

THE $(2k + 1)$ –SECTOR PROBLEM IN RIGHT-ANGLED TRIANGLES

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Abstract

This paper presents a method for analyzing the division of angles and segment ratios in right-angled triangles. Specifically, we consider a right-angled triangle ABC with $\angle A = 90^\circ$, where the angle $\angle ACB$ is divided into $n + 1$ equal angles by points $D_1, D_2, \dots, D_n$ on the segment AB , with n being an even positive integer. By using the Angle Bisector Theorem and reflective symmetry, the main result establishes a direct relationship between the segment ratios along AB , without knowing the lengths of $\overline{AC}$, $\overline{BC}$, or the measure of $\angle ACB$. This approach significantly simplifies the computations compared to the traditional methods, which often rely on trigonometric identities and Pythagorean Theorem.

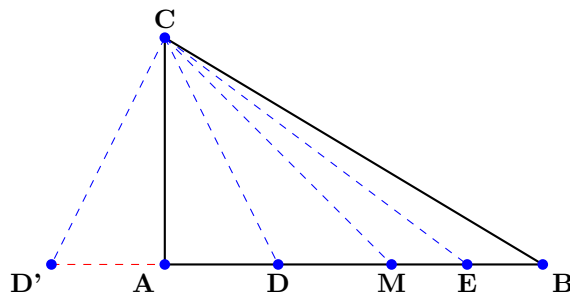
The Angle Bisector Theorem is a fundamental result in Euclidean geometry, which states that in any triangle, the bisector of an interior angle divides the opposite side into two segments whose lengths are proportional to the lengths of the two other sides of the triangle. In other words, if a triangle $\triangle ABC$ has the angle bisector of $\angle C$ intersecting AB at D then $\frac{AD}{DB} = \frac{CA}{CB}$. Furthermore, the Angle Bisector Theorem can be extended to the general case where CD is not necessarily the angle bisector of $\angle C$, in which the segment ratio is given by $\frac{AD}{DB} = \frac{CA \cdot \sin \angle DCB}{CB \cdot \sin \angle DCA}$. In this article, we analyze a different extension of the classical Angle Bisector Theorem, where we look at the case of multiple angle divisions. This leads to the $(n + 1)$ -sector problem, where the angle $\angle C$ is divided into $n + 1$ equal angles by lines intersecting the opposite side AB at points $D_1, D_2, \dots, D_n$. The segment ratios are defined by the following equalities $\gamma_1 AD_1 = \gamma_2 D_1 D_2 = \gamma_3 D_2 D_3 = \dots = \gamma_n D_{n-1} D_n = \gamma_{n+1} D_n B$. Moreover, unless mentioned, we assume $\triangle ABC$ is right-angled in $\angle A$ and we aim to find a recurrence relation between γ_i 's without using the usual computational approach, which would require the lengths of $\overline{AC}$, $\overline{BC}$, or the measure of $\angle ACB$.

Before delving into the main result, we first present the motivation for this article, a problem proposed in the Romanian District Olympiad, involving the trisector of an angle. Readers are encouraged to attempt solving the problem beforehand, as the proof relies on the construction of a key point, which, once identified, makes the solution immediate.

Problem 1. (Problem 2, 9th Grade) Consider a right-angled triangle ABC , $\angle A = 90^\circ$ and points D and E on the segment AB such that $\angle ACD \equiv \angle DCE \equiv \angle ECB$. Prove that if $3\overrightarrow{AD} = 2\overrightarrow{DE}$ and $\overrightarrow{CD} + \overrightarrow{CE} = 2\overrightarrow{CM}$ then $\overrightarrow{AB} = 4\overrightarrow{AM}$.

Proof. First, note that M is the midpoint of the segment DE . Let D' be the reflection of D with respect to A . Since $AD' = AD$ and $\angle A = 90^\circ$ then $CD' = CD$. By applying the Angle Bisector Theorem in $\triangle CDB$ we obtain

$$\frac{CD}{CB} = \frac{DE}{EB}. \quad (1)$$



Moreover, since $\triangle CD'D$ is isosceles and $CA \perp D'D$ we get $\angle D'CA = \angle ACD$. Therefore, it follows that $\angle D'CD = 2\angle ACD = 2\angle DCE = \angle DCB$, in particular CD is the angle bisector of $\angle D'CB$. Thus, by applying the Angle Bisector Theorem in $\triangle CD'B$ we obtain

$$\frac{CB}{CD'} = \frac{DB}{DD'}. \quad (2)$$

By multiplying the ratios from (1) and (2) we have

$$\frac{CD}{CB} \cdot \frac{CB}{CD'} = \frac{DE}{EB} \cdot \frac{DB}{DD'}.$$

By using $CD = CD'$ it holds $\frac{DE}{EB} \cdot \frac{DB}{DD'} = 1$, in particular $\frac{DB}{EB} = \frac{2AD}{DE} = \frac{4}{3}$, since $D'D = 2AD$ and $3AD = 2DE$. Therefore, since $\frac{DB}{EB} = \frac{4}{3}$, we find $\frac{DE}{EB} = \frac{1}{3}$, which implies $EB = 6DM$. Finally, by using $3AD = 4DM$, we conclude

$$AM = AD + DM = \frac{7}{4}DM \text{ and } AB = AD + DE + EB = \frac{28}{3}DM = 7AM,$$

in particular $\overrightarrow{AB} = 4\overrightarrow{AM}$, since $M \in (AB)$, hence the result. $\square$

Remark 1. In the official solution, the segments $AD = 2, DE = 3$ are chosen for convenience. This choice ensures that one can explicitly compute AB and AC by using the Pythagorean Theorem in $\triangle ACE, \triangle ABC$, respectively. However, note that the presented proof does not rely on the lengths of AB or AC and instead computes the ratio $\frac{DE}{EB}$ given $\frac{AD}{DE}$. Thus, given the ratio $\gamma_1 := \frac{AD}{DE}$ we can precisely determine E so that $\angle C$ is divided into three equal parts by finding $\gamma_2 := \frac{DE}{EB}$, which depends only in terms of γ_1 .

Now that we have proved Problem 1, we delve into the main result of this article, where we use the same approach of reflective symmetry and the Angle Bisector Theorem.

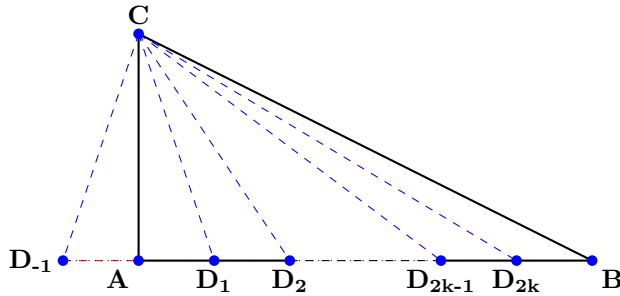
Problem 2. Let ABC be a right-angled triangle with $\angle A = 90^\circ$, and let $\angle ACB$ be divided into $n + 1$ equal angles by points $D_1, D_2, \dots, D_n$ on segment AB , such that

$$\angle ACD_1 = \angle D_1CD_2 = \dots = \angle D_{n-1}CD_n = \angle D_nCB,$$

where $n = 2k$. Moreover, suppose that the segment ratios are given by the following equalities $\gamma_1 AD_1 = \gamma_2 D_1 D_2 = \gamma_3 D_2 D_3 = \dots = \gamma_n D_{n-1} D_n = \gamma_{n+1} D_n B$. Prove that

$$\gamma_{n+1} = \gamma_1 \cdot \frac{2 + \sum_{i=2}^k \frac{\gamma_1}{\gamma_i} - \left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right]}{\left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right] \left[\sum_{i=k+1}^{2k} \frac{\gamma_1}{\gamma_i} \right]}. \quad (3)$$

Proof. Similarly to Problem 1, let D_{-1} be the reflection of D_1 with respect to A . For clarity and consistency, we use the following notation $D_0 = A, D_{2k+1} = B$.



By applying the Angle Bisector Theorem in $\triangle CD_{2i-1}D_{2i+1}$ for $0 \leq i \leq k$ we have

$$\frac{D_{2i-1}D_{2i}}{D_{2i}D_{2i+1}} = \frac{CD_{2i-1}}{CD_{2i+1}}. \quad (4)$$

By multiplying correspondingly the ratios from (4) it follows

$$\prod_{i=0}^k \frac{D_{2i-1}D_{2i}}{D_{2i}D_{2i+1}} = \prod_{i=0}^k \frac{CD_{2i-1}}{CD_{2i+1}} = \frac{CD_{-1}}{CB}. \quad (5)$$

Since $\frac{D_{-1}D_0}{D_0D_1} = 1$ and $\frac{D_{2i-1}D_{2i}}{D_{2i}D_{2i+1}} = \frac{\gamma_{2i+1}}{\gamma_{2i}}$ for all $1 \leq i \leq k$ we obtain

$$\prod_{i=0}^k \frac{D_{2i-1}D_{2i}}{D_{2i}D_{2i+1}} = \prod_{i=1}^k \frac{\gamma_{2i+1}}{\gamma_{2i}} \quad (6)$$

Furthermore, since $\triangle CD_{-1}D_1$ is isosceles we have $\angle D_{-1}CD_0 = \angle D_0CD_1$, in particular $\angle D_{-1}CB$ is divided into $n + 2$ equal angles by the points $D_0, D_1, \dots, D_n$. Thus, by observing that $\angle D_{-1}CB = (2k + 2)\angle D_0CD_1$ and

$\angle D_{-1}CD_k = (k+1)\angle D_0CD_1$ we conclude $(CD_k$ is the angle bisector of $\angle D_{-1}CB$. Therefore, by applying the Angle Bisector Theorem in $\triangle CD_{-1}B$ it follows

$$\frac{CD_{-1}}{CB} = \frac{D_{-1}D_k}{D_kB}. \quad (7)$$

Hence, by combining (5), (6), (7) we get

$$\frac{D_{-1}D_k}{D_kB} = \prod_{i=1}^k \frac{\gamma_{2i+1}}{\gamma_{2i}} = \frac{\gamma_{2k+1}}{\gamma_1} \left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right] \quad (8)$$

Also, note that the following equalities hold

$$\frac{D_{-1}D_k}{AD_1} = \frac{D_{-1}D_1}{AD_1} + \sum_{i=2}^k \frac{D_{i-1}D_i}{AD_1} = 2 + \sum_{i=2}^k \frac{\gamma_1}{\gamma_i}, \quad (9)$$

$$\frac{D_kB}{AD_1} = \sum_{i=k+1}^{2k+1} \frac{D_{i-1}D_i}{AD_1} = \sum_{i=k+1}^{2k+1} \frac{\gamma_1}{\gamma_i} = \frac{\gamma_1}{\gamma_{2k+1}} + \sum_{i=k+1}^{2k} \frac{\gamma_1}{\gamma_i}. \quad (10)$$

Thus, by the previous equalities, it follows

$$\frac{D_{-1}D_k}{D_kB} = \frac{2 + \sum_{i=2}^k \frac{\gamma_1}{\gamma_i}}{\frac{\gamma_1}{\gamma_{2k+1}} + \sum_{i=k+1}^{2k} \frac{\gamma_1}{\gamma_i}}. \quad (11)$$

Therefore, by (7), (8), (11) we conclude

$$\frac{\gamma_{2k+1}}{\gamma_1} \left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right] = \frac{2 + \sum_{i=2}^k \frac{\gamma_1}{\gamma_i}}{\frac{\gamma_1}{\gamma_{2k+1}} + \sum_{i=k+1}^{2k} \frac{\gamma_1}{\gamma_i}}. \quad (12)$$

By rewriting (12) we obtain

$$\left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right] + \frac{\gamma_{2k+1}}{\gamma_1} \left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right] \cdot \left[\sum_{i=k+1}^{2k} \frac{\gamma_1}{\gamma_i} \right] = 2 + \sum_{i=2}^k \frac{\gamma_1}{\gamma_i},$$

or equivalently

$$\gamma_{2k+1} = \gamma_1 \cdot \frac{2 + \sum_{i=2}^k \frac{\gamma_1}{\gamma_i} - \left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right]}{\left[\prod_{i=1}^k \frac{\gamma_{2i-1}}{\gamma_{2i}} \right] \left[\sum_{i=k+1}^{2k} \frac{\gamma_1}{\gamma_i} \right]},$$

hence the result. $\square$

Remark 2. By substituting $n = 2$, $\gamma_1 = 3$, $\gamma_2 = 2$ into (3), we have $\gamma_3 = \gamma_1 \left(2 - \frac{\gamma_1}{\gamma_2} \right) / \left(\frac{\gamma_1}{\gamma_2} \right)^2 = \frac{2}{3}$. This result is consistent with Problem 1, where we established that $EB = 6DM$ and $3AD = 4DM$, leading to $3AD = \frac{2}{3}EB$.

Remark 3. Note that we only treat the case where n is even. A similar relation as in (3) can be found when n is odd. However, in this case, there is no need to construct D_{-1} or to assume $\triangle ABC$ is right-angled. Indeed, if $n = 2k - 1$, where k is a positive integer and denote $D_0 := A$, $D_{2k} := B$ by applying the Angle Bisector Theorem in $\triangle ABC$ and $\triangle CD_{2i-2}D_{2i}$ for $1 \leq i \leq k$ we obtain a similar recurrence as in (3), in particular γ_{2k} can be written as a function of γ_j 's for $j < 2k$.

Remark 4. (Construction of $D_1, D_2, \dots, D_n$) If we assume that $\gamma_1, \gamma_2, \dots, \gamma_n$ are known, then γ_{n+1} can be explicitly computed using the expression derived above. This immediate observation allows us to view Problem 2 from an inductive perspective, regardless of whether n is even or odd.

To illustrate this, fix $\triangle CAD_2$ and assume γ_1 and γ_2 are known. Then D_3 is the point that divides $\angle ACD_3$ into three equal angles. By applying Problem 2 for $n = 2$, we can precisely determine D_3 without knowing CD_3 , since γ_3 depends only on γ_1 and γ_2 . Moreover, by Remark 3, γ_4 can be determined from $\gamma_1, \gamma_2, \gamma_3$. Since γ_3 itself can be expressed in terms of γ_1 and γ_2 , it follows that γ_4 depends only on γ_1 and γ_2 . Thus, by continuing this reasoning inductively, we conclude that γ_{n+1} can be derived as a function of γ_1 and γ_2 . Hence, this inductive approach provides a systematic way to construct the points $D_1, D_2, \dots, D_n$ and compute the corresponding segment ratios by knowing only the initial ratio $\frac{AD_1}{D_1D_2}$.

Finally, an immediate question that arises is whether we can apply the same approach when $\triangle ABC$ is not necessarily right-angled. Unfortunately, the reflective symmetry construction used in the right-angled case does not directly extend to arbitrary triangles. Indeed, when we consider D_{-1} , we rely on the fact that $\triangle CD_{-1}A$ is the mirror image of $\triangle CAD_1$ with respect to CA . This reflective symmetry only works in the right-angled case, where CA and BA are perpendicular. In a general triangle, this symmetry is lost, and the construction of D_{-1} does not simplify the problem the same way as in the right-angled case. Thus, the $(2k+1)$ -sector problem for general triangles would be an interesting topic for future research since it is not clear whether we can express γ_{n+1} as a function of γ_i 's without knowing the lengths of AC, BC or the measure of the angle $\angle ACB$.

Overall, this article shows the elegance of reflective symmetry in solving the $(2k+1)$ -sector problem for right-angled triangles. We established a direct, recursive relationship between the segment ratios $\gamma_1, \gamma_2, \dots, \gamma_{n+1}$ which cannot be achieved using the computational method from Remark 1 for the general case.