

ANNOUNCEMENT. The readers who guessed the most authors's actual names are: Theo Koupelis, Li Zhou, Taes Padhiary, Nicușor Zlota, Corneliu Mănescu-Avram, Eduard-Ștefan Dragomir, Alexander George Stoicescu, Kousik Sett, and Katherine Merrill. Congratulations!

Junior Problems

J715. Let a, b, c, d be real numbers with $a > \max(b, c, d)$ such that not all solutions to the equation

$$(a - b)x^3 + (b - c)x^2 + (c - d)x + d - a = 0$$

are real. Prove that $a + c > b + d$.

Proposed by Adrian Andreescu, Dallas, USA

J716. Solve in real numbers

$$4x^4 + 8x^3 + 3x^2 + 2x = -\frac{1}{4}.$$

Proposed by Alessandro Ventullo, Milan, Italy

J717. Find the least prime that can be written as

$$2pq(q - 2p) + 2qr(r - 2q) + 2rp(p - 2r) + 7pqr,$$

where p, q, r are also primes.

Proposed by Titu Andreescu, Dallas, USA

J718. Let a, b, c be nonnegative real numbers such that $ab + bc + ca + 2abc = 1$. Prove that

$$a + b + c \geq \frac{3}{2}.$$

Proposed by Mircea Becheanu, Canada

J719. Let a, b, c be positive real numbers. Prove that

$$\sqrt{2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} + 4abc \geq (a + b)(b + c)(c + a).$$

Proposed by Mihai Opincariu, Brad, Romania

J720. Let a, b, c be positive real numbers such that at most one of them is less than 1 and $a + b + c = 3$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2025}{1 + 968abc}.$$

Proposed by Titu Andreescu, Dallas, USA and Marius Stănean, Zalău, Romania

Senior Problems

S715. Let x, y, z be positive real numbers and

$$a = \frac{x}{y+z}, \quad b = \frac{y}{z+x}, \quad c = \frac{z}{x+y},$$

Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq a+b+c.$$

Proposed by An Zhenping, Xianyang Normal University, China

S716. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^4 \geq \frac{(a+b+c)^5}{abc(abc+2)}.$$

Proposed by Marin Chirciu, Pitești, Romania

S717. Let ABC be a triangle. Prove that

$$am_a + bm_b + cm_c \geq s(h_a + h_b + h_c - 3r).$$

Proposed by Dan-Ștefan Marinescu, Hunedoara, Romania

S718. Let ABC be a triangle with incenter I and center of gravity G . Prove that if $I \neq G$, there is no triangle for which the conditions $\cos A + \cos B = 1$ and $IG \parallel BC$ simultaneously hold.

Proposed by Mihaela Berindeanu, Bucharest, Romania

S719. Let $x \geq y \geq z \geq 0$ be real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$6 + xy + yz + zx + (3\sqrt{2} - 4)(x - z)^2 \geq 3(x + y + z).$$

When does equality hold?

Proposed by Marius Stănean, Zalău, Romania

S720. Given a triangle ABC with the centroid G . M is an arbitrary point inside the triangle. The lines passing through M and parallel to the median lines intersect the sides BC, CA, AB at X, Y, Z respectively. Prove that

$$XY^2 + YZ^2 + ZX^2 = 3(MX^2 + MY^2 + MZ^2) - \frac{9}{4}MG^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Undergraduate Problems

U715. Let α and β be real numbers. Define the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{e^{(x+y)^2} - 1 - (x+y)^2}{(|x| + |y|)^\alpha (x^2 + y^2)^\beta} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Determine for which real parameters α and β the function f is continuous at the origin.

Proposed by Alessandro Ventullo, Milan, Italy

U716. Let A and B be two $n \times n$ real matrices such that $\text{rank}(A + iB) = 1$ and $\text{Tr}(A^2 + B^2) = 0$. Prove that $AB = BA$.

Proposed by Mihai Opincariu, Brad, Romania

U717. Find all positive real constant k such that

$$9(a^2 + k)(b^2 + k)(c^2 + k) \geq (1 + k)^3(a + b + c)^2$$

for all nonnegative real numbers a, b, c with $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

U718. (a) Find all nonnegative integers x such that $5^x + (x + 10)^2$ is a perfect square.

(b) Let p be a prime, and let f be a polynomial with integer coefficients. Prove that $p^x + f^2(x)$ is a perfect square for at most finitely many nonnegative integers x .

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

U719. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x^{2k-1})$ and $f(x^{2k})$ are periodic functions, for some positive integer k .

Proposed by Dan-Ștefan Marinescu, Hunedoara, Romania

U720. Evaluate

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1}{2n+3} - \cdots \right) - \frac{1}{4n} \right]^2.$$

Proposed by Paolo Perfetti, Università di Tor Vergata, Rome, Italy

Olympiad Problems

- O715.** Prove that there are infinitely many triples (a, b, c) of distinct positive integers such that $a^3 + b^3 + c^3 - 3abc$ is a perfect square but $a + b + c$ is not.

Proposed by Gabriel Dospinescu, École normale supérieure, Lyon, France

- O716.** Let $a_1, \dots, a_n$ be positive integers such that any two of them are relatively prime, and let $b_1, \dots, b_n$ be positive integers. Prove that there are infinitely many positive integers N such that, for every $i \in \{1, \dots, n\}$, the number $\frac{N}{b_i}$ is a power (of a positive integer) with exponent a_i .

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

- O717.** Let $a_1, a_2, \dots, a_n$ be positive real numbers such that at most one of them is less than 1 and $a_1 + a_2 + \dots + a_n = n$. Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} \geq a_1^3 + a_2^3 + \dots + a_n^3.$$

Proposed by Vasile Cîrtoaje, Oil-Gas University, Ploiești, Romania

- O718.** Find all strictly monotonic functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f\left(a - \frac{1}{b}\right) + f\left(b - \frac{1}{c}\right) + f\left(c - \frac{1}{a}\right) + f\left(a - \frac{1}{b}\right)f\left(b - \frac{1}{c}\right)f\left(c - \frac{1}{a}\right) = 0$$

for all real numbers a, b, c satisfying $abc = 1$.

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

- O719.** Let x, y, z be nonnegative real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$x + y + z \geq 2 + \frac{(9 + 3\sqrt{3})xyz}{4 + (5 + 3\sqrt{3})xyz}.$$

When does the equality hold?

Proposed by Marius Stănean, Zalău, Romania

- O720.** Find all pairs (a, b) of positive integers such that for all sufficiently large positive integers n , each of the numbers $(a^n + 1)(b^n + 1)$ and $((a + b)^n + 1)((a - b)^n + 1)$ can be written as the sum of squares of two integers.

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France