \geq 0$, where

$$f_1(a) = a^k + a^{k/2} + 1 - 3 \left(\frac{a + 1}{2} \right)^k.$$

If $f_1(a)$ is increasing, then $f_1(a) \geq f_1(1) = 0$. Thus, it suffices to show that $f_1'(a) \geq 0$, that is

$$2a^{k-1} + a^{k/2-1} - 3 \left(\frac{a + 1}{2} \right)^{k-1} \geq 0, \quad 2 + a^{-k/2} - 3 \left(\frac{a + 1}{2a} \right)^{k-1} \geq 0.$$

Using the substitution $a = 1/x$, we need to show that $x \in (0, 1]$ involves $f_2(x) \geq 0$, where

$$f_2(x) = 2 + x^{k/2} - 3 \left(\frac{x + 1}{2} \right)^{k-1}.$$

We have

$$f_2'(x) = \frac{k}{2} x^{k/2-1} - \frac{3(k-1)}{2} \left(\frac{x + 1}{2} \right)^{k-2} = \frac{1}{2} x^{k/2-1} f_3(x),$$

where

$$f_3(x) = k - 3(k-1)(g(x))^{2-k}, \quad g(x) = \frac{2\sqrt{x}}{x+1}.$$

From $g'(x) = \frac{1-x}{\sqrt{x}(x+1)^2} \geq 0$, it follows that $g(x)$ is increasing and $f_3(x)$ is decreasing on $[0, 1]$. Since $f_3(0) = k$ and $f_3(1) = -2k + 3 < 0$, there is $x_1 \in (0, 1)$ such that $f_3(x) \geq 0$ for $x \in (0, x_1]$ and $f_3(x) \leq 0$

for $x \in [x_1, 1]$, $f_2(x)$ is increasing on $(0, x_1]$ and decreasing on $[x_1, 1]$. Since $f_2(0) = f_2(1) = 0$, we have $f_2(x) \geq 0$ for $x \in [0, 1]$.

The proof is completed. For $k = k_0$, the equality occurs when $a = b = c = d = 1$.

Lemma. Let a, b, c, d be nonnegative real numbers and $k > \frac{3}{2}$ such that

$$(a + c)(b + d) = 4, \quad a^k + c^k \geq b^k + d^k, \quad a \geq c, \quad b \geq d.$$

For fixed k , a and c , the expression

$$F(a, b, c, d) = (ab)^k + (ac)^k + (ad)^k + (bc)^k + (bd)^k + (cd)^k$$

has the minimum value when $d = b$.

Proof. According to the given constraint $ab + bc + cd + da = 4$, we may consider that b and F are functions of d . Let $F(a, b, c, d) := f(d)$. If $f'(d) \leq 0$, then $f(d)$ is decreasing and has the minimum value when d is maximum, hence when $d = b$. So, it suffices to show that $f'(d) \leq 0$. From the hypothesis $(a + c)(b + d) = 4$, we get $b' = -1$, hence

$$\begin{aligned} \frac{1}{k}f'(d) &= (a^k + c^k + d^k)b^{k-1}b' + (a^k + c^k + b^k)d^{k-1} = -(a^k + c^k + d^k)b^{k-1} + (a^k + c^k + b^k)d^{k-1} \\ &= b^{k-1}d^{k-1}(b - d) - (a^k + c^k)(b^{k-1} - d^{k-1}) \leq b^{k-1}d^{k-1}(b - d) - (b^k + d^k)(b^{k-1} - d^{k-1}) \\ &= -b^{2k-1} - 2b^{k-1}d^k + 2b^kd^{k-1} + d^{2k-1}. \end{aligned}$$

So, it suffices to show that

$$-b^{2k-1} - 2b^{k-1}d^k + 2b^kd^{k-1} + d^{2k-1} \leq 0.$$

Due to homogeneity, we may set $b = 1$. Thus, we need to show that $g(d) \leq 0$ for $d \in [0, 1]$, where

$$g(d) = d^{2k-1} + 2d^{k-1} - 2d^k - 1.$$

We have

$$g'(d) = (2k - 1)d^{2k-2} + 2(k - 1)d^{k-2} - 2kd^{k-1} = d^{k-2}h(d),$$

where

$$h(d) = (2k - 1)d^k + 2(k - 1) - 2kd.$$

From

$$h'(d) = k(2k - 1)d^{k-1} - 2k,$$

it follows that $h'(d) \leq 0$ for $d \in [0, d_0]$ and $h'(d) \geq 0$ for $d \in [d_0, 1]$, where

$$d_0^{k-1} = \frac{2}{2k - 1}, \quad 0 < d_0 < 1.$$

So, $h(d)$ is decreasing on $[0, d_0]$ and increasing on $[d_0, 1]$, hence

$$h(d) \geq h(d_0) = (2k - 1)d_0^k + 2(k - 1) - 2kd_0 = 2d_0 + 2(k - 1) - 2kd_0 = 2(k - 1)(1 - d_0) > 0,$$

$g'(d) > 0$, $g(d)$ is increasing, $g(d) \leq g(1) = 0$.

Also solved by Theo Koupelis, Clark College, Washington, USA.

U713. Let $f(x) = x^5 - 3x^3 + 1 \in \mathbb{Q}[x]$. Determine the Galois group of $f(x)$ over $\mathbb{Q}$ and then find whether the equation $f(x) = 0$ is solvable by radicals.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

First, observe that if $f(x)$ were reducible over $\mathbb{Z}$, then it would also be reducible modulo 2. Now, consider

$$\bar{f}(x) = x^5 + x^3 + 1 \in \mathbb{F}_2[x].$$

We will show that $\bar{f}(x)$ is irreducible over $\mathbb{F}_2$.

Assuming that $\bar{f}(x)$ is reducible over $\mathbb{F}_2$, it must factor into polynomials of degrees adding up to 5. Since there are no roots in $\mathbb{F}_2$ (as $f(0) = 1$, $f(1) = 1$), it cannot have linear factors. We need to check if $\bar{f}(x)$ factors into an irreducible quadratic and an irreducible cubic. The only monic irreducible quadratic over $\mathbb{F}_2$ is $x^2 + x + 1$, and the monic irreducible cubics are $x^3 + x + 1$ and $x^3 + x^2 + 1$.

Observe that

$$(x^2 + x + 1)(x^3 + x + 1) = x^5 + x^4 + 2x^3 + 2x^2 + 2x + 1 \equiv x^5 + x^4 + 1 \pmod{2}$$

and

$$(x^2 + x + 1)(x^3 + x^2 + 1) = x^5 + 2x^4 + 2x^3 + 2x^2 + x + 1 \equiv x^5 + x + 1 \pmod{2}.$$

None of these polynomials is $\bar{f}(x)$, so $\bar{f}(x)$ is irreducible over $\mathbb{F}_2$.

Since $\bar{f}(x)$ is irreducible over $\mathbb{F}_2$ and $f(x)$ is irreducible over $\mathbb{Z}$ and since $f(x)$ is monic, by Gauss's Lemma $f(x)$ is irreducible over $\mathbb{Q}$.

Now, let K be the splitting field of $f(x)$ over $\mathbb{Q}$, and let $G = \text{Gal}(K/\mathbb{Q})$. Since $f(x)$ is an irreducible polynomial of degree 5, then $f(x)$ has no multiple roots and so G is isomorphic to a subgroup of S_5 . Moreover, $f(x)$ has exactly three distinct real roots (by easy calculus) and so it has two non-real roots. The minimal field extension generated by a root α of $f(x)$ is $\mathbb{Q}(\alpha)$, with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$. Since

$$|G| = [K : \mathbb{Q}] = [K : \mathbb{Q}(\alpha)] \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}],$$

then $|G|$ is divisible by 5. By Cauchy's Theorem, G has an element of order 5. But the only elements of S_5 having order 5 are the 5-cycles. Therefore G contains a 5-cycle. Complex conjugation is a $\mathbb{Q}$ -automorphism of $\mathbb{C}$, and therefore induces a $\mathbb{Q}$ -automorphism of K . This leaves the three real roots of f fixed, while transposing the two non-real roots. Therefore G contains a 2-cycle. Without loss of generality we can assume that G contains the 2-cycle (12) and the 5-cycle (12345) . But then $\langle (12), (12345) \rangle \leq G$ and since $\langle (12), (12345) \rangle = S_5$, we conclude that $G \cong S_5$.

Since the Galois group $G \cong S_5$ is not a solvable group (because it contains the simple non-abelian group A_5), the equation $f(x) = 0$ is not solvable by radicals.

U714. Evaluate

$$\int_0^\infty \frac{\arctan x}{\sqrt[4]{x^8 + 1}} dx.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania

Solution 1 by Daniel Pascuas, Barcelona, Spain

The substitution $x = \frac{1}{y}$ yields that

$$\int_0^\infty \frac{\arctan x}{\sqrt[4]{x^8 + 1}} dx = \int_0^\infty \frac{\arctan(1/y)}{\sqrt[4]{y^8 + 1}} dy = \int_0^\infty \frac{\arctan(1/x)}{\sqrt[4]{x^8 + 1}} dx,$$

and so

$$\int_0^\infty \frac{\arctan x}{\sqrt[4]{x^8 + 1}} dx = \frac{1}{2} \int_0^\infty \frac{\arctan x + \arctan(1/x)}{\sqrt[4]{x^8 + 1}} dx = \frac{\pi}{4} \int_0^\infty \frac{dx}{\sqrt[4]{x^8 + 1}},$$

since $\arctan x + \arctan(1/x) = \frac{\pi}{2}$, for every $x > 0$. Now perform the substitution $y = x^8$ to get

$$\int_0^\infty \frac{dx}{\sqrt[4]{x^8 + 1}} = \frac{1}{8} \int_0^\infty \frac{y^{\frac{1}{8}-1}}{(1+y)^{\frac{1}{4}}} dy = \frac{1}{8} B\left(\frac{1}{8}, \frac{1}{8}\right) = \frac{1}{8} \frac{\Gamma(\frac{1}{8})^2}{\Gamma(\frac{1}{4})},$$

since

$$\int_0^\infty \frac{y^{s-1}}{(1+y)^{s+t}} dy = B(s, t) \quad (s, t > 0),$$

where $B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$ is Euler's Beta function. Therefore we finally obtain that

$$\int_0^\infty \frac{\arctan x}{\sqrt[4]{x^8 + 1}} dx = \frac{\pi}{32} \frac{\Gamma(\frac{1}{8})^2}{\Gamma(\frac{1}{4})}.$$

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Using the substitutions $y = 1/x$ and $t = x^8$ we get

$$\begin{aligned} \int_0^\infty \frac{\arctan(x)}{\sqrt[4]{x^8 + 1}} dx &= \int_0^1 \frac{\arctan(x)}{\sqrt[4]{x^8 + 1}} dx + \int_1^\infty \frac{\arctan(x)}{\sqrt[4]{x^8 + 1}} dx \\ &= \int_0^1 \frac{\arctan(x)}{\sqrt[4]{x^8 + 1}} dx - \int_1^0 \frac{\arctan(1/y)}{\sqrt[4]{y^8 + 1}} dy \\ &= \int_0^1 \frac{\arctan(x) + \arctan(1/x)}{\sqrt[4]{x^8 + 1}} dx \stackrel{(1)}{=} \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt[4]{x^8 + 1}} dx \\ &= \frac{\pi}{16} \int_0^1 \frac{1}{t^{7/8}(1+t)^{1/4}} dt \stackrel{(2)}{=} \frac{\pi}{16} \cdot \frac{\Gamma(\frac{1}{8})\Gamma(1)}{\Gamma(\frac{9}{8})} \cdot {}_2F_1\left(\frac{1}{4}, \frac{1}{8}; \frac{9}{8}; -1\right) \\ &\stackrel{(3)}{=} \frac{\pi}{2} \cdot \frac{\Gamma(\frac{1}{8})\Gamma(1)}{\Gamma(\frac{1}{8})} \cdot \frac{\Gamma^2(\frac{9}{8})}{\Gamma(\frac{5}{4})\Gamma(1)} = \frac{\pi}{2} \cdot \frac{\Gamma^2(\frac{9}{8})}{\Gamma(\frac{5}{4})} \stackrel{(4)}{=} \frac{\pi^{3/2}}{2^{5/4}} \cdot \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{8})}. \end{aligned}$$

In the above we used the well-known equality

$$\arctan(x) + \arctan(1/x) = \pi/2, \quad \forall x > 0, \quad (1),$$

and the integral expression for the hypergeometric function

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt, \quad (2)$$

where $c > b > 0$, $z < 1$, and $\Gamma(z)$ is the Gamma function. We also used Kummer's theorem,

$${}_2F_1(a, b; 1 + a - b; -1) = \frac{\Gamma(1 + a - b)\Gamma\left(1 + \frac{a}{2}\right)}{\Gamma(1 + a)\Gamma\left(1 + \frac{a}{2} - b\right)}, \quad (3)$$

and the Legendre duplication formula

$$\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{9}{8}\right) = \frac{\sqrt{\pi}}{2^{1/4}}\Gamma\left(\frac{5}{4}\right). \quad (4)$$

Also solved by Anita Tufisi, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Joshua Pit  , Cambridge Rindge and Latin School, MA, USA; Se  n M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; The Ring Lords, Baden-W  rttemberg, Germany; Xiao Jiang, Shanghai, China.

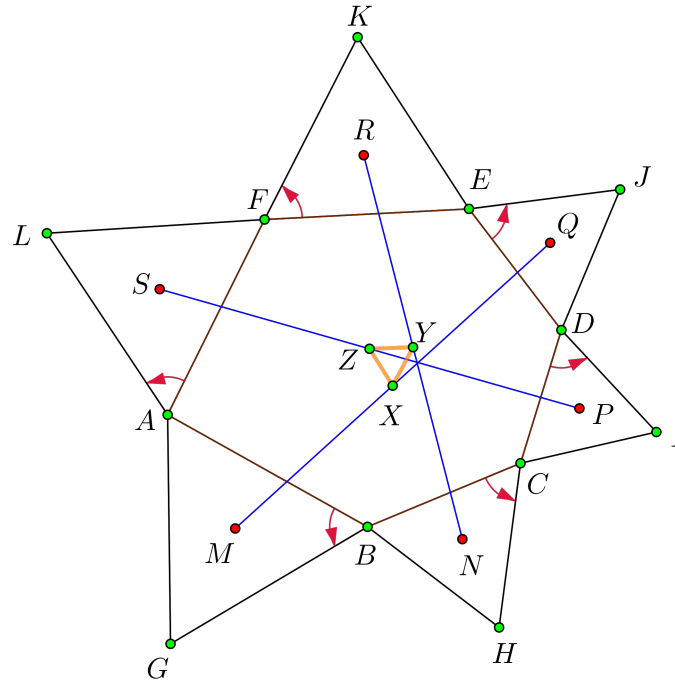
Olympiad problems

O709. Let $ABCDEF$ be a hexagon in the plane, and let M, N, P, Q, R, S be the centers of the equilateral triangles constructed on its sides AB, BC, CD, DE, EF , and FA respectively such that either all these triangles are built outside the hexagon, or they are all towards its interior. Let X, Y, Z be the midpoints of the line segments MQ, NR , and PS respectively. Prove that the triangle XYZ is equilateral.

Proposed by Hayk Sedrakian and Maximilian Wolfensberger

Solution by Kousik Sett, India

Consider the complex plane with its origin at O . In the accompanying figure, the coordinate of a point denoted by an uppercase letter (e.g., A) is represented by the corresponding lowercase letter (e.g., a). The imaginary unit is denoted by $\mathbf{i}$.



Let

$$\epsilon = \cos \frac{\pi}{3} + \mathbf{i} \sin \frac{\pi}{3}.$$

Then we have

$$\epsilon^2 = \cos \frac{2\pi}{3} + \mathbf{i} \sin \frac{2\pi}{3} \quad \text{and} \quad \epsilon^2 = -1 + \epsilon.$$

We assume all equilateral triangles are drawn externally. Then the rotation formula gives

$$\begin{aligned} g &= b + \epsilon(a - b), & h &= c + \epsilon(b - c), & i &= d + \epsilon(c - d), \\ j &= e + \epsilon(d - e), & k &= f + \epsilon(e - f), & l &= a + \epsilon(f - a). \end{aligned}$$

Then the coordinates of the centers of the equilateral triangles are

$$\begin{aligned} m &= \frac{a+b+g}{3} = \frac{a+2b}{3} + \epsilon \left(\frac{a-b}{3} \right), \\ n &= \frac{b+c+h}{3} = \frac{b+2c}{3} + \epsilon \left(\frac{b-c}{3} \right), \\ p &= \frac{c+d+i}{3} = \frac{c+2d}{3} + \epsilon \left(\frac{c-d}{3} \right), \\ q &= \frac{d+e+j}{3} = \frac{d+2e}{3} + \epsilon \left(\frac{d-e}{3} \right), \\ r &= \frac{e+f+k}{3} = \frac{e+2f}{3} + \epsilon \left(\frac{e-f}{3} \right), \\ s &= \frac{f+a+l}{3} = \frac{f+2a}{3} + \epsilon \left(\frac{f-a}{3} \right). \end{aligned}$$

The coordinates of X , Y , and Z are

$$\begin{aligned} x &= \frac{a+2b+d+2e}{6} + \epsilon \left(\frac{a-b+d-e}{6} \right), \\ y &= \frac{b+2c+e+2f}{6} + \epsilon \left(\frac{b-c+e-f}{6} \right), \\ z &= \frac{c+2d+f+2a}{6} + \epsilon \left(\frac{c-d+f-a}{6} \right). \end{aligned}$$

Thus

$$\begin{aligned} y-x &= \frac{-a-b+2c-d-e+2f}{6} + \epsilon \left(\frac{-a+2b-c-d+2e-f}{6} \right), \\ z-x &= \frac{a-2b+c+d-2e+f}{6} + \epsilon \left(\frac{-2a+b+c-2d+e+f}{6} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \epsilon(y-x) &= \left(\frac{-a-b+2c-d-e+2f}{6} \right) \epsilon + (-1+\epsilon) \left(\frac{-a+2b-c-d+2e-f}{6} \right) \\ &= \left(\frac{a-2b+c+d-2e+f}{6} \right) + \epsilon \left(\frac{-2a+b+c-2d+e+f}{6} \right). \end{aligned}$$

Therefore, $z-x = \epsilon(y-x)$ and XYZ is equilateral. If all equilateral triangles are drawn internally, then the similar approach gives the same conclusion.

Also solved by Specialized Mathematical Gymnasium, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA.

O710. Let $n \geq 3$ be a positive integer. A finite set S of positive integers is called *balanced* if there exist two disjoint, nonempty subsets A and B of S such that

$$\sum_{a \in A} a = \sum_{b \in B} b.$$

Let S be a set of 2^n positive integers, each of which is at most 2^{2^n-n-1} . Prove that S is balanced.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

Let S be a set of 2^n positive integers and assume $s \leq 2^{2^n-n-1}$ for every $s \in S$. For any subset $T \subseteq S$, define $\sigma(T) := \sum_{t \in T} t$. Since S has 2^n elements, there are $2^{|S|} - 1 = 2^{2^n} - 1$ nonempty subsets of S . For any nonempty subset $T \subseteq S$ we have

$$1 \leq \sigma(T) = \sum_{t \in T} t \leq |T| \cdot 2^{2^n-n-1} \leq 2^n \cdot 2^{2^n-n-1} = 2^{2^n-1}.$$

Thus the possible values of $\sigma(T)$ lie in the set $\{1, 2, \dots, 2^{2^n-1}\}$, so there are at most 2^{2^n-1} distinct subset sums. Since for $n \geq 2$ we have $2^{2^n} - 1 > 2^{2^n-1}$, by the Pigeonhole Principle there are two distinct nonempty subsets A and B of S such that $\sigma(A) = \sigma(B)$. However, these two subsets might not be disjoint.

Now, let $X = A \setminus B$ and $Y = B \setminus A$. Then X and Y are disjoint subsets of S . Moreover, we have

$$\sigma(A) = \sigma(X) + \sigma(A \cap B), \quad \sigma(B) = \sigma(Y) + \sigma(A \cap B).$$

Since $\sigma(A) = \sigma(B)$, it follows that $\sigma(X) = \sigma(Y)$.

Finally, we claim that X and Y are both nonempty. Suppose, for instance, that $X = \emptyset$. Then $A \subseteq B$. Since $A \neq B$ and all elements of S are positive integers, it follows that $\sigma(A) < \sigma(B)$, contradicting $\sigma(A) = \sigma(B)$. Hence $X \neq \emptyset$, and the same argument shows that $Y \neq \emptyset$.

Thus, we have found two disjoint nonempty subsets X and Y with equal sum and we are done.

Also solved by Specialized Mathematical Gymnasium, Prishtine, Kosovo; Ayaan Alam, IIT Delhi.

O711. Let $x, y, z > 0$ be real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$x^2 + y^2 + \frac{2(x + y + z - 2)}{2 - z} \geq 4.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

This condition leads us to the following substitutions

$$x = 2 \cos A, \quad y = 2 \cos B, \quad z = 2 \cos C$$

where ABC is a nonobtuse triangle, including the possibility of having a degenerate triangle. The inequality becomes

$$2 \cos^2 A + 2 \cos^2 B + \frac{\cos A + \cos B + \cos C - 1}{1 - \cos C} \geq 2.$$

Let $a = \sqrt{v+w}$, $b = \sqrt{w+u}$, $c = \sqrt{u+v}$, be the side lengths of a triangle ABC , where $u, v, w \geq 0$. The inequality becomes

$$\frac{2u^2}{(u+v)(u+w)} + \frac{2v^2}{(u+v)(v+w)} + \frac{u\sqrt{v+w} + v\sqrt{u+w}}{\sqrt{(u+v)(v+w)(w+u)} - w\sqrt{u+v}} \geq 3,$$

or

$$\frac{u\sqrt{v+w} + v\sqrt{u+w}}{\sqrt{(u+v)(v+w)(w+u)} - w\sqrt{u+v}} \geq 1 + \frac{2wu(v+w) + 2wv(w+u)}{(u+v)(v+w)(w+u)},$$

or

$$\frac{(u\sqrt{v+w} + v\sqrt{u+w})(\sqrt{(u+v)(v+w)(w+u)} + w\sqrt{u+v})}{(u+v)(v+w)(w+u) - w^2(u+v)} \geq 1 + \frac{2wu(v+w) + 2wv(w+u)}{(u+v)(v+w)(w+u)},$$

or

$$\frac{\sqrt{(u+v)(u+w)} + \sqrt{(u+v)(v+w)}}{u+v} \geq 1 + \frac{2wu(v+w) + 2wv(w+u)}{(u+v)(v+w)(w+u)},$$

or

$$\sqrt{(u+v)(u+w)} + \sqrt{(u+v)(v+w)} \geq u+v + \frac{2wu}{w+u} + \frac{2wv}{v+w},$$

or

$$\sqrt{(u+v)(u+w)} + \sqrt{(u+v)(v+w)} \geq u+v + \frac{(u+w)^2 - (u-w)^2}{2(u+w)} + \frac{(v+w)^2 - (v-w)^2}{2(v+w)},$$

or

$$\frac{(u-w)^2}{2(u+w)} + \frac{(v-w)^2}{2(v+w)} \geq \left(\frac{u+v}{2} - \sqrt{(u+v)(v+w)} + \frac{v+w}{2} \right) + \left(\frac{u+v}{2} - \sqrt{(u+v)(u+w)} + \frac{u+w}{2} \right),$$

or

$$\frac{(u-w)^2}{u+w} + \frac{(v-w)^2}{v+w} \geq (\sqrt{u+v} - \sqrt{v+w})^2 + (\sqrt{u+v} - \sqrt{u+w})^2,$$

or

$$\begin{aligned} & \frac{(\sqrt{u+v} - \sqrt{v+w})^2 (\sqrt{u+v} + \sqrt{v+w})^2}{u+w} + \frac{(\sqrt{u+v} - \sqrt{u+w})^2 (\sqrt{u+v} + \sqrt{u+w})^2}{v+w} \\ & \geq (\sqrt{u+v} - \sqrt{v+w})^2 + (\sqrt{u+v} - \sqrt{u+w})^2, \end{aligned}$$

which is clearly true because

$$(\sqrt{u+v} + \sqrt{v+w})^2 \geq u+w$$

and

$$(\sqrt{u+v} + \sqrt{u+w})^2 \geq v+w.$$

The equality holds when $x = y = z = 1$ or $x = 2, y = z = 0$, or $x = 0, y = 2, z = 0$.

Solution 2 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam

Let $x = 2 \cos A, y = 2 \cos B, z = 2 \cos C$ for a triangle with $A + B + C = \pi$. Then

$$x^2 + y^2 = 4(1 - \cos C \cos(A - B)), \quad \frac{2(x + y + z - 2)}{2 - z} = \frac{\cos A + \cos B + \cos C - 1}{\sin^2 \frac{C}{2}}.$$

Put $u = \frac{A - B}{2}$ and $s = \sin \frac{C}{2}$. Using $\cos A + \cos B = 2s \cos u$ and $\cos C = 1 - 2s^2$, we reduce the claim to

$$\frac{\cos u}{s} - 1 \geq 2(1 - 2s^2) \cos 2u. \quad (*)$$

Let $a = \cos u \in [s, 1]$. Then $\cos 2u = 2a^2 - 1$ and

$$G(a) := \frac{a}{s} - 1 - 2(1 - 2s^2)(2a^2 - 1)$$

is a quadratic in a . Since $1 - 2s^2 = \cos C = z/2 > 0$, on the interval $[s, 1]$ the minimum of G is attained at an endpoint. Evaluating, we find

$$G(1) = \frac{1}{s} + 4s^2 - 3 = \frac{(2s - 1)^2(2 + s)}{s} \geq 0, \quad G(s) = 2(1 - 2s^2)^2 \geq 0.$$

Hence $G(a) \geq 0$ for all $a \in [s, 1]$, proving $(*)$ and therefore the desired inequality.

Also solved by Arkady Alt, San Jose, California, USA; Specialized Mathematical Gymnasium, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Xiao Jiang, Shanghai, China.

O712. Let a, b, c, d be nonnegative real numbers such that at most one is less than 1 and $ab + bc + cd + da = 4$.
Prove that

$$\frac{1}{ab+3} + \frac{1}{ac+3} + \frac{1}{ad+3} + \frac{1}{bc+3} + \frac{1}{bd+3} + \frac{1}{cd+3} \geq \frac{3}{2}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania

Solution by the author

Assume that $a \leq 1$ and $b, c, d \geq 1$, denote $x = \frac{b+d}{2} \geq 1$ and write the equality constraint as

$$(a+c)x = 2.$$

Since

$$\begin{aligned} \frac{1}{ab+3} + \frac{1}{ad+3} &\geq \frac{4}{(ab+3) + (ad+3)} = \frac{2}{ax+3}, \\ \frac{1}{bc+3} + \frac{1}{cd+3} &\geq \frac{4}{(bc+3) + (cd+3)} = \frac{2}{cx+3} \end{aligned}$$

and

$$\frac{1}{bd+3} \geq \frac{1}{x^2+3},$$

it suffices to show that

$$\frac{1}{ac+3} + \frac{2}{ax+3} + \frac{2}{cx+3} + \frac{1}{x^2+3} \geq \frac{3}{2}.$$

Since

$$\frac{1}{ax+3} + \frac{1}{cx+3} = \frac{(a+c)x+6}{acx^2+3(a+c)x+9} = \frac{8}{acx^2+15},$$

the inequality becomes

$$\frac{1}{ac+3} + \frac{16}{acx^2+8} + \frac{1}{x^2+3} \geq \frac{3}{2}.$$

From $(a-1)(c-1) \leq 0$, we get

$$ac \leq a+c-1 = \frac{2}{x} - 1 = \frac{2-x}{x},$$

therefore

$$\frac{1}{ac+3} \geq \frac{1}{(2-x)/x+3} = \frac{x}{2(x+1)}$$

and

$$\frac{16}{acx^2+15} \geq \frac{16}{(2-x)x+15} = \frac{16}{15+2x-x^2}.$$

So, it suffices to show that

$$\frac{x}{2(x+1)} + \frac{16}{15+2x-x^2} + \frac{1}{x^2+3} \geq \frac{3}{2},$$

which is equivalent to

$$2x^5 - x^4 - 14x^2 + 22x - 9 \geq 0, \quad (x-1)^3(2x^2+5x+9) \geq 0.$$

The proof is finished. The equality occurs for $a = b = c = d = 1$.

Remark. Note that 3 is the largest positive value of k such that

$$\frac{1}{ab+k} + \frac{1}{ac+k} + \frac{1}{ad+k} + \frac{1}{bc+k} + \frac{1}{bd+k} + \frac{1}{cd+k} \geq \frac{6}{1+k}$$

for any nonnegative real numbers a, b, c, d with at most one of them less than 1 and $ab + bc + cd + da = 4$.

To show this, assume that $c = 1$, $b = d \in [1, 2]$ and $a = (2 - d)/d$, when the constraints are satisfied, while the inequality can be written as follows:

$$\begin{aligned}
& \frac{1}{a+k} + \frac{2}{ad+k} + \frac{2}{d+k} + \frac{1}{d^2+k} \geq \frac{6}{1+k}, \\
& \frac{d}{(k-1)d+2} + \frac{2}{k+2-d} + \frac{2}{d+k} + \frac{1}{d^2+k} \geq \frac{6}{1+k}, \\
& \left[\frac{d}{(k-1)d+2} - \frac{1}{1+k} \right] + 2 \left(\frac{1}{k+2-d} - \frac{1}{1+k} \right) + 2 \left(\frac{1}{d+k} - \frac{1}{1+k} \right) + \left(\frac{1}{d^2+k} - \frac{1}{1+k} \right) \geq 0, \\
& \frac{2(d-1)}{(k-1)d+2} + \frac{2(d-1)}{k+2-d} - \frac{2(d-1)}{d+k} - \frac{d^2-1}{d^2+k} \geq 0, \\
& \frac{2}{(k-1)d+2} + \frac{2}{k+2-d} - \frac{2}{d+k} - \frac{d+1}{d^2+k} \geq 0, \\
& 2 \left[\frac{1}{(k-1)d+2} - \frac{1}{1+k} \right] + 2 \left(\frac{1}{k+2-d} - \frac{1}{1+k} \right) - 2 \left(\frac{1}{d+k} - \frac{1}{1+k} \right) - \left(\frac{d+1}{d^2+k} - \frac{2}{1+k} \right) \geq 0, \\
& \frac{-2(k-1)(d-1)}{(k-1)d+2} + \frac{2(d-1)}{k+2-d} + \frac{2(d-1)}{d+k} + \frac{(d-1)(2d-k+1)}{d^2+k} \geq 0, \\
& \frac{-2(k-1)}{(k-1)d+2} + \frac{2}{k+2-d} + \frac{2}{d+k} + \frac{2d-k+1}{d^2+k} \geq 0.
\end{aligned}$$

For $d \rightarrow 1$, we get the necessary condition $k \leq 3$.

Also solved by Theo Koupelis, Clark College, Washington, USA; Zon Zon, Kweilin, China.

O713. Prove that the equation $x^2 + x^2y + 2y^2 - yz^2 + 3 = 0$ has no integer solutions.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Substituting

$$(y, x, z) = \left(\frac{t}{2}, \frac{v-u}{2}, \frac{u+v}{2} \right)$$

then

$$12 + (u-v)^2 - 2uv + 2t^2 = 0.$$

Thus, we shall prove that

$$12 + (u \pm v)^2 - 2uv + 2t^2 = 0 \tag{1}$$

has no integer solution.

Assume we have a solution such that $|u| + |v| + |t|$ is minimal. Assume $u \geq v \geq 0$ and clearly $t \geq 0$. First assume that $2t^2 \geq u^2$. Then

$$2uv + 2t^2 + (u \pm v)^2 + 12 \leq 2t^2 + 4u^2 + 12 \leq 10t^2 + 12 \leq 12t^2$$

if $t > 2$ (the cases $t = 0, 1, 2$ can be checked separately). Thus,

$$\frac{uv}{6} \leq t.$$

On the other hand, the equation has another solution

$$t' = uv - t = \frac{12 + (u \pm v)^2}{2t} \geq 0.$$

Considering u, v, t we have $t' \geq t$, that is, $uv - t \geq t$. Multiplying (1) by 2 and rewriting, we get

$$(uv - 2t)^2 - (uv)^2 + 2(u \pm v)^2 + 24 = 0,$$

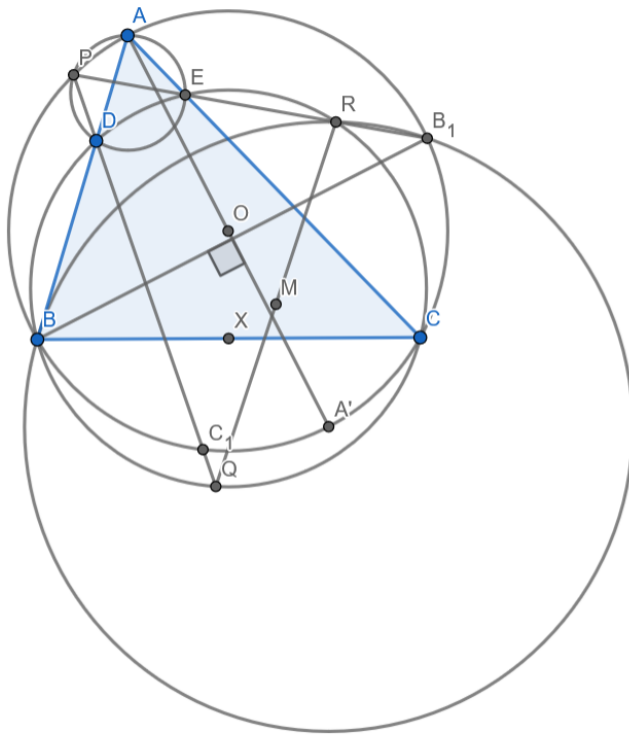
but this is only possible for small u, v . This finishes the case $2t^2 \geq u^2$. Note that we did this by considering the equation as quadratic in t . In the same way, the case $u^2 \geq 2t^2$ can also be done.

Also solved by Anita Tufisi, Romania; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Specialized Mathematical Gymnasium, Prishtine, Kosovo.

- O714. Consider a triangle ABC and its circumcircle Ω . Let there be a point D on the line AB and let E be the intersection of the circumcircle of $\triangle BCD$, denoted ω , with line AC . Consider P the intersection of the circumcircle of $\triangle ADE$ with Ω , $P \neq A$, and Q and R the intersections of lines PD and PE with circle ω . Let M be the midpoint of PQ . Prove that, as D is moving along the line AB , M is moving along a circle.

Proposed by Iancu-Ioan Sandea, Cluj-Napoca, Romania

Solution 1 by the author



Let O be the circumcenter of triangle $\triangle ABC$ and A' be the reflection of A over O .

We will prove that the circle M is moving along is the circle centered in A' and passing through the midpoint X of BC .

Lemma 1. D and E can be uniquely defined when P is moving along Ω .

To prove this, we just have to prove this next claim.

Lemma 2. PD and PE meet Ω in fixed points C_1 and B_1 , where B_1 and C_1 are the reflections of B and C over AA' .

Proof. Since $\widehat{ABB_1} = \widehat{ADE} = \widehat{ACB}$ and $\widehat{BAA'} = 90^\circ - \widehat{ACB} \implies BB_1 \perp AA'$, but AA' is a diameter, so AA' passes through the midpoint of BB_1 , so B_1 is the reflection of B over AA' . So D and E can be uniquely defined as $PC_1 \cap AB$ and $PB_1 \cap AC$. $\square$

Lemma 3. R is moving along the circle centered in A' passing through B .

Proof. To prove this claim we will redefine R as the intersection of the circle centered in A' passing through B and line PB_1 and prove that $R \in \omega$ (the other point of intersection is B_1 , so there is no confusion).

We will use complex bashing for this: Consider O the origin of the plane and Ω the unit circle, and WLOG $A(1)$, $B(b)$, $C(c)$, $P(p)$.

Then $a' = -1$.

Line OA is the line of real numbers, so $b_1 = \bar{b} = \frac{1}{b}$ and $c_1 = \bar{c} = \frac{1}{c}$.

$$D = PC_1 \cap AB \implies d = \frac{\frac{p}{c}(b+1) - b\left(p + \frac{1}{c}\right)}{\frac{p}{c} - b} = \frac{pb + p - bcp - b}{p - bc}.$$

$$R \in PB_1 \implies \frac{r - b_1}{p - b_1} = \frac{\bar{r} - \bar{b}_1}{\bar{p} - \bar{b}_1} \implies p\bar{r} - pb = 1 - br \implies \bar{r} = \frac{1 - br + pb}{p}.$$

$$R \text{ is on the circle centered in } A' \text{ passing through } B \implies |r - a_1| = |b - a_1| \implies |r + 1|^2 = |b + 1|^2 \implies r\bar{r} + r + \bar{r} + 1 = 2 + b + \frac{1}{b} \implies r \cdot \frac{1 - br + pb}{p} + r + \frac{1 - br + pb}{p} = 1 + b + \frac{1}{b} \implies br^2 - r(1 + p + pb - b) + p + \frac{p}{b} - 1 = 0.$$

$$\text{Since this quadratic equation has the solutions } \frac{1}{b} \text{ and } r, \text{ by Viète we have } r + \frac{1}{b} = \frac{1 + p + pb - b}{b} \implies \implies r = \frac{p + pb - b}{b}.$$

$$\text{We now want to prove that } R \in \omega \iff \frac{b-r}{c-r} : \frac{b-d}{c-d} \in R \iff \frac{(b-r)(c-d)}{(c-r)(b-d)} \in R.$$

$$\begin{aligned} b-r &= b - \frac{p+pb-b}{b} = \frac{b^2+b-p-bp}{b} = \frac{(b+1)(b-p)}{b} \\ b-d &= b - \frac{pb+p-bcp-b}{p-bc} = \frac{pb-b^2c-pb-p+bc p+b}{p-bc} = \frac{(p-b)(bc-1)}{p-bc} \\ c-d &= c - \frac{pb+p-bcp-b}{p-bc} = \frac{pc-bc^2-pb-p+bc p+b}{p-bc} = \frac{(bc+b-p-pb)(1-c)}{p-bc} \\ c-r &= c - \frac{p+pb-b}{b} = \frac{bc+b-p-pb}{b} \end{aligned}$$

$$\text{So } \frac{(b-r)(c-d)}{(c-r)(b-d)} \in R \iff \frac{\frac{(b+1)(b-p)}{b} \cdot \frac{(bc+b-p-pb)(1-c)}{p-bc}}{\frac{bc+b-p-pb}{b} \cdot \frac{(p-b)(bc-1)}{p-bc}} \in R \iff \frac{(b+1)(c-1)}{bc-1} \in R \iff$$

$$\iff \frac{(b+1)(c-1)}{bc-1} = \frac{\left(\frac{1}{b} + 1\right)\left(\frac{1}{c} - 1\right)}{\frac{1}{bc} - 1}, \text{ which is true.}$$

So R is on the circle centered in A' which passes through B . □

Now we just want to prove that $|m - a'| = |x - a'|$.

$$\text{Similarly, we have } q = \frac{p+pc-c}{c}, \text{ so } m = \frac{1}{2} \left(\frac{p+pb-b}{b} + \frac{p+pc-c}{c} \right) = \frac{2pbc + pb + pc - 2bc}{bc}.$$

$$\text{Since } X \text{ is the midpoint of } BC, \text{ then } x = \frac{b+c}{2}.$$

$$\begin{aligned} \text{Moreover, } |m - a'| = |x - a'| &\iff \left| \frac{2pbc + pb + pc - 2bc}{2bc} + 1 \right| = \left| \frac{b+c}{2} + 1 \right| \iff \frac{1}{2} \left| \frac{p}{bc} \right| \cdot |b+c+2bc| = \\ \frac{1}{2} |b+c+2| &\iff |b+c+2bc| = |b+c+2| \iff |b+c+2bc|^2 = |b+c+2|^2, \text{ which is true by } |z|^2 = z\bar{z}. \end{aligned}$$

This concludes our proof.

Solution 2 by the author

Suppose $AB < AC$.

Using the second claim of the first solution, we know that P, D, C_1, Q are collinear, and P, E, B_1, R are collinear.

Since $BB_1 \perp AA'$ and $CC_1 \perp AA' \implies BB_1 \parallel CC_1 \implies \widehat{BC_1} = \widehat{CB_1} \implies \widehat{BPC_1} = \widehat{CPB_1}$, so the angles $\widehat{BPB_1}$ and $\widehat{CPC_1}$ share the same bisector.

$$\text{Also, } \widehat{PBD} = \widehat{PCE} \implies \triangle PBD \sim \triangle PCE \implies PB \cdot PE = PC \cdot PD.$$

Consider the inversion of center P and power $PB \cdot PE = PC \cdot PD$ and the symmetry across the common

bisector of angles $\widehat{BPB_1}$ and $\widehat{CPC_1}$.

This sends E in B , B in E , D in C and C in D , so this keeps the circle $(BCDE)$ in place.

This means that the power of the inversion is the same as the power of point P to the circle $(BCDE)$, which implies that $PB \cdot PE = PR \cdot PE \implies PR = PB$, and similarly $PQ = PC$.

We also see that, since B_1 is the reflection of B over line $AA' \implies \widehat{BA'} = \widehat{A'B_1} \implies \widehat{BPA'} = \widehat{A'PB_1} \implies PA'$ is the bisector of angle $\widehat{BPB_1}$.

But $PB = PR$, so PA' is the bisector of segment BR and similarly of segment CQ .

This implies that triangles $\triangle A'BC$ and $\triangle A'QR$ are congruent, so $A'M = A'X$, so M is on the circle centered in A' and passing through the midpoint of BC .

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