

Junior Problems

J727. Find all real numbers x and y such that

$$(y + 2x)^3 + (x + 2y)^3 - 9(x^3 + y^3) = 9(x^2 + xy + y) = 18.$$

Proposed by Alessandro Ventullo, Milan, Italy

J728. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1 - a + a^2} + \frac{1}{1 - b + b^2} + \frac{1}{1 - c + c^2} \leq 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J729. Solve in integers the equation

$$x^4 - \frac{7}{2}xy + y^4 = 2026.$$

Proposed by Adrian Andreescu, Dallas, USA

J730. Let x, y, z be nonnegative real numbers such that $4x + 3y + z = 1$. Find the minimum and maximum of

$$x + \sqrt{x^2 + y^2 + z^2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J731. Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = a + b + c$. Prove that

$$ab + bc + ca \leq abc + \frac{2}{3}(a + b + c).$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

J732. Let ABC be an acute triangle, with $AB \neq AC$, M be the midpoint of BC and D, E, F the feet of altitudes from A, B , and C , respectively. A line parallel to EF passes through D and cuts the line AC in X and the line AB in Y . The circumcircle of $\triangle MXY$ cuts the line BC in $Z \neq M$. Show that points E, F, Z are collinear.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Senior Problems

S727. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\sqrt[5]{\frac{a^5 + b^5}{2}} + \sqrt[5]{\frac{b^5 + c^5}{2}} + \sqrt[5]{\frac{c^5 + a^5}{2}} \leq \sqrt[5]{\frac{a^4}{b^3} + \frac{b^4}{c^3} + \frac{c^4}{a^3}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S728. Let p, q, r be primes such that $p \equiv q \equiv 1 \pmod{r}$ and $p^3 - q^3 > 3pqr + r^3$. Prove that

$$\frac{p^3 - q^3}{2} \geq 3pqr + 4r^3.$$

Proposed by Alessandro Ventullo, Milan, Italy

S729. Let ABC be an acute triangle with circumcircle $\Gamma(O, R)$. The perpendicular bisector of side BC meets Γ at X , on the minor arc BC , and let Y be the intersection point of AX and the segment BC . Let D and E be the feet of the perpendiculars from Y and C to line AB , respectively, and let Z be the foot of the perpendicular from C to line AX . Let T be the intersection point of lines DZ and BC , K be the intersection point of lines AT and CE and S be the circumcenter of $\triangle BKC$. Prove that

$$\overrightarrow{AS} = \overrightarrow{AK} + \overrightarrow{AO}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

S730. Let a, b, c be nonnegative real numbers. Prove that

$$3\sqrt{(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)} + 9abc \geq 2(a + b + c)(ab + bc + ca).$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

S731. Let a, b, c, d, e, f be real numbers such that $a + b + c + d + e + f = 0$. Prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \geq 2(ab + bc + cd + de + ef + fa).$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

S732. Find all positive integers n with the following property: there is a positive integer k and 3-variable polynomials $f_1, \dots, f_k$ with real coefficients such that for all real numbers a, b, c

$$a(a + b)^n + b(b + c)^n + c(c + a)^n = f_1(a, b, c)^2 + \dots + f_k(a, b, c)^2.$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

Undergraduate Problems

U727. Let $\{a_n\}_{n \geq 1}$ be a sequence defined by $a_1 = 2$ and $a_n = a_{n-1}^2 - a_{n-1} + 1$ for $n \geq 2$. Evaluate

$$\lim_{n \rightarrow \infty} \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U728. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x)f(-x) = 4$ for all $x \in \mathbb{R}$. Evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(1 + 3 \sin^2 x)(2 + f(x))} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

U729. Determine all natural numbers $n \geq 2$ for which the following statement holds:

For every $A, B \in M_n(\mathbb{C})$ such that $A^n = O_n$ and $\det(A + B) = \det B = 0$, it follows that $BAB = O_n$.

Proposed by Florin Stănescu, Găești, Romania

U730. Prove that 2 is the least value of the exponent k such that

$$\frac{a^4 + b^4 + c^4}{3} \leq \left(\frac{a^3 + b^3 + c^3}{3} \right)^k$$

for any nonnegative real numbers a, b, c with at most one of them less than 1 and $a + b + c = 3$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

U731. Prove

$$\int_0^1 \frac{\arcsin x}{4 - x^2} dx = \frac{\pi}{12} \log(9 + 6\sqrt{3}) - \frac{2}{3}G.$$

Here $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant.

Proposed by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

U732. (a) Find all finite groups G having exactly 5 distinct subgroups $H_1, H_2, H_3, H_4, H_5 \leq G$ of the same order $m \geq 2$ such that $H_i \cap H_j = \{1\}$ for all $i \neq j$ and $G = \bigcup_{i=1}^5 H_i$.

(b) Find all finite groups G having exactly 6 distinct subgroups $H_1, H_2, H_3, H_4, H_5, H_6 \leq G$ of the same order $m \geq 2$ such that $H_i \cap H_j = \{1\}$ for all $i \neq j$ and $G = \bigcup_{i=1}^6 H_i$.

Proposed by Alessandro Ventullo, Milan, Italy

Olympiad Problems

O727. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{b+c} + \frac{b}{c+a}\right) \left(\frac{b}{c+a} + \frac{c}{a+b}\right) \left(\frac{c}{a+b} + \frac{a}{b+c}\right) + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

O728. Find all positive integers n such that $1 + 2 + 2^2 + \cdots + 2^n$ divides $(1 + 2)(1 + 2^2) \cdots (1 + 2^n) - 1$.

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

O729. Find all prime numbers p such that $\frac{2^{p-1} - 1}{p}$ is a perfect power.

Proposed by Alessandro Ventullo, Milan, Italy

O730. Let $f(x) = \frac{x^3}{3} + x$ and let a, b, c be distinct real numbers such that $abc = 1$ and

$$f\left(a - \frac{1}{b}\right) + f\left(b - \frac{1}{c}\right) + f\left(c - \frac{1}{a}\right) = 0.$$

Prove that

$$f\left(a - \frac{1}{a}\right) + f\left(b - \frac{1}{b}\right) + f\left(c - \frac{1}{c}\right) = \left(a^2 - \frac{1}{a^2}\right) + \left(b^2 - \frac{1}{b^2}\right) + \left(c^2 - \frac{1}{c^2}\right).$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

O731. Let $P(x)$ be a polynomial of degree d .

- (a) Let c be an integer such that for all integers $c \leq x \leq c + d$, $P(x)$ is an integer. Prove that for all integers n , $P(n)$ is an integer.
- (b) Prove that for each positive integer d , there is a polynomial $P(x)$ of degree d for which there is an integer c such that for all integers $c \leq x \leq c + d - 1$, $P(x)$ is an integer, but for any other integer n , $P(n)$ is irrational.

Proposed by Mohammadreza Ghelichkhani, Shahed University, Tehran, Iran

O732. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^2bc \geq 1$, and let

$$F(a, b, c) = \frac{1}{\sqrt[3]{abc}} - \sqrt{\frac{3}{ab + bc + ca}}.$$

Prove that

$$F(a, b, c) \leq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

Proposed by Vasile Cîrtoaje, Romania and Vasile Mircea Popa, Romania