

HIDDEN CONNECTIONS BETWEEN WONDERFUL ELEMENTS IN THE GEOMETRY OF A TRIANGLE

PART 2: WHY-POINTS

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This paper is the second part of the research that investigates properties and relationships between particular unique objects in the geometry of a triangle. The reader is recommended to familiarise themselves with the first part [1] about the mixtilinear circles to get acquainted with the notations and definitions used, which I omit in this paper.

In this part, I will investigate the properties of the so-called why-points, which, allegedly, first appeared in ISL 2011 G4 and were previously researched in 2021 by Muricaaaaaaa [2]. In addition, I will show the vague relationship between why-points and mixtilinear incircles. However, I will reveal stronger links between those in the next two papers through other unique objects, including but not limited to *cuddling circles*.

Note: Many problems were taken from the Telegram channel Olympiad Geometry [3]. At the time of writing, the channel's owner, Fedor Bakharev, had published an article [4] in the Kvant journal, in particular about the why-points. Some of the solutions in this paper may overlap with that work but were derived independently.

Keywords — why-point, pester point, Euler line, X(25).

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1 Definition

Let the tangent at A to the circumcircle of $\triangle ABC$ meet BC at P , and let D be the antipode of A wrt (ABC) . Denote the second intersection point of PD and (ABC) by W_A (so that W_A and A are in different half-planes wrt BC). This point is called the *why-point* relative to the vertex A , or the A -whypoint. We will now investigate some properties of the why-points.

2 Angle chasing

Although very basic, this method solves most of the problems involving why-point. However, we must first prove one substantial fact, and then many other problems will quickly crumble.

2.1 Key observation

Problem. Let G be the centroid of $\triangle ABC$, and let A_1 be a foot of the perpendicular from A to BC . Prove that points G , A_1 , W_A are collinear.

Proof. From our definitions, we have

$$\angle AW_AP = 90^\circ = \angle AA_1P,$$

so APW_AA_1 is cyclic. Therefore,

$$\angle PA_1W_A = \angle PAW_A = \angle ADW_A = \angle ADP,$$

implying that W_AA_1FD is also cyclic. Now let A' be the reflection of A over the perpendicular bisector of $\overline{BC}$. Note that as

$$\angle GMA_1 = \angle GAA' \text{ and } \frac{GM}{GA} = \frac{1}{2} = \frac{A_1M}{AA'},$$

we infer that $\triangle A_1GM \sim \triangle A'GA$, from which it follows that A_1 , G , A' are collinear. Thus, we may conclude after

$$\angle A_1W_AD = \angle AFA_1 = \angle FAA' = \angle A'W_AD.$$

2.3 Corollary 2.

Problem. Let AA_1 intersect (ABC) at $E \neq A$. Prove that $(W_A A_1 E)$ touches BC at A_1 .

Proof. It suffices to show that EA_1 is the diameter in $(W_A A_1 E)$, which is obvious since

$$\angle EW_A A_1 = \angle EAA' = 90^\circ.$$

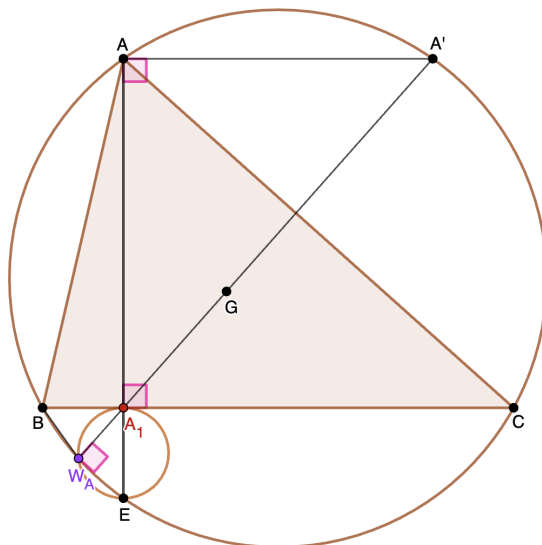


Figure 3: Another tangency

q.e.d.

2.4 Corollary 3.

Problem. Let M be the midpoint of $\overline{BC}$, and let AM intersect (ABC) at $J \neq A$. Prove that points W_A, A_1, M, J are concyclic.

Proof. Again, a straightforward angle chase gives us the desired:

$$\angle MJW_A = \angle AJW_A = \angle AA'W_A = \angle AA'A_1 = \angle A'A_1M.$$

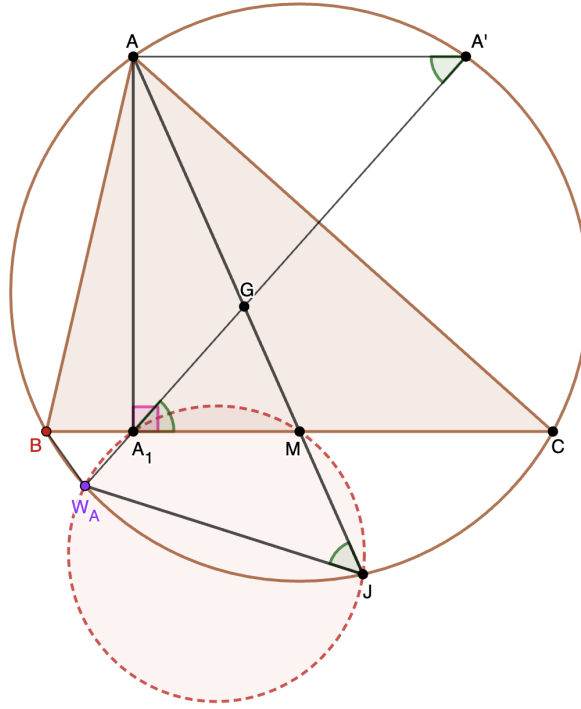


Figure 4: Corollary on concyclicity

q.e.d.

3 Three altitudes

Denote the orthocentre of $\triangle ABC$ by H , and define points B_1 and C_1 similarly to A_1 . We will now investigate interconnections between the introduced points and the why-points.

Let BC and B_1C_1 meet at Q . This point has many beautiful properties, but here we look only at those that have to do with the A -whypoint. Let us begin.

3.1 H-sidepoint

Problem. Points Q, H, A_1, W_A are concyclic.

Proof. Let AQ intersect (ABC) at $R \neq A$. Then

$$QR \cdot QA = QB \cdot QC = QC_1 \cdot QB_1,$$

so RAB_1HC_1 is cyclic. Hence, we obtain that

$$\angle HRA = 90^\circ = \angle DRA,$$

so D, H, R are collinear and RHA_1Q is cyclic. But then

$$\angle A_1HD = \angle HDA' = \angle RDA' = \angle RW_AA',$$

from which the desired follows.

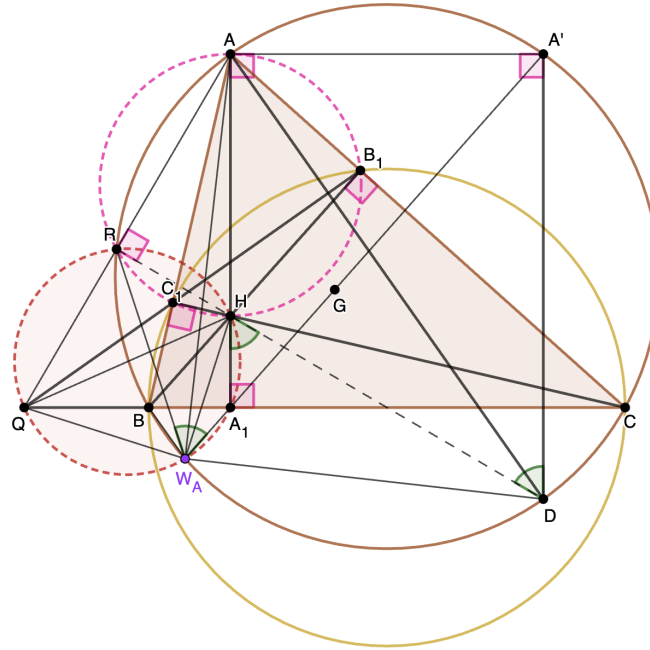


Figure 5: H-sidepoint concyclicity

q.e.d.

3.2 More radical axes

Problem. Points Q , W_A , J are collinear.

Proof. By **2.2**, W_AA_1MJ is cyclic. Also $MA_1C_1B_1$ is cyclic as all four points lie on the nine-point circle of $\triangle ABC$. Hence, as from **3.1**, points A , R , Q are collinear, we have that

$$QR \cdot QA = QC_1 \cdot QB_1 = QA_1 \cdot QM,$$

so A_1RAM is also cyclic. Then, the desired immediately follows since JW_A , BC , AR meet at the radical centre of (AB_1HC_1R) , (ABC) , (W_AA_1MJ) .

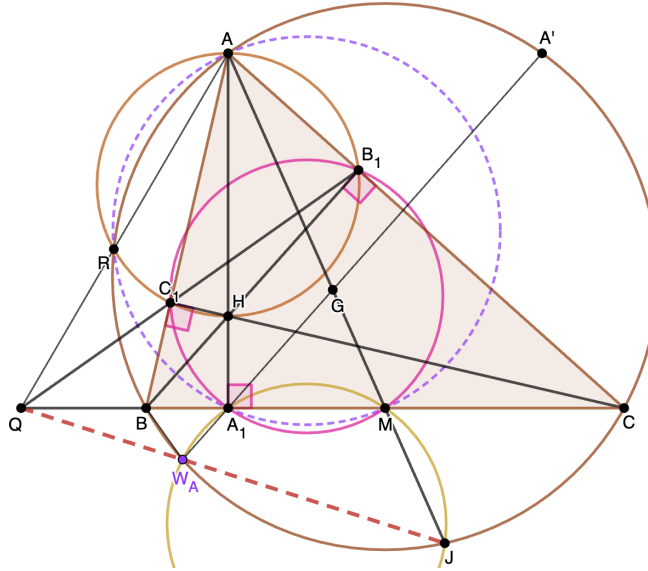


Figure 6: H-sidepoint collinearity

q.e.d.

3.3 Questionable bisector

Problem. Let N be the midpoint of $\overline{QA_1}$. Prove that points N, E, W_A are collinear.

Proof. Let BC and $A'D$ meet at I . Since $\angle EW_A A' = 90^\circ$, it suffices to show $\angle BW_A A' = 90^\circ$, or $A'IT_A N$ is cyclic, which will follow if $A_1 N \cdot A_1 I = A_1 T_A \cdot A_1 A'$, or if $A_1 N \cdot A_1 I = A_1 B \cdot A_1 C$. By cross-ratio criteria, $(Q, A_1; B, C) = -1$. So,

$$\begin{aligned}
 (!) \quad A_1 N \cdot A_1 I &= A_1 B \cdot A_1 C \iff (!) \quad \frac{A_1 I}{A_1 B} = \frac{A_1 C}{A_1 N} \iff (!) \quad \frac{A_1 C - A_1 B}{A_1 B} = \frac{2A_1 C}{A_1 Q} \\
 (!) \quad \frac{A_1 C}{A_1 B} &= \frac{2A_1 C + A_1 Q}{A_1 Q} \iff (!) \quad \frac{A_1 C}{A_1 B} = \frac{A_1 C + QC}{A_1 Q} \iff (!) \quad \frac{A_1 Q}{A_1 B} = \frac{A_1 C + QC}{A_1 C} \\
 (!) \quad \frac{QB}{A_1 B} &= \frac{QC}{A_1 C} \iff (!) \quad \frac{A_1 C}{A_1 B} = \frac{QC}{QB},
 \end{aligned}$$

which is true, so we may conclude

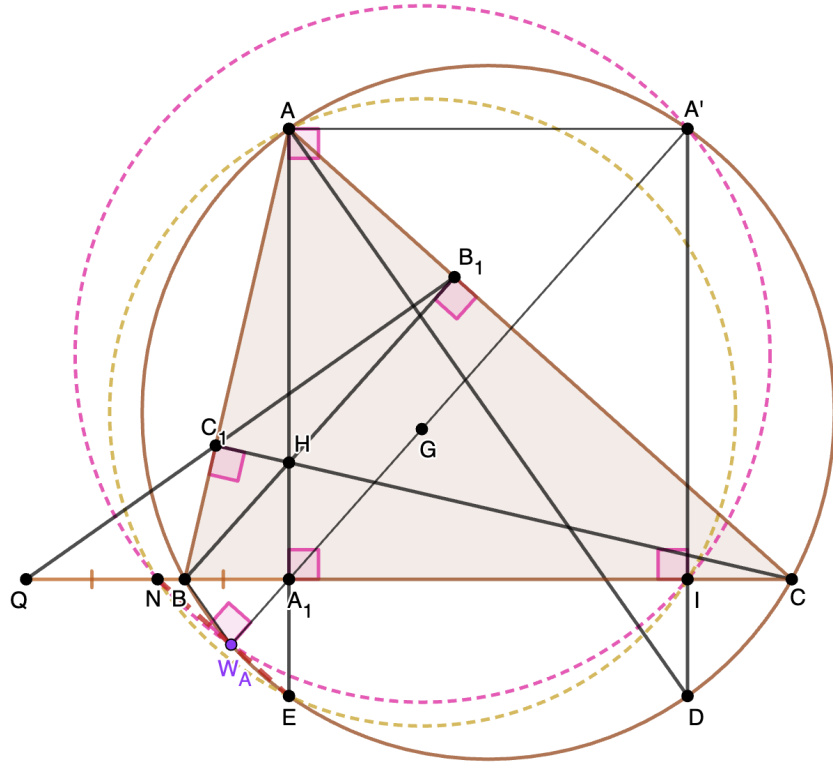


Figure 7: Questionable bisector

q.e.d.

3.4 The fourth and fifth altitudes

Problem. Let H_1 be a foot of the perpendicular from A_1 to B_1C_1 , and let M_1 be the midpoint of $\overline{A_1H_1}$. Also, let H_0 be a foot of the perpendicular from H to B_1C_1 . Prove that points A, H_0, M_1, W_A are collinear.

Proof. Note that since

$$\angle QH_0H = 90^\circ = \angle QA_1H \text{ and } \angle NW_AA_1 = \angle EAA' = 90^\circ = \angle NM_1A_1,$$

we have, combining with **3.1**, that $QH_0HA_1W_A$ and $NM_1A_1W_A$ are cyclic. Also, we can see that

$$\angle B_1CQ = \angle CAA' \text{ and } \angle QB_1C = 180^\circ - \angle ABC = \angle AA'C,$$

from which $\triangle QB_1C \sim \triangle CAA'$, so $\angle B_1QC = \angle ACA'$. Points H_0, M_1, W_A are collinear because

$$\angle M_1W_AA_1 = \angle M_1BA_1 = \angle H_0QA_1 = \angle H_0W_AA_1.$$

We are left to prove that A, H_0, W_A are collinear, which is true as

$$\angle H_0W_AA_1 = \angle H_0QA_1 = \angle ACA' = \angle AW_AA' = \angle AW_AA_1.$$

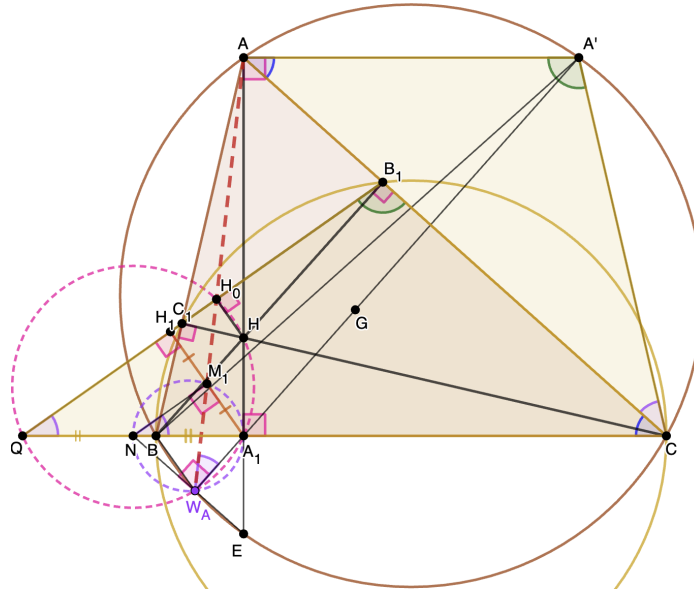


Figure 8: More altitudes

q.e.d.

4 More questions

Here, we will have a look at two unique constructions, each of which gives substantial insight into the why-point's nature. The first one will come very handy in the next part of the research about cuddling circles.

4.1 Why-homothety (ISL 2011/G4)

Problem. Let K and L be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Prove that (W_AKL) touches (ABC) at W_A .

Proof. Let W_AK and W_AL intersect (ABC) again at U and V , respectively. Note that since $\widehat{AB} = \widehat{A'C}$, we have $\angle AW_AB = \angle A'W_AC$ and, from the isogonality lemma for $\angle BW_AC$ and the rays W_AU and W_AG , we conclude that

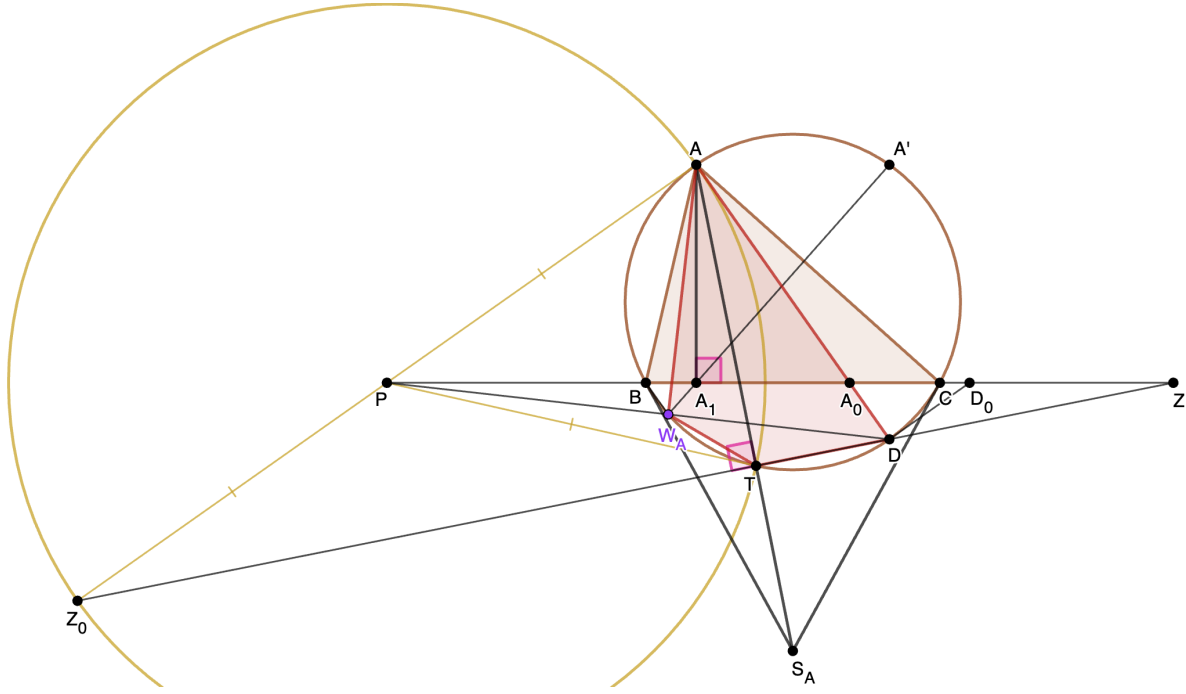


Figure 10: Why-harmony

q.e.d.

5 Pester point

Define W_B and W_C similarly to W_A . Let us see how the three why-points are interconnected.

5.1 Relation between the why-points

Problem. Prove that the lines AW_A , BW_B , CW_C are concurrent at some point W (the whywhywhy-point), which we will call the pester point.

Proof. Note that, by the law of sines,

$$\frac{\sin \angle BAW_A}{\sin \angle W_A AC} = \frac{\sin \angle BA'W_A}{\sin \angle W_A A'C} = \frac{c}{b} \cdot \frac{BA_1}{A_1 C}.$$

Similarly,

$$\frac{\sin \angle ACW_C}{\sin \angle W_C CB} = \frac{b}{a} \cdot \frac{AC_1}{C_1 B} \quad \text{and} \quad \frac{\sin \angle CBW_B}{\sin \angle W_B BA} = \frac{a}{c} \cdot \frac{CB_1}{B_1 A}.$$

Hence, the desired follows after the converse of Ceva's theorem because

$$\frac{\sin \angle BAW_A}{\sin \angle W_A AC} \cdot \frac{\sin \angle ACW_C}{\sin \angle W_C CB} \cdot \frac{\sin \angle CBW_B}{\sin \angle W_B BA} = \frac{BA_1}{A_1 C} \cdot \frac{AC_1}{C_1 B} \cdot \frac{CB_1}{B_1 A} = 1.$$

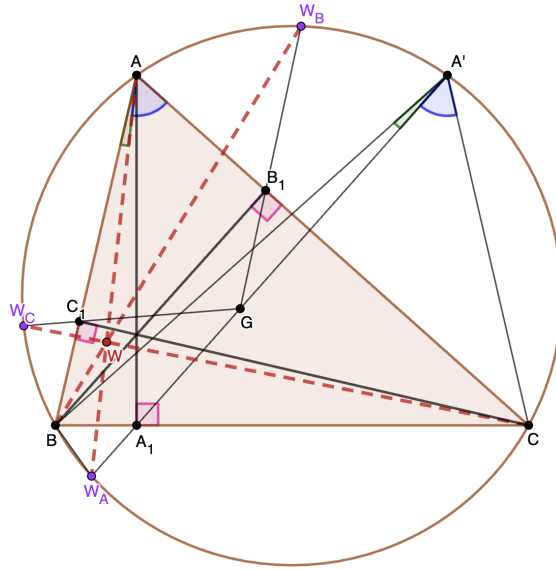


Figure 11: The pester point

q.e.d.

5.2 Deeper insights

Problem. Prove that points W_C, C_1, A_1, W_A are concyclic (similarly for W_A, A_1, B_1, W_B and W_B, B_1, C_1, W_C), and that points W_C, C_1, M, J are also concyclic (similarly for W_B, B_1, M, J).

Proof. Define C' similarly to A' . From **2.1**, we know that $\frac{GA_1}{GA'} = \frac{1}{2}$ and by analogy $\frac{GC_1}{GC'} = \frac{1}{2}$. So, $\frac{GA_1}{GA'} = \frac{GC_1}{GC'}$, and thus $A_1C_1 \parallel A'C'$.

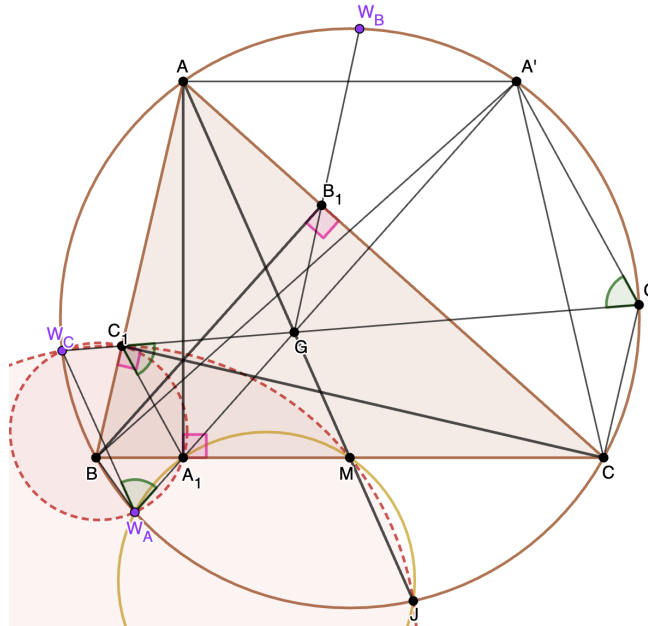


Figure 12: A few concyclicities

Then

$$\angle A_1C_1C' = \angle A'C'C_1 = \angle A'W_AW_C,$$

from which $W_C C_1 A_1 W_A$ is cyclic as desired. By **2.2**, $W_A A_1 M J$ is cyclic, so the cyclicity of points W_C, C_1, M, J follows as

$$GM \cdot GJ = GA_1 \cdot GW_A = GC_1 \cdot GW_C.$$

q.e.d.

6 Poles and polars burst

In this subsection, we will look at two wonderful configurations. The first one is directly connected to cuddling circles, as will be explained later on in the third part of the research, and the second one allows us to demonstrate the relation between why-points and other well-known lines in $\triangle ABC$. Poles and polars will help us out a lot in proofs. Without further ado:

6.1 Stretching the picture

Problem. Let the tangent to (ABC) through A meet the tangents through B and C at S_C and S_B , respectively. Let the lines parallel to AB and AC through S_C and S_B meet at S . Prove that points S, A, W_A are collinear.

Proof. We first prove the following lemma.

Lemma. We are given $\triangle ABC$ with centroid G , circumcentre O , medians BN and CM , and an altitude AH . Point P is the foot of the perpendicular from O to MN . Then H, G, P are collinear.

Proof. Let AH and AO intersect (ABC) again at H' and A' , respectively, and let AP and BC meet at Q . Then, as $\angle AH'A' = 90^\circ$, we have $A'H' \parallel BC$. Also, since P lies on the midline of $\triangle ABC$, we have $AP = AQ$ and thus, the line OP is the midline in $\triangle A'AQ$, so $A'Q \parallel OP$. Then by symmetry, $CQ = BH$. So,

$$\frac{1}{2} = \frac{PN}{QC} = \frac{PN}{BH} \Rightarrow 2PN = BH,$$

from which, as $2GN = GB$, we conclude that $\triangle PGN \sim \triangle BGH$, so H, G, P are collinear as desired.

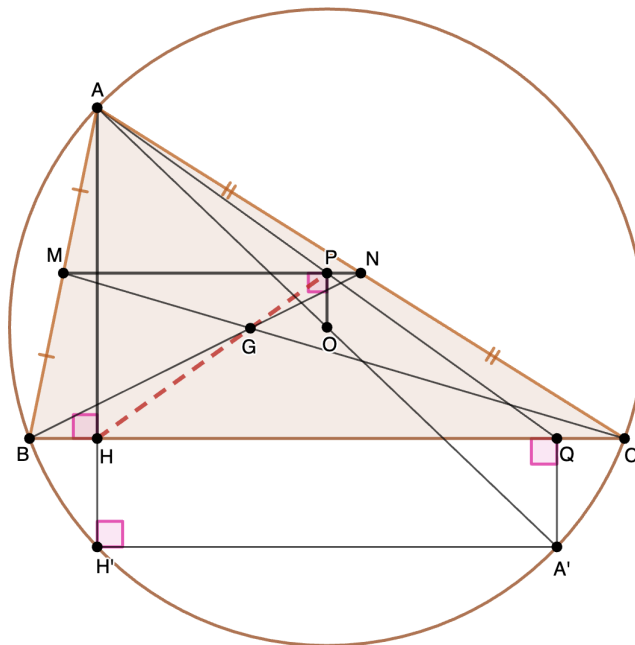


Figure 13: Handy lemma

Now let us get back to the main problem. Let O be the circumcentre of $\triangle ABC$. It is well-known that AB is the polar of S_C wrt (ABC) , and K lies on it. Therefore, S_C lies on the polar of K . Since $OS_C \perp S_C S$, we can infer that the line SS_C is the actual polar of K , and, similarly, the line SS_B is the polar of L . This, in turn, means that, by the poles and polars reciprocity principle ¹, K and L lie on the polar of S , so point S is the pole of the line KL . Consider inversion around (ABC) . Then

$$S \iff H_0 \text{ where } H_0 = KL \cap OS.$$

We must prove that AH_0OW_A is cyclic. Let W_0 be the midpoint of $\widehat{BAC}$. Then

$$\angle AW_A H_0 = \angle AW_A A' = \angle ACA' = 2\angle ACW_0 = \angle AOW_0 = \angle AOH_0,$$

and we may conclude.

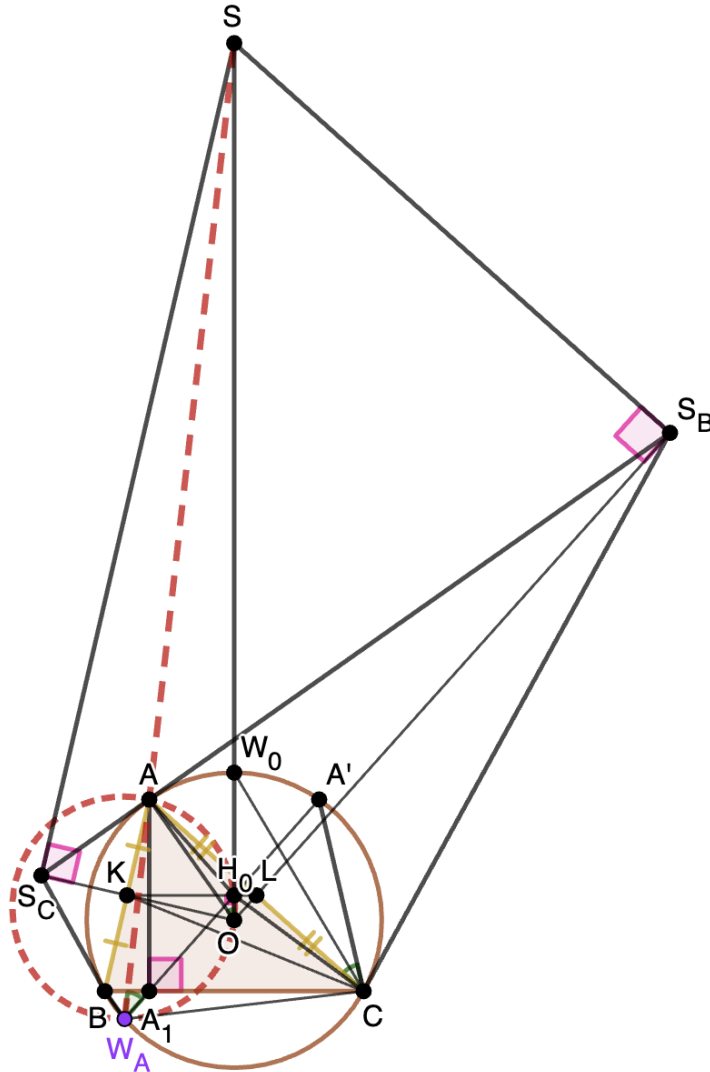


Figure 14: The stretched picture

q.e.d.

¹This refers to the well-know fact that if X lies on the polar of Y , then Y lies on the polar of X

6.2 Redefining the why-point

Problem. Let KL meet the tangent through A to (ABC) at Q_0 . Prove that Q_0W_A is tangent to (ABC) .

Proof. Let KL intersect (ABC) at K_0 and L_0 as shown in the picture below. It is clear that (AKL) and (ABC) are homothetic around A , thus (AKL) is tangent to the line AQ_0 as well. So we have:

$$pow(Q_0, (ABC)) = QK_0 \cdot QL_0 = Q_0 A^2 = QK \cdot QL = pow(Q_0, (KLW_A)).$$

Hence, Q_0 lies on the radical axis of (ABC) and (KLW_A) , which, by **4.1**, is the tangent to (ABC) at W_A , so we may conclude.

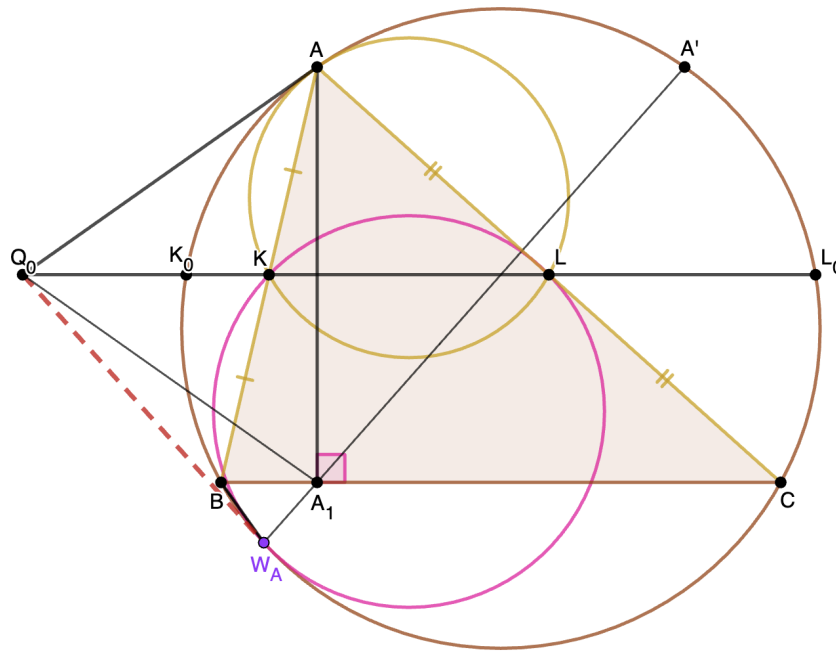


Figure 15: Redefining the why-point

q.e.d.

6.3 Immediate consequence (TUYMAADA 2024/J4)

Problem. Let AI_0 be the bisector in $\triangle AKL$. Prove that points A, I_0, A_1, W_A are concyclic.

Proof. From the previous problem, we know that $Q_0A = Q_0W_A$. Also, $Q_0A = Q_0A_1$ by symmetry over KL . The desired follows because we can easily obtain that $Q_0A = Q_0I_0$ as the violet angle is equal to the sum of the green and blue angles, so we conclude.

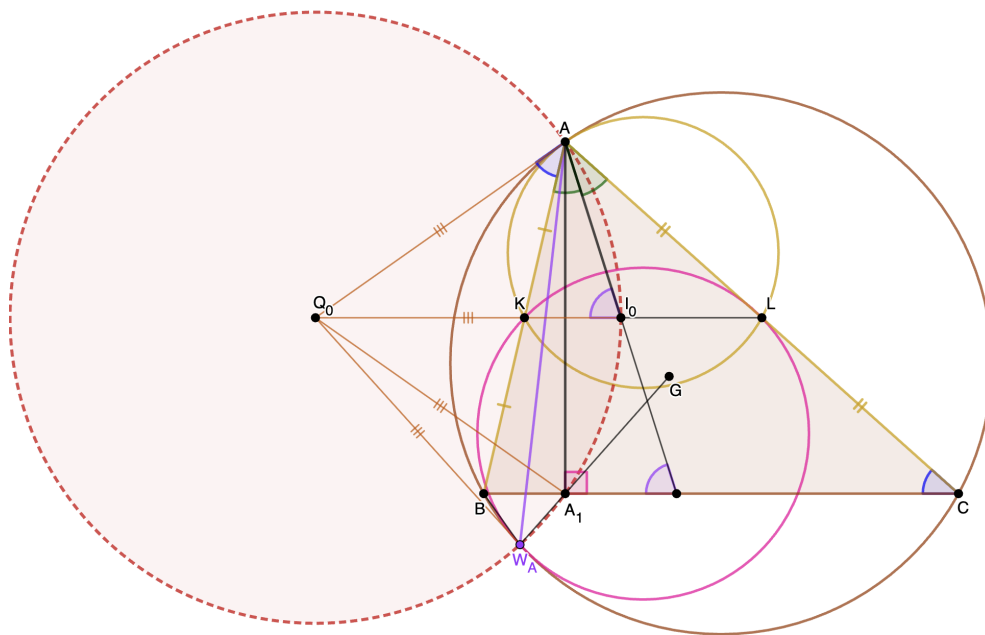


Figure 16: Another concyclicity

q.e.d.

6.4 Relation to the Euler line

Problem. Prove that point W lies on the Euler line of $\triangle ABC$.

Proof. We begin by proving the following crucial lemma and claim.

Lemma. Let AD, BE, CF be the altitudes of $\triangle ABC$ and denote its orthocentre and circumcentre by H and O , respectively. Let EF and BC , DF and AC , DE and AB meet, respectively, at P, Q, R . Then OH is perpendicular to the line that passes through P, Q, R . To put it more simply, the Euler line of the triangle is perpendicular to its orthic axis, which we will call l .

Proof. Clearly $AEDB, BFEC, AFDC$ are cyclic. Consider (DEF) , which is the nine-point circle of $\triangle ABC$, and denote its circumcentre by O_9 . It is well-known that O_9 lies on the Euler line of $\triangle ABC$, so it suffices to show that $OO_9 \perp l$.

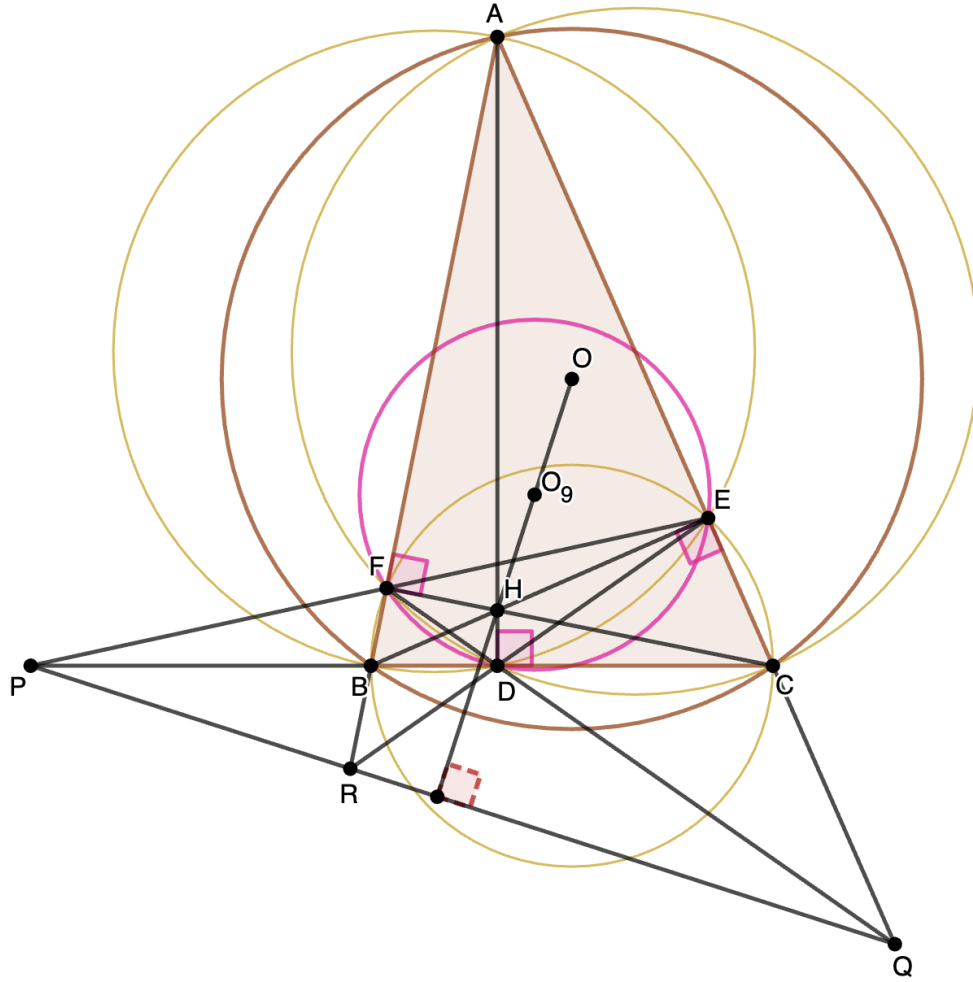


Figure 17: The Euler's line lemma

Note that

$$\text{pow}(P, (DEF)) = PF \cdot PE = PB \cdot PC = \text{pow}(P, (ABC))$$

and

$$\text{pow}(Q, (DEF)) = QD \cdot QF = QC \cdot QA = \text{pow}(Q, (ABC)),$$

thus we conclude that l is the radical axis of (DEF) and (ABC) , from which the desired follows.

Claim. Point W is the pole of the orthic axis l of $\triangle ABC$ wrt (ABC) .

Proof. Notice that A_1, C_1, K, B_1, L all lie on the nine-point circle of $\triangle ABC$, and recall that (AKL) is tangent to (ABC) . Then

$$\text{pow}(Q_0, (A_1B_1C_1)) = QK \cdot QL = Q_0A^2 = \text{pow}(Q_0, (ABC)),$$

which means that Q_0 lies on the radical axis of (ABC) and $(A_1B_1C_1)$, thus by *lemma*, on the line l . On the other hand, from **6.2**, we know that Q_0 is the pole of AW_A . We have proved that Q_0 lies on the orthic axis of $\triangle ABC$. Then, similarly, the poles of the lines BW_B and CW_C must as well lie on the orthic axis. Recall that the lines AW_A, BW_B, CW_C are concurrent at W . Then, by the pole and polar reciprocity principle, we conclude that the polar of W is the orthic axis as desired.

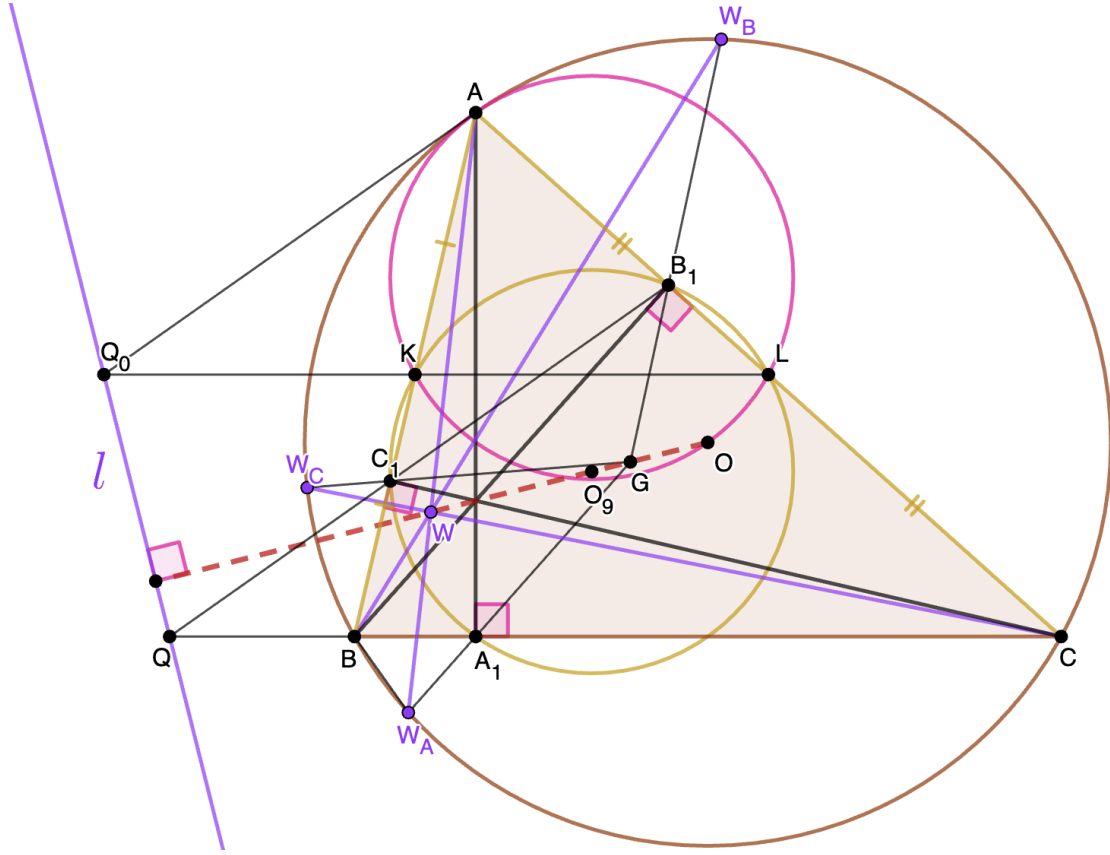


Figure 18: Pester the Euler's line

Now, we can return to the main problem. By *claim*, we know point W and the orthic axis of $\triangle ABC$ are in pole-polar relationship wrt (ABC) . Hence $OW \perp l$, which, combined with *lemma*, yields the desired.

q.e.d.

6.5 Why X(25)?

Define S_A similarly to S_B and S_C . We will prove that the point W actually coincides with X(25) — this is the point with number 25 in the Encyclopedia of Triangle Centres [5] and is defined as the homothetic centre of orthic and tangential triangle.

Problem. Prove that the lines S_AA_1 , S_BB_1 , S_CC_1 are also concurrent at W .

Proof. By similarity, it suffices to show that the line S_AA_1 passes through W , which will follow, by **6.3 lemma**, if A_1 and S_A lie on the polar of Q wrt (ABC) . Let us prove this. It is indeed obvious that QA_1 is the polar of S_A , hence S_A lies on the polar of Q , so we are left to show that A_1 lies on the polar of Q wrt (ABC) , or that $S_AA_1 \perp OQ$, or that point A_1 is the orthocentre of $\triangle QOS_A$, or that $\angle S_AA_1M = \angle QOM$, or that $\triangle QOM \sim \triangle S_AA_1M$, or that $\frac{MQ}{MO} = \frac{MS_A}{MA_1}$. Since $BOCS_A$ is clearly cyclic, we have $MO \cdot MS_A = MB \cdot MC$. Therefore,

$$\begin{aligned} (!) \quad QM \cdot A_1M &= MB \cdot MC \iff (!) \quad (QB + \frac{BC}{2})(\frac{BC}{2} - A_1B) = \frac{BC^2}{4} \\ (!) \quad \frac{QB \cdot BC}{2} &= A_1B \cdot (\frac{BC}{2} + QB) \iff (!) \quad QB \cdot (A_1B + A_1C) = A_1B \cdot (QC + QB) \\ (!) \quad QB \cdot A_1C &= A_1B \cdot QC \iff (!) \quad \frac{QB}{QC} = \frac{A_1B}{A_1C}, \end{aligned}$$

which is true since $(Q, A_1; B, C) = -1$.

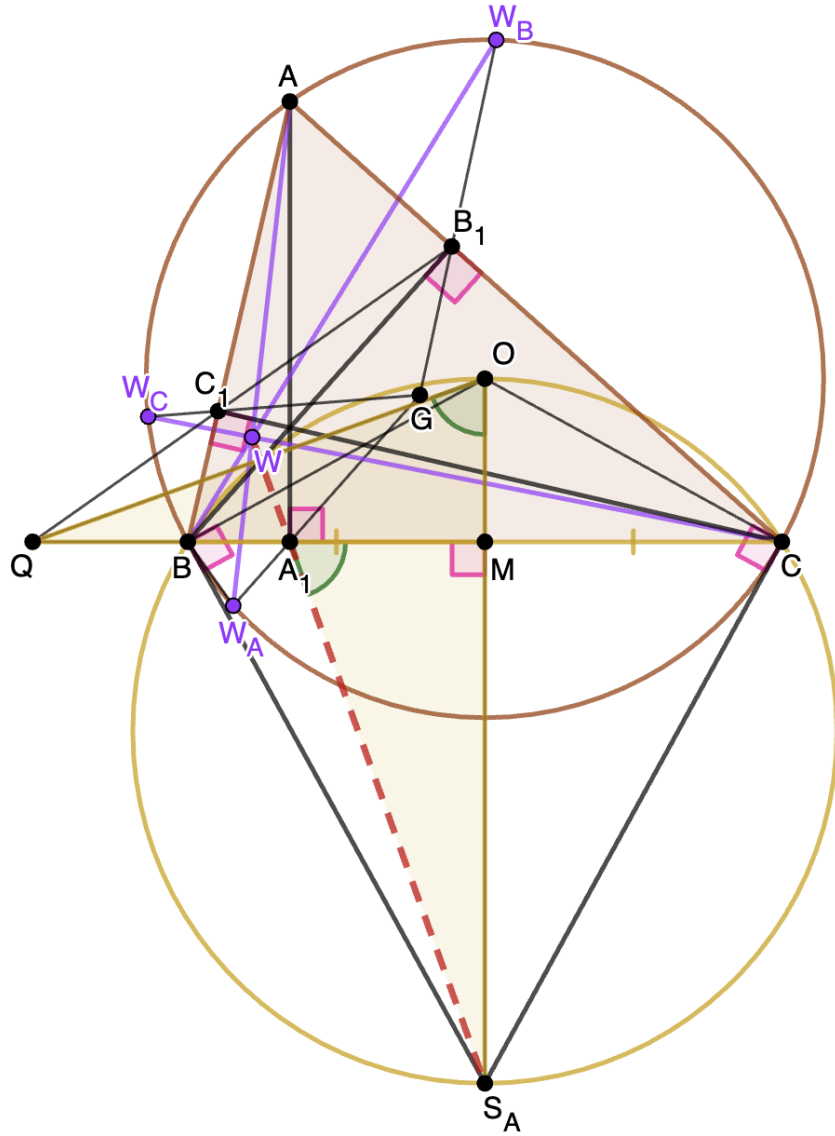


Figure 19: The pester X(25)

q.e.d.

7 Relation to a mixtilinear incircle

Let us have a look at an unexpected relation between the A whypoint and the A -mixtilinear incircle. We will use some points notations from the previous section.

7.1 Mixti-sidepoint and why-point

Problem. Prove that points X , X_S , A_1 , W_A are concyclic.

Proof. Since $\angle IX_SA = 90^\circ$, we have that X_S , I , D are collinear. Since $\angle DA'A = 90^\circ$, we have $A'D \parallel AA_1$, so if we let AA_1 and $X_S I$ meet at Y , then

$$\angle X_S W_A A_1 = \angle X_S D A' = \angle X_S Y A,$$

from which $X_S Y A_1 W_A$ is cyclic. Obviously, since

$$\angle XX_S Y = 90^\circ = \angle XA_1 Y,$$

we obtain that $XX_S Y A_1$ is also cyclic, and we may conclude.

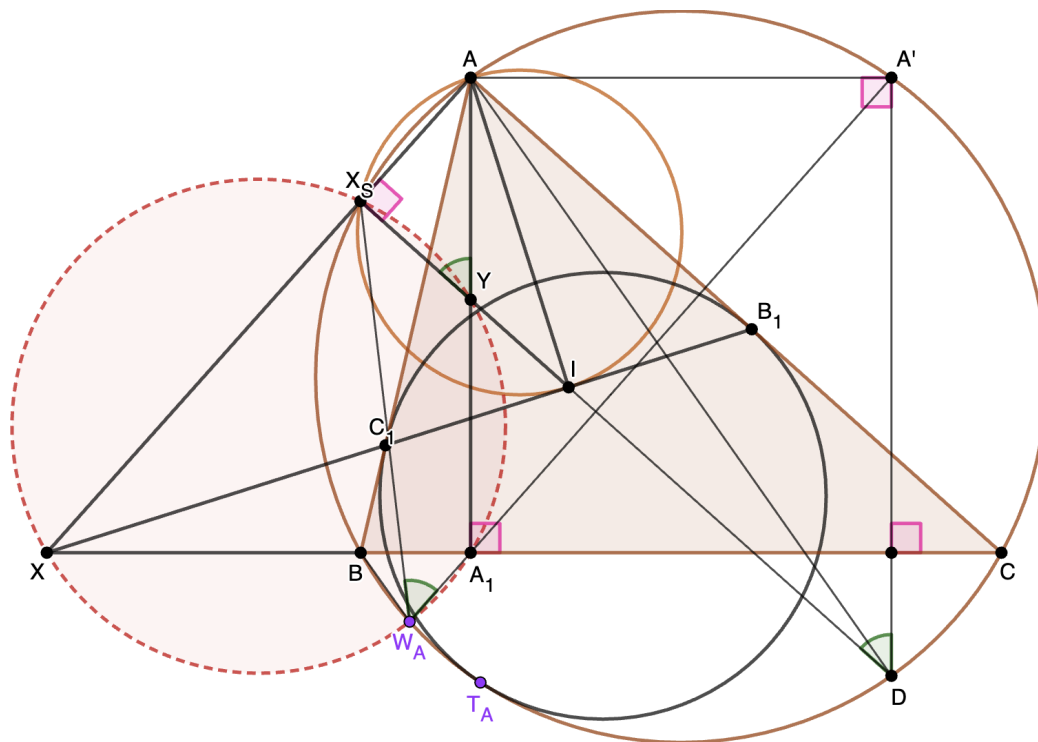


Figure 20: Mixti-sharky-why

q.e.d.

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