

Junior Problems

J721. Let a, b, c be real numbers such that $ab + bc + ca = 3$ and

$$\frac{a^3 + b^3 + c^3}{3} = (a - 1)(b - 1)(c - 1) + 1.$$

Find all possible values of $a + b + c$.

Proposed by Adrian Andreescu, Dallas, USA

J722. Let a, b, c be positive real numbers such that $11a + 5b + c \leq 24$. Prove that

$$\frac{3}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

J723. Let a, b, c be the side-lengths of an acute triangle. Prove that

$$\frac{abc}{a^2 + b^2 - c^2} + \frac{abc}{b^2 + c^2 - a^2} + \frac{abc}{c^2 + a^2 - b^2} \geq a + b + c.$$

Proposed by Titu Andreescu, Dallas, USA

J724. Let a, b, c be positive real numbers. Prove that

$$(a^3 + bc)(b^3 + ca)(c^3 + ab) \geq (abc)^2(a + 1)(b + 1)(c + 1).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J725. Prove that there are no integer numbers x, y, z such that

$$x^2y + xy + y^2z + 2x + 2z + 1 = 0.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

J726. Let ABC be a triangle with orthocenter H and let A_1, B_1 be the midpoints of sides BC and AC , respectively. Line d_1 passes through point A , line d_2 passes through point B , and $d_1 \cap d_2 = \{X\}$. The projection of point H onto d_1 is $Z \neq A$ and onto d_2 is $Y \neq B$. If $A_1Z = A_1A$ and $B_1Y = B_1B$, show that CX is a symmedian of $\triangle ABC$.

Proposed by Mihaela Berindeanu, Bucharest, România

Senior Problems

S721. Find all pairs of integers (m, n) such that $m - 1$, $n - 2$, $n - 2m$, and $2n - 3m - 1$ are all perfect squares.

Proposed by Alessandro Ventullo, Milan, Italy

S722. Solve in real numbers the system of equations

$$\begin{aligned}x(x^2 + 10xy + 5y^2)^2 &= 8(33 + 8\sqrt{2}), \\y(5x^2 + 10xy + y^2)^2 &= 8(33 - 8\sqrt{2}).\end{aligned}$$

Proposed by Titu Andreescu, Dallas, USA

S723. Let ABC be an acute triangle with side lengths $BC = a$, $AC = b$, $AB = c$ and circumcircle $\Gamma(O, R)$. The lines AO , BO , CO cut the circumcircles of triangles OBC , OAC and OAB at points D , E , and F , respectively. If $OD \cdot OE \cdot OF \leq \frac{8R(a^2 + b^2 + c^2)}{9}$, prove that $\triangle ABC$ is equilateral.

Proposed by Mihaela Berindeanu, Bucharest, România

S724. Let ABC be a nonisosceles triangle, Ω the center of its nine point circle, and L be its symmedian point (also called Lemoine point). Prove that if the line ΩL is parallel to the longest side of ABC then the triangle has a right angle.

Proposed by Titu Zvonaru, Comănești and Neculai Stanciu, Buzău, România

S725. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$a\sqrt{bc+1} + b\sqrt{ca+1} + c\sqrt{ab+1} \geq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

S726. Prove that in any triangle

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} \geq \sqrt{\frac{3}{2Rr}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Undergraduate Problems

U721. For every positive integer n , let $S_n = 1 + 2 + \cdots + n$ and

$$P_n = \frac{S_2}{S_2 - 1} \cdot \frac{S_3}{S_3 - 1} \cdot \cdots \cdot \frac{S_n}{S_n - 1}.$$

Evaluate $\lim_{n \rightarrow \infty} P_n$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U722. Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq 1 \geq d$ and $ab + bc + cd = 3$. Prove that

$$\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} + \frac{1}{d^2 + 2} \geq \frac{4}{3}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, România

U723. Evaluate

$$\int_0^\pi \frac{x \sin 2x \cos x}{1 + \cos^6 x} dx.$$

Proposed by Sorin Monel Budișan, Beclean, România

U724. Let $a > 0$ and $f : [-a, a] \rightarrow \mathbb{R}$ be a function such that $f(x) + f(-x) = \frac{1}{a}$ for all $x \in [-a, a]$. Show that

$$\int_{-a}^a \frac{(x^4 + 1)f(x)}{x^6 + 1} dx < 1 + \frac{a^2}{3}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

U725. Prove that the improper integral $I(a) = \int_a^\infty \frac{\cos^2 x}{x^2(x^2 + 1)} dx$ is convergent for every real number $a > 0$ and evaluate $\lim_{a \rightarrow 0} I(a)$.

Proposed by Sorin Monel Budișan, Beclean, România

U726. Let $f(x) = x^8 - 10x^4 + 1$.

- (a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
- (b) Determine the splitting field of $f(x)$ over $\mathbb{Q}$.
- (c) Determine the Galois group of $f(x)$ over $\mathbb{Q}$.

Proposed by Alessandro Ventullo, Milan, Italy

Olympiad Problems

O721. Find all pairs (m, n) of positive integers for which there is an integer s such that

$$(m-1)! + 3^{3n-1} = (s-87)^2 \quad \text{and} \quad (m+1)! + 3^{3n+1} = (s+87)^2.$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

O722. Find all positive integers k for which

$$\varphi(\varphi(n) + k) = n - k$$

has infinitely many integer solutions.

Proposed by Alessandro Ventullo, Milan, Italy

O723. Solve in positive integers the equation

$$x^8 - 56x^4y^4 + y^8 = 2026.$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

O724. Let ABC be a triangle with $AB \neq AC$ and let G be its centroid, K be its symmedian point and Γ be its circumcircle. Let G_a be the foot of the perpendicular drawn from G to BC and M_a be the midpoint of BC . Let D, E be points of Γ such that $AC \parallel BD, AB \parallel CE$. Knowing that DG_aM_aE is a cyclic quadrilateral, prove that K is the midpoint of GG_a .

Proposed by Todor Zaharinov, Sofia, Bulgaria

O725. Let $a, b, c > 0$ such that $ab + bc + ca + abc = 4$. Prove that

$$\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + (5 + 4\sqrt{2})abc \geq 8 + 4\sqrt{2}.$$

Proposed by Marius Stănean, Zalău, România

O726. Prove that $\frac{n-2}{n-1}$ is the least positive value of k such that

$$ka_1 + a_2 + \cdots + a_{n-1} \geq k + n - 2$$

for any sequence of real numbers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ with $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$, where $n \geq 3$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, România