

Junior problems

J727. Find all real numbers x and y such that

$$(y + 2x)^3 + (x + 2y)^3 - 9(x^3 + y^3) = 9(x^2 + xy + y) = 18.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by Theo Koupelis, Clark College, Washington, USA

Using the identity

$$(y + 2x)^3 + (x + 2y)^3 - 9(x^3 + y^3) = 18xy(x + y),$$

and setting $x(x + y) = z$, the given system of equations is now equivalent to

$$\begin{aligned} zy &= 1, \\ z + y &= 2. \end{aligned}$$

Thus, z, y are solutions of the quadratic $t^2 - 2t + 1 = 0$, and therefore $y = z = 1$. From $x(x + 1) = 1$ we get $x = (-1 \pm \sqrt{5})/2$. Thus, the solutions of the given system of equations are

$$(x, y) = \left(\frac{-1 \pm \sqrt{5}}{2}, 1 \right).$$

Solution 2 by Daniel Pascuas, Barcelona, Spain

Since $(y + 2x)^3 + (x + 2y)^3 - 9(x^3 + y^3) = 18xy(x + y)$, we have to find all real numbers x and y that satisfy the following two equations:

$$xy(x + y) = 1 \tag{1}$$

$$x(x + y) + y = 2. \tag{2}$$

Multiplying (2) by y and taking into account (1) we get that $0 = y^2 - 2y + 1 = (y - 1)^2$, so $y = 1$. Therefore $x(x + 1) = 1$, that is, $x^2 + x - 1 = 0$, and so $x = \frac{-1 \pm \sqrt{5}}{2}$. Hence the solutions to equations (1) and (2) are

$$(x, y) = \left(\frac{-1 - \sqrt{5}}{2}, 1 \right) \quad \text{and} \quad (x, y) = \left(\frac{-1 + \sqrt{5}}{2}, 1 \right).$$

Solution 3 by Ervin Ramirez, Richland Center, Wisconsin, USA

Expanding the first equation gives $18x^2y + 18xy^2 = 18$, hence

$$xy(x + y) = 1.$$

In particular, $x, y \neq 0$, so

$$x + y = \frac{1}{xy}.$$

From $x^2 + xy + y = 2$, we obtain

$$x(x + y) + y = 2.$$

Substituting $x(x + y) = \frac{1}{y}$ yields $\frac{1}{y} + y = 2$, thus $(y - 1)^2 = 0$, and therefore $y = 1$.

Returning to $xy(x + y) = 1$, we get $x(x + 1) = 1$, that is, $x^2 + x - 1 = 0$. Hence $x = \frac{-1 \pm \sqrt{5}}{2}$.

Therefore,

$$(x, y) = \left(\frac{-1 + \sqrt{5}}{2}, 1 \right), \quad \left(\frac{-1 - \sqrt{5}}{2}, 1 \right).$$

Also solved by Aarnav Siva, AMMOC, USA; Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Debarun Das, CMC, Hyderabad, India; G. C. Greubel, Newport News, VA, USA; José Luis Díaz-Barrero, Barcelona, Spain; Kalkin Gandhi, AMMOC Bengaluru, India; Kousik Sett, India; Luke Paluso, Blinn College, Brenham, TX, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Sahel Koley, Tarakeswar, West Bengal, India; Sundares H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Aaditya Bhaumik Sahu, AMMOC, India; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Diar Gashi, University of Primorska, Slovenia; Ganghun Kim, Fort Lee High School, NJ, USA; Kristine Lee, Mount Juliet High School, TN, USA; Loukya Ghanta, AMMOC; Parth Andhare, AMMOC; Polyahedra, Polk State College, FL, USA; Shaurya Patil, AMMOC; Shilong Zhuang, California, USA; Vibhu Konduru, AMMOC; Vidur Konduru, AMMOC; William Han, Horace Mann School, Bronx, New York, USA; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

J728. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1-a+a^2} + \frac{1}{1-b+b^2} + \frac{1}{1-c+c^2} \leq 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

Since $abc = 1$, there exist positive real numbers x, y, z such that

$$a = \frac{y}{x}, b = \frac{z}{y}, c = \frac{x}{z}.$$

Then the inequality becomes

$$\frac{x^2}{x^2 - xy + y^2} + \frac{y^2}{y^2 - yz + z^2} + \frac{z^2}{z^2 - zx + x^2} \leq 3.$$

It's easy to see that

$$x^2 - xy + y^2 \geq \sqrt{\frac{x^4 + y^4}{2}}.$$

Hence

$$\frac{x^2}{x^2 - xy + y^2} \leq \sqrt{\frac{2x^4}{x^4 + y^4}}.$$

Establishing two similar inequalities and adding them up we get

$$\sum_{\text{cyc}} \frac{x^2}{x^2 - xy + y^2} \leq \sqrt{\frac{2x^4}{x^4 + y^4}} + \sqrt{\frac{2y^4}{y^4 + z^4}} + \sqrt{\frac{2z^4}{z^4 + x^4}}.$$

Now we use Cauchy-Schwarz inequality as follows

$$\begin{aligned} \sum_{\text{cyc}} \sqrt{\frac{2x^4}{x^4 + y^4}} &= \sum_{\text{cyc}} \sqrt{(x^4 + z^4) \frac{2x^4}{(x^4 + y^4)(x^4 + z^4)}} \\ &\leq \sqrt{4(x^4 + y^4 + z^4) \sum_{\text{cyc}} \frac{x^4}{(x^4 + y^4)(x^4 + z^4)}} \\ &= \sqrt{\frac{8(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4)}{(x^4 + y^4)(y^4 + z^4)(z^4 + x^4)}} \\ &\leq 3, \end{aligned}$$

where the last inequality is equivalent to

$$\begin{aligned} &9(x^4 + y^4)(y^4 + z^4)(z^4 + x^4) \geq 8(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4) \\ \iff &9((x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4) - x^4y^4z^4) \geq 8(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4) \\ \iff &(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4) \geq 9x^4y^4z^4, \end{aligned}$$

which follows from the AM-GM Inequality.

Solution 2 by Polyhedra, Polk State College, FL, USA

If $x, y, z > 0$ and $xyz = 1$, then

$$(1 + x + x^2)(1 + y + y^2)(1 + z + z^2) - \sum_{\text{cyc}} (1 + x + x^2)(1 + y + y^2) = xy + yz + zx - x^2 - y^2 - z^2 \leq 0,$$

and thus $\sum_{cyc} \frac{1}{1+x+x^2} \geq 1$ (which is also known and can be found in T. Andreescu & A. Ganesh, *109 Inequalities from AMSP*, XYZ Press, 2015, p. 9). Since $x = 1/a^2$, $y = 1/b^2$ and $z = 1/c^2$ also satisfy $xyz = 1$, we have

$$1 \leq \sum_{cyc} \frac{1}{1+x+x^2} = \sum_{cyc} \frac{a^4}{1+a^2+a^4}.$$

Using $(1-a+a^2)(1+a+a^2) = 1+a^2+a^4$ we have

$$\begin{aligned} 3 &\geq 6 - \sum_{cyc} \frac{1}{1+a+a^2} - 2 \sum_{cyc} \frac{a^4}{1+a^2+a^4} = \sum_{cyc} \left(2 - \frac{1-a+a^2+2a^4}{1+a^2+a^4} \right) \\ &= \sum_{cyc} \frac{1+a+a^2}{1+a^2+a^4} = \sum_{cyc} \frac{1}{1-a+a^2}. \end{aligned}$$

Solution 3 by Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania

We have

$$\sum_{cyc} \frac{1}{1-a+a^2} \leq 3 \iff \sum_{cyc} \left(\frac{4}{3} - \frac{1}{1-a+a^2} \right) \geq 1.$$

But

$$\frac{4}{3} - \frac{1}{1-a+a^2} = \frac{4(1-a+a^2)-3}{3(1-a+a^2)} = \frac{(2a-1)^2}{3(1-a+a^2)}.$$

Thus it is enough to prove

$$\sum_{cyc} \frac{(2a-1)^2}{1-a+a^2} \geq 3.$$

By Titu's lemma, we get

$$\sum_{cyc} \frac{(2a-1)^2}{1-a+a^2} \geq \frac{\left(\sum_{cyc} (2a-1) \right)^2}{\sum_{cyc} (1-a+a^2)}.$$

Let $p = a+b+c$, $q = ab+bc+ca$, $abc = 1$. Then

$$\sum_{cyc} (2a-1) = 2p-3$$

and

$$\sum_{cyc} (1-a+a^2) = 3-p + \sum_{cyc} a^2 = 3-p+p^2-2q.$$

Therefore,

$$\sum_{cyc} \frac{(2a-1)^2}{1-a+a^2} \geq \frac{(2p-3)^2}{p^2-2q-p+3}.$$

It remains to prove that

$$\frac{(2p-3)^2}{p^2-2q-p+3} \geq 3.$$

This is equivalent to

$$(2p-3)^2 \geq 3(p^2-2q-p+3) \iff p^2-9p+6q \geq 0.$$

Now, since

$$q^2 = (ab+bc+ca)^2 \geq 3abc(a+b+c) = 3p,$$

we have, by AM-GM,

$$p^2+6q = p^2+3q+3q \geq 3\sqrt[3]{9p^2q^2} \geq 3\sqrt[3]{27p^3} = 9p.$$

Hence $p^2-9p+6q \geq 0$. Equality holds when $a=b=c=1$.

Solution 4 by Theo Koupelis, Clark College, Washington, USA

Substituting $c = 1/(ab)$, and setting $p = a + b$ and $r = ab$, the desired inequality becomes equivalent to

$$\frac{p^2 - 2r - p + 2}{p^2 + r^2 - pr - p - r + 1} + \frac{r^2}{r^2 - r + 1} \leq 3.$$

Clearing denominators we get the following quadratic in p :

$$p^2(r^2 - 2r + 2) - (2r^3 - 2r^2 + r + 2)p + 2r^4 - 3r^3 + 4r^2 - 2r + 1 \geq 0.$$

The discriminant of the above quadratic is $\Delta = -4r^6 + 20r^5 - 48r^4 + 68r^3 - 59r^2 + 28r - 4$. We have $r^2 - 2r + 2 = (r - 1)^2 + 1 > 0$, and using AM-GM we get $p \geq \sqrt{4r}$. Thus, it suffices to show that

$$\sqrt{4r} \geq \frac{2r^3 - 2r^2 + r + 2 + \sqrt{\Delta}}{2(r^2 - 2r + 2)}.$$

Squaring and simplifying we get

$$16(r - 1)^2(r^2 - 2r + 2)^2(r^2 + r + 1)[(2r^2 - 2r + 1)^2 + r(r + 1)] \geq 0,$$

which is obvious for $r > 0$. Equality occurs when $(p, r) = (2, 1)$, and thus when $a = b = c = 1$.

Solution 5 by Ervin Ramirez, Richland Center, Wisconsin, USA

By the uvw principle, it suffices to consider the case

$$b = c = t, \quad a = \frac{1}{t^2}.$$

Then

$$\sum \frac{1}{1 - a + a^2} = \frac{t^4}{t^4 - t^2 + 1} + \frac{2}{t^2 - t + 1}.$$

Let

$$S = \frac{t^4}{t^4 - t^2 + 1} + \frac{2}{t^2 - t + 1}.$$

We compute $S - 3$:

$$S - 3 = \frac{t^4(t^2 - t + 1) + 2(t^4 - t^2 + 1) - 3(t^4 - t^2 + 1)(t^2 - t + 1)}{(t^4 - t^2 + 1)(t^2 - t + 1)}.$$

Expanding the numerator and factoring, we get

$$-2t^6 + 2t^5 + 3t^4 - 3t^3 - 2t^2 + 3t - 1 = -(t - 1)^2(2t^4 + 2t^3 - t^2 - t + 1).$$

Hence

$$S - 3 = -\frac{(t - 1)^2(2t^4 + 2t^3 - t^2 - t + 1)}{(t^4 - t^2 + 1)(t^2 - t + 1)}.$$

Since $(t^4 - t^2 + 1)(t^2 - t + 1) > 0$ and $2t^4 + 2t^3 - t^2 - t + 1 = 2(t^2 - 1/4)^2 + 2t(t - 1/2)^2 + (16t^2 - 12t + 7)/8 > 0$ for all $t > 0$, it follows that $S - 3 \leq 0$.

Therefore,

$$\frac{1}{1 - a + a^2} + \frac{1}{1 - b + b^2} + \frac{1}{1 - c + c^2} \leq 3.$$

Equality holds when $t = 1$, that is, $a = b = c = 1$.

Editor's Comment. Daniel Pascuas observed that this problem appeared as Problem 1.46 (with two different solutions) in Vasile Cîrtoaje's book *Mathematical Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities*, published by LAP LAMBERT Academic Publishing in 2021.

Also solved by Adela Llapashtica, *Specialized Mathematical Gymnasium, Prishtine, Kosovo*; Arkady Alt, *San Jose, California, USA*; Cao Minh Quang, *Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam*; José Luis Díaz-Barrero, *Barcelona, Spain*; Prodromos Fotiadis, *University of Crete, Greece*; Saarang Agarwal, *AMMOC*; Sahel Koley, *Tarakeswar, West Bengal, India*; The Ring Lords, *Baden-Württemberg, Germany*; Aaditya Bhaumik Sahu, *AMMOC, India*; Daniel Pascuas, *Barcelona, Spain*; Parth Andhare, *AMMOC*; Shilong Zhuang, *California, USA*.

J729. Solve in integers the equation

$$x^4 - \frac{7}{2}xy + y^4 = 2026.$$

Proposed by Adrian Andreescu, Dallas, USA

Solution 1 by Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA

Let $p = |xy|$. By the AM-GM Inequality, $x^4 + y^4 \geq 2x^2y^2 = 2p^2$. Hence

$$2026 = x^4 - \frac{7}{2}xy + y^4 \geq 2p^2 - \frac{7}{2}p.$$

It follows that $p \leq 32$, and therefore

$$\left| \frac{7}{2}xy \right| \leq \frac{7}{2} \cdot 32 = 112.$$

Thus,

$$1914 = 2026 - 112 \leq x^4 + y^4 = 2026 + \frac{7}{2}xy \leq 2026 + 112 = 2138.$$

Therefore, $x^4 + y^4 \in [1914, 2138]$. Since $7^4 = 2401 > 2138$, we must have $\max(|x|, |y|) \leq 6$. The possible fourth powers are 0, 1, 16, 81, 256, 625, 1296. The only way for the sum of two such fourth powers to lie in the interval $[1914, 2138]$ is $x^4 + y^4 = 625 + 1296 = 1921$. Thus, $\{|x|, |y|\} = \{5, 6\}$.

Substituting $x^4 + y^4 = 1921$ into the original equation gives

$$1921 - \frac{7}{2}xy = 2026 \implies xy = -30.$$

Therefore x and y have opposite signs, and the integer solutions are

$$(x, y) \in \{(-6, 5), (-5, 6), (5, -6), (6, -5)\}.$$

Solution 2 by Polyhedra, Polk State College, FL, USA

Let $f(x, y)$ be the polynomial on the left side of the equation. If $|x| \geq 7$, or $|y| \geq 7$, or $|x| = |y| = 6$, then

$$f(x, y) \geq x^4 - \frac{7}{4}(x^2 + y^2) + y^4 > x^2(x^2 - 2) + y^2(y^2 - 2) > 2026.$$

If $|x|, |y| \leq 5$, then

$$f(x, y) \leq 5^4 + 4 \cdot 5^2 + 5^4 < 2026.$$

If $|x| = 6$ but $|y| \leq 4$, then

$$f(x, y) \leq 6^4 + 21 \cdot 4 + 4^4 < 2026.$$

Therefore, we must have $\{|x|, |y|\} = \{5, 6\}$.

It is then easy to check that the solutions are $(x, y) = (5, -6), (-5, 6), (6, -5)$, and $(-6, 5)$.

Solution 3 by The Ring Lords, Baden-Württemberg, Germany

First, observe that at least one of x, y must be even, otherwise $\frac{7}{2}xy$ is not an integer.

Furthermore, note that if n is even, then $n^4 \equiv 0 \pmod{16}$ and if n is odd, then $n^4 \equiv 1 \pmod{16}$. Assume that x is even and consider the equation modulo 16.

1. If y is odd then $-\frac{7}{2}xy + 1 \equiv 2026 \pmod{16}$, which implies $\frac{1}{2}xy \equiv 1 \pmod{16}$.
2. If y is even then $-\frac{7}{2}xy \equiv 2026 \pmod{16}$, which implies $\frac{1}{2}xy \equiv 10 \pmod{16}$.

We next derive a bound for xy . By the AM-GM Inequality,

$$x^4 + y^4 \geq 2x^2y^2.$$

Substituting into the original equation yields

$$2x^2y^2 - \frac{7}{2}xy \leq 2026.$$

Let $t = xy$. Then

$$2t^2 - \frac{7}{2}t - 2026 \leq 0.$$

The roots of this quadratic are approximately -30.965 and 32.715 , hence $-30 \leq xy \leq 32$. From the modular constraints $\frac{1}{2}xy \equiv 1 \pmod{16}$ or $\frac{1}{2}xy \equiv 10 \pmod{16}$, we obtain $xy \in \{-30, -12, 2, 20\}$.

Since $x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2$, we rewrite the equation as

$$(x^2 + y^2)^2 = 2026 + 2x^2y^2 + \frac{7}{2}xy.$$

Let

$$f(t) = 2026 + 2t^2 + \frac{7}{2}t, \quad t = xy.$$

Evaluate:

$$f(-30) = 3721 = 61^2, \quad f(-12) = 2272, \quad f(2) = 2041, \quad f(20) = 2896.$$

Only $t = -30$ yields a perfect square. Thus,

$$xy = -30, \quad x^2 + y^2 = 61.$$

We compute

$$(x - y)^2 = x^2 - 2xy + y^2 = 121, \quad (x + y)^2 = x^2 + 2xy + y^2 = 1.$$

Thus

$$\begin{cases} x - y &= \pm 11 \\ x + y &= \pm 1 \end{cases}$$

Solving gives the integer pairs $(x, y) \in \{(5, -6), (6, -5), (-5, 6), (-6, 5)\}$.

We verify that these are all solutions to the given equation.

Solution 4 by Theo Koupelis, Clark College, Washington, USA

Solution #1

For solutions in the integers we must have $2 \mid (xy)$ and thus at least one of the variables is even. Clearly if (x, y) is a solution then (y, x) , $(-x, -y)$, and $(-y, -x)$ are also solutions. Also, $(x, y) = 1$ and $xy \neq 0$ because the number $2026 = 2 \cdot 1013$ does not have a square factor. Without loss of generality, let $x = 2z$ be a positive, even integer. Substituting into the given equation we get

$$16z^4 - 7zy + y^4 = 2026.$$

Using AM-GM, from the above equation we get

$$(4z^2 + y^2)^2 = 2026 + 8z^2y^2 + 7zy \leq 2026 + \frac{(4z^2 + y^2)^2}{2} + \frac{7(4z^2 + y^2)}{4},$$

and thus

$$4z^2 + y^2 \leq 65 \implies z \in \{1, 2, 3, 4\}.$$

(i) For $z = 1$ we get $y^2 \leq 61$ and $y^4 - 7y = 2010$. Working modulo 7 we get $y \equiv \pm 1 \pmod{7}$, and thus $y \in \{\pm 1, \pm 6\}$. None of these values satisfies the given equation.

(ii) For $z = 2$ we get $y^2 \leq 49$ and $y^4 - 14y = 1770$. Working modulo 7 we get no integer solutions to this equation.

(iii) For $z = 3$ we get $y^2 \leq 29$ and $y^4 - 21y = 730$. Working modulo 7 we get $y \equiv \pm 2 \pmod{7}$, and thus $y \in \{\pm 2, \pm 5\}$. Only the value $y = -5$ satisfies the given equation, and thus $(x, y) = (6, -5)$ is an acceptable solution.

(iv) For $z = 4$ we get $y^2 \leq 1$ (and thus $y = \pm 1$) and $y^4 - 28y = -2070$, which is not possible.

Therefore, the integer solutions of the given equation are $(x, y) = (6, -5)$, $(-5, 6)$, $(-6, 5)$, and $(5, -6)$.

Solution #2

Let $p = x + y$ and $r = xy$. The given equation is equivalent to

$$4r^2 - (8p^2 + 7)r + 2p^4 - 4052 = 0,$$

whose solutions are

$$r_{\pm} = \frac{8p^2 + 7 \pm \sqrt{32p^4 + 112p^2 + 64881}}{8}.$$

From AM-GM we get $r \leq p^2/4$, and thus only r_- is acceptable and

$$32p^4 + 112p^2 + 64881 \geq (6p^2 + 7)^2 \iff p^4 - 7p^2 - 16208 \leq 0 \implies -11 \leq p \leq 11.$$

Because r_- is a function of p^2 , it suffices to examine only the values $p \in \{0, 1, \dots, 11\}$. By substitution we get that only for $p = 1$ the quantity $32p^4 + 112p^2 + 64881$ is a square. Thus, the acceptable solutions over the integers are $p = \pm 1$ and $r = -30$. Therefore, the integer solutions of the given equation are $(x, y) = (6, -5)$, $(-5, 6)$, $(-6, 5)$, and $(5, -6)$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Anita Tufisi, Romania; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; G. C. Greubel, Newport News, VA, USA; José Luis Díaz-Barrero, Barcelona, Spain; Sundaresh H. R., Shivamogga, Karnataka, India; Aaditya Bhaumik Sahu, AMMOC, India; Daniel Pascuas, Barcelona, Spain; Diar Gashi, University of Primorska, Slovenia; Ervin Ramirez, Richland Center, Wisconsin, USA; Loukya Ghanta, AMMOC; Parth Andhare, AMMOC; Shaurya Patil, AMMOC; Shilong Zhuang, California, USA; Vibhu Konduru, AMMOC; Vidur Konduru, AMMOC; William Han, Horace Mann School, Bronx, New York, USA; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

J730. Let x, y, z be nonnegative real numbers such that $4x + 3y + z = 1$. Find the minimum and maximum of

$$x + \sqrt{x^2 + y^2 + z^2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution 1 by Polyhedra, Polk State College, FL, USA

First, $x + \sqrt{x^2 + y^2 + z^2} \leq x + \sqrt{(3x + 3y + z)^2} = 4x + 3y + z = 1$, with equality when $(x, y, z) = (0, 0, 1)$, so the maximum is 1. Next, by the Cauchy-Schwarz inequality,

$$x + \sqrt{x^2 + y^2 + z^2} = x + \frac{\sqrt{(3^2 + 12^2 + 4^2)(x^2 + y^2 + z^2)}}{13} \geq x + \frac{3x + 12y + 4z}{13} = \frac{4}{13},$$

with equality when $(x, y, z) = \left(\frac{3}{52}, \frac{3}{13}, \frac{1}{13}\right)$, so the minimum is $\frac{4}{13}$.

Solution 2 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Denote

$$F = x + \sqrt{x^2 + y^2 + z^2}.$$

Since $x, y, z \geq 0$, we have $\sqrt{x^2 + y^2 + z^2} \leq x + y + z$. Therefore,

$$F \leq x + (x + y + z) = 2x + y + z = (4x + 3y + z) - 2x - 2y = 1 - 2x - 2y \leq 1.$$

Hence, $F \leq 1$ and equality holds for $(x, y, z) = (0, 0, 1)$. Thus, the maximum value is 1.

From $4x + 3y + z = 1$ we get $3y + z = 1 - 4x$. By Cauchy-Schwarz,

$$(3y + z)^2 \leq (3^2 + 1^2)(y^2 + z^2) = 10(y^2 + z^2).$$

Hence

$$y^2 + z^2 \geq \frac{(1 - 4x)^2}{10}.$$

Thus

$$F = x + \sqrt{x^2 + y^2 + z^2} \geq x + \sqrt{x^2 + \frac{(1 - 4x)^2}{10}}.$$

We now compare the expression under the square root with $\left(\frac{4}{13} - x\right)^2$. Indeed,

$$x^2 + \frac{(1 - 4x)^2}{10} - \left(\frac{4}{13} - x\right)^2 = \frac{(52x - 3)^2}{1690} \geq 0.$$

Therefore

$$x + \sqrt{x^2 + \frac{(1 - 4x)^2}{10}} \geq x + \left(\frac{4}{13} - x\right) = \frac{4}{13}.$$

So, $F \geq \frac{4}{13}$. Equality occurs when $52x - 3 = 0$, that is, $x = \frac{3}{52}$. Equality must also hold in Cauchy-Schwarz, so $y : z = 3 : 1$. Since

$$3y + z = 1 - 4x = 1 - \frac{12}{52} = \frac{10}{13},$$

we get $y = \frac{3}{13}$ and $z = \frac{1}{13}$. Thus the minimum value is $\frac{4}{13}$.

Solution 3 by Daniel Pascuas, Barcelona, Spain

We have to find the the minimum and maximum values of the function

$$f(x, y, z) := x + \sqrt{x^2 + y^2 + z^2}$$

on the set $T := \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, 4x + 3y + z = 1\}$, which is the triangle with vertices at the points $p_1 = (\frac{1}{4}, 0, 0)$, $p_2 = (0, \frac{1}{3}, 0)$, and $p_3 = (0, 0, 1)$.

Note that $f = g + h$, where $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean norm and $h(x, y, z) = x$. Since g is a strictly convex function while h is a linear function, f is also a strictly convex function. Therefore

$$\max_{p \in T} f(p) = \max\{f(p_1), f(p_2), f(p_3)\} = \max\{\frac{1}{2}, \frac{1}{3}, 1\} = 1,$$

and this maximum value is only assumed on T at $p_3 = (0, 0, 1)$.

In order to find the minimum value of f on T , we write

$$T = \bigcup_{0 \leq a \leq \frac{1}{4}} I_a, \quad \text{where} \quad I_a := \{(x, y, z) \in T : x = a\}.$$

Note that, for $0 \leq a < \frac{1}{4}$, I_a is the closed line segment with endpoints $P_a = (a, 0, 1-4a)$ and $Q_a = (a, \frac{1-4a}{3}, 0)$, that is,

$$I_a = \{P_a + t \overrightarrow{P_a Q_a} : 0 \leq t \leq 1\}.$$

Then the plane $\frac{1-4a}{3}y + (4a-1)z = 0$ passes through the origin and is orthogonal to the straight line containing I_a (because $\overrightarrow{P_a Q_a} = (0, \frac{1-4a}{3}, 4a-1)$). Since this plane cuts I_a at the point

$$\tilde{P}_a = (a, \frac{3}{10}(1-4a), \frac{1}{10}(1-4a)),$$

$\tilde{P}_a$ is the closest point on I_a to the origin, so

$$\min_{p \in T_a} f(p) = g(a), \quad \text{where} \quad g(a) = f(\tilde{P}_a) = a + \sqrt{a^2 + \frac{1}{10}(1-4a)^2}.$$

Since $T_{\frac{1}{4}} = \{p_1\}$, the above formula also holds for $a = \frac{1}{4}$. It follows that

$$\min_{p \in T} f(p) = \min_{0 \leq a \leq \frac{1}{4}} g(a).$$

Since

$$g'(a) = 1 + \frac{a - \frac{2}{5}(1-4a)}{\sqrt{a^2 + \frac{1}{10}(1-4a)^2}} = 1 + \frac{13a-2}{5\sqrt{a^2 + \frac{1}{10}(1-4a)^2}}$$

the critical points of g are the solutions of the equation

$$5\sqrt{a^2 + \frac{1}{10}(1-4a)^2} = 2 - 13a.$$

Any solution to that equation must satisfy that $2 - 13a > 0$, i.e. $a < \frac{2}{13}$. For those a 's, by squaring we get the equivalent equation

$$25(a^2 + \frac{1}{10}(1-4a)^2) = (2-13a)^2,$$

which can be written as $208a^2 - 64a + 3 = 0$, after simplifying terms. The solutions to this quadratic equation are $a = \frac{1}{4}$ and $a = \frac{3}{52}$. Since $\frac{3}{52} < \frac{8}{52} = \frac{2}{13} < \frac{1}{4}$, $a = \frac{3}{52}$ is the only critical point of g , and it lies in the interval $(0, \frac{1}{4})$. Therefore

$$\min_{p \in T} f(p) = \min_{0 \leq a \leq \frac{1}{4}} g(a) = \min\{g(0), g(\frac{1}{4}), f(\frac{3}{52})\} = \min\{\frac{1}{\sqrt{10}}, \frac{1}{2}, \frac{4}{13}\} = \frac{4}{13},$$

because $\frac{4}{13} < \frac{1}{\sqrt{10}} < \frac{1}{2}$.

Hence we have just proved that the maximum and minimum values of

$$x + \sqrt{x^2 + y^2 + z^2} \quad (x, y, z \geq 0, 4x + 3y + z = 1)$$

are 1 and $\frac{4}{13}$, respectively. They are only assumed at the points $p_3 = (0, 0, 1)$ and $\tilde{P}_{\frac{3}{52}} = (\frac{3}{52}, \frac{3}{13}, \frac{1}{13})$, respectively.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Ervin Ramirez, Richland Center, Wisconsin, USA; Ganghun Kim, Fort Lee High School, NJ, USA; Loukya Ghanta, AMMOC; Parth Andhare, AMMOC; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

J731. Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = a + b + c$. Prove that

$$ab + bc + ca \leq abc + \frac{2}{3}(a + b + c).$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

Solution 1 by Polyhedra, Polk State College, FL, USA

Since $(a - 1)(b - 1)(c - 1) = abc - (ab + bc + ca) + a^2 + b^2 + c^2 - 1 \geq abc - 1 \geq -1$, we have

$$\begin{aligned} abc - (ab + bc + ca) + \frac{2}{3}(a + b + c) &= (a - 1)(b - 1)(c - 1) + 1 - \frac{a + b + c}{3} \\ &= (a - 1)(b - 1)(c - 1) + \frac{(a - 1)^2 + (b - 1)^2 + (c - 1)^2}{3} \\ &\geq \sqrt[3]{(a - 1)^2(b - 1)^2(c - 1)^2} \left(\sqrt[3]{(a - 1)(b - 1)(c - 1)} + 1 \right) \\ &\geq 0, \end{aligned}$$

where the penultimate inequality follows by the AM-GM Inequality.

Solution 2 by Nguyen Viet Hung, Hanoi University of Science, Vietnam

The inequality can be rewritten in homogeneous form as follows

$$\frac{(ab + bc + ca)(a^2 + b^2 + c^2)}{a + b + c} \leq abc + \frac{2(a^2 + b^2 + c^2)^2}{3(a + b + c)}.$$

This is equivalent to each of the following

$$\begin{aligned} 3(ab + bc + ca)(a^2 + b^2 + c^2) &\leq 3abc(a + b + c) + 2(a^2 + b^2 + c^2)^2, \\ \sum_{\text{cyc}} 3ab(a^2 + b^2) &\leq 2(a^4 + b^4 + c^4) + 4(a^2b^2 + b^2c^2 + c^2a^2), \\ \sum_{\text{cyc}} 3ab(a - b)^2 &\leq \sum_{\text{cyc}} (a^2 - b^2)^2, \\ 0 &\leq \sum_{\text{cyc}} (a^2 - ab + b^2)(a - b)^2. \end{aligned}$$

The last inequality is true because

$$a^2 - ab + b^2 = \frac{1}{4}(a + b)^2 + \frac{3}{4}(a - b)^2 \geq 0.$$

The proof is complete.

Solution 3 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$. Since $a^2 + b^2 + c^2 = a + b + c = p$, we also have

$$p^2 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = p + 2q.$$

Hence

$$q = \frac{p^2 - p}{2}.$$

By Schur's inequality, $p^3 + 9r \geq 4pq$. Substituting $q = \frac{p^2 - p}{2}$, we get

$$9r \geq 4p \cdot \frac{p^2 - p}{2} - p^3 = 2p(p^2 - p) - p^3 = p^3 - 2p^2.$$

Therefore,

$$r \geq \frac{p^2(p-2)}{9}.$$

Now

$$r + \frac{2}{3}p - q \geq \frac{p^2(p-2)}{9} + \frac{2}{3}p - \frac{p^2-p}{2}.$$

Multiplying by 18, we obtain $18\left(r + \frac{2}{3}p - q\right) \geq 2p^2(p-2) + 12p - 9(p^2-p)$. Thus

$$18\left(r + \frac{2}{3}p - q\right) \geq 2p^3 - 13p^2 + 21p = p(3-p)(7-2p).$$

It remains to note that $0 \leq p \leq 3$. Indeed, by Cauchy-Schwarz,

$$p^2 = (a+b+c)^2 \leq 3(a^2+b^2+c^2) = 3p.$$

Since $p \geq 0$, this gives $p \leq 3$. Hence $p(3-p)(7-2p) \geq 0$. Therefore,

$$r + \frac{2}{3}p - q \geq 0$$

and we are done. Equality holds when $p = 0$ or $p = 3$, that is, when $(a, b, c) = (0, 0, 0)$ or $(1, 1, 1)$.

Also solved by Andrea Fanchini, Cantù, Italy; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Debarun Das, CMC, Hyderabad, India; José Luis Díaz-Barrero, Barcelona, Spain; Kousik Sett, India; Prodromos Fotiadis, University of Crete, Greece; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Daniel Pascuas, Barcelona, Spain; Ervin Ramirez, Richland Center, Wisconsin, USA; Ganghun Kim, Fort Lee High School, NJ, USA; Kristine Lee, Mount Juliet High School, TN, USA; Parth Andhare, AMMOC; Polyhedra, Polk State College, FL, USA; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania.

J732. Let ABC be an acute triangle, with $AB \neq AC$, M be the midpoint of BC and D, E, F the feet of altitudes from A, B , and C , respectively. A line parallel to EF passes through D and cuts the line AC in X and the line AB in Y . The circumcircle of $\triangle MXY$ cuts the line BC in $Z \neq M$. Show that points E, F, Z are collinear.

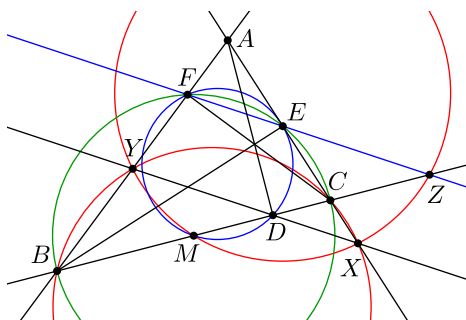
Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Polyhedra, Polk State College, FL, USA

Since $\triangle AXY \sim \triangle AEF \sim \triangle ABC$, the points B, X, C, Y lie on a circle. By the power of D , $MD \cdot DZ = YD \cdot DX = BD \cdot DC$. Therefore,

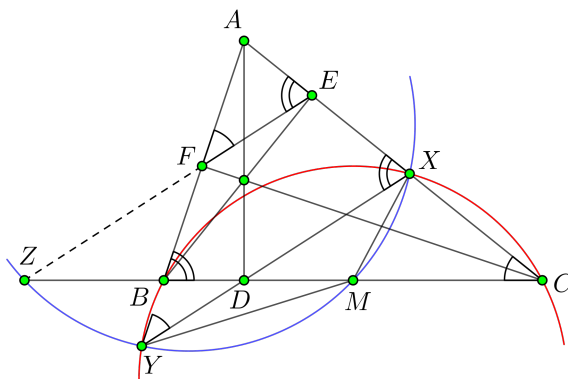
$$\begin{aligned} MD \cdot MZ &= MD(MD + DZ) = MD^2 + BD \cdot DC \\ &= MD^2 + (BM + MD)(MC - MD) = MC^2, \end{aligned}$$

so $Z = f(D)$, where f is the inversion centered at M with radius MC . Since f maps the nine-point circle through M, D, E, F to the line EF , Z is on EF .



Solution 2 by Kousik Sett, India

Assume without loss of generality that $AB < AC$. Since D, E, F the feet of altitudes from A, B , and C , respectively, we have quadrilateral $BCEF$ is cyclic. So $\angle AEF = \angle FBC$ and $\angle AFE = \angle BCE = \angle BCX$. Since $EF \parallel XY$, we have $\angle AFE = \angle FYX = \angle BYX$. Thus, $\angle BCX = \angle BYX$. Therefore, quadrilateral $CXBY$ is cyclic. Also, since the circumcircle of $\triangle MXY$ cuts the line BC in $Z \neq M$, we have quadrilateral $XMYZ$ is cyclic.



Therefore, by the intersecting chords theorem, we obtain

$$YD \cdot DX = BD \cdot DC = ZD \cdot DM.$$

Now,

$$\begin{aligned} BD \cdot DC &= ZD \cdot DM = (ZB + BD)(DC - BC/2) \\ \Rightarrow ZB &= \frac{BD \cdot BC}{2DC - BC} = \frac{BD \cdot BC}{2(BC - BD) - BC} = \frac{BD \cdot BC}{BC - 2BD}, \end{aligned}$$

and

$$ZC = ZB + BC = \frac{BC(BC - BD)}{BC - 2BD} = \frac{BC \cdot DC}{BC - 2BD}.$$

Hence, we obtain

$$\frac{ZB}{ZC} = \frac{BD}{DC}.$$

Since, AD , BE , and CF are concurrent, Ceva's theorem yields

$$\frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD}{DC} = 1 \Rightarrow \frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{ZB}{ZC} = 1.$$

Therefore, by the converse of Menelaus' theorem, we conclude that E , F , and Z are collinear.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Andrea Fanchini, Cantù, Italy; Theo Koupelis, Clark College, Washington, USA.

Senior problems

S727. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\sqrt[5]{\frac{a^5 + b^5}{2}} + \sqrt[5]{\frac{b^5 + c^5}{2}} + \sqrt[5]{\frac{c^5 + a^5}{2}} \leq \sqrt[5]{\frac{a^4}{b^3} + \frac{b^4}{c^3} + \frac{c^4}{a^3}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

We apply Holder's inequality to obtain

$$\begin{aligned} \sum_{\text{cyc}} \sqrt[5]{\frac{a^5 + b^5}{2}} &= \sum_{\text{cyc}} \sqrt[5]{\frac{a+b}{2} (a^4 - a^3b + a^2b^2 - ab^3 + b^4)} \\ &= \sum_{\text{cyc}} \sqrt[5]{\frac{a+b}{2} \cdot b \cdot b \cdot b \left(\frac{a^4}{b^3} - \frac{a^3}{b^2} + \frac{a^2}{b} - a + b \right)} \\ &\leq \sqrt[5]{(a+b+c)^4 \left(\frac{a^4}{b^3} + \frac{b^4}{c^3} + \frac{c^4}{a^3} - \frac{a^3}{b^2} - \frac{b^3}{c^2} - \frac{c^3}{a^2} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \right)}. \end{aligned}$$

It suffices to show that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \leq \frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2}.$$

Indeed, using the AM-GM inequality we have

$$\left(\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \right) (a + b + c) \geq \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \right)^2,$$

and

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

Multiplying these two inequalities we get the desired result and the proof is completed.

Solution 2 by Alessandro Ventullo, Milan, Italy

Let

$$L = \sqrt[5]{\frac{a^5 + b^5}{2}} + \sqrt[5]{\frac{b^5 + c^5}{2}} + \sqrt[5]{\frac{c^5 + a^5}{2}}.$$

Since $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$, we have

$$\sqrt[5]{\frac{a^5 + b^5}{2}} = \sqrt[5]{\frac{a+b}{2} \cdot b \cdot b \cdot b \left(\frac{a^4}{b^3} - \frac{a^3}{b^2} + \frac{a^2}{b} - a + b \right)}.$$

Applying Hölder's inequality to the three cyclic terms, we obtain

$$\begin{aligned} L &\leq \sqrt[5]{\left(\sum_{\text{cyc}} \frac{a+b}{2} \right) \left(\sum_{\text{cyc}} b \right)^3 \sum_{\text{cyc}} \left(\frac{a^4}{b^3} - \frac{a^3}{b^2} + \frac{a^2}{b} - a + b \right)} \\ &= \sqrt[5]{\sum_{\text{cyc}} \frac{a^4}{b^3} - \sum_{\text{cyc}} \frac{a^3}{b^2} + \sum_{\text{cyc}} \frac{a^2}{b}}, \end{aligned}$$

because $a + b + c = 1$, and $\sum_{\text{cyc}}(-a + b) = 0$.

It remains to show that

$$\sum_{\text{cyc}} \frac{a^3}{b^2} \geq \sum_{\text{cyc}} \frac{a^2}{b}.$$

Indeed, by Titu's Lemma,

$$\sum_{\text{cyc}} \frac{a^3}{b^2} = \sum_{\text{cyc}} \frac{\left(\frac{a^2}{b}\right)^2}{a} \geq \frac{\left(\sum_{\text{cyc}} \frac{a^2}{b}\right)^2}{a + b + c} = \left(\sum_{\text{cyc}} \frac{a^2}{b}\right)^2 \geq \left(\frac{(a + b + c)^2}{a + b + c}\right)^2 = 1.$$

Therefore,

$$\sum_{\text{cyc}} \frac{a^3}{b^2} \geq \left(\sum_{\text{cyc}} \frac{a^2}{b}\right)^2 \geq \sum_{\text{cyc}} \frac{a^2}{b}.$$

Hence,

$$\sum_{\text{cyc}} \frac{a^4}{b^3} - \sum_{\text{cyc}} \frac{a^3}{b^2} + \sum_{\text{cyc}} \frac{a^2}{b} \leq \sum_{\text{cyc}} \frac{a^4}{b^3},$$

and so

$$L \leq \sqrt[5]{\sum_{\text{cyc}} \frac{a^4}{b^3}}.$$

Editor's Comment. Theo Koupelis noticed that for both solutions the final step can be considered the direct result of using the weighted power mean in the form

$$\left(b \frac{a^3}{b^3} + c \frac{b^3}{c^3} + a \frac{c^3}{a^3}\right)^{1/3} \geq \left(b \frac{a^2}{b^2} + c \frac{b^2}{c^2} + a \frac{c^2}{a^2}\right)^{1/2} \geq \left(b \frac{a}{b} + c \frac{b}{c} + a \frac{c}{a}\right)^{1/1} = 1.$$

Also solved by Adela Llapashtica, *Specialized Mathematical Gymnasium, Prishtine, Kosovo*; The Ring Lords, *Baden-Württemberg, Germany*; Theo Koupelis, *Clark College, Washington, USA*; Aaditya Bhaumik Sahu, *AMMOC, India*; Ganghun Kim, *Fort Lee High School, NJ, USA*; Parth Andhare, *AMMOC*.

S728. Let p, q, r be primes such that $p \equiv q \equiv 1 \pmod{r}$ and $p^3 - q^3 > 3pqr + r^3$. Prove that

$$\frac{p^3 - q^3}{2} \geq 3pqr + 4r^3.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Clearly, $p > q$, so $p - r \geq q$. It must be $p - r \neq q$, otherwise we would have

$$p^3 - q^3 = p^3 - (p - r)^3 = 3p^2r - 3pr^2 + r^3 = 3pr(p - r) + r^3 = 3pqr + r^3,$$

which contradicts the given inequality. So $p - 2r \geq q$ and

$$p^3 - q^3 \geq p^3 - (p - 2r)^3 = 6p^2r - 12pr^2 + 8r^3 = 6pr(p - 2r) + 8r^3 \geq 6pqr + 8r^3,$$

which gives the desired conclusion.

Remark

If $r = 2$, we have Problem S241 of Mathematical Reflections 5/2012, proposed by Titu Andreescu.

Solution 2 by Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA

The inequality $p^3 - q^3 > 3pqr + r^3$ can be rewritten as $p^3 - q^3 - r^3 - 3pqr > 0$. Using the identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

with $(a, b, c) = (p, -q, -r)$, we obtain

$$p^3 - q^3 - r^3 - 3pqr = (p - q - r)(p^2 + q^2 + r^2 + pq + pr - qr).$$

Hence,

$$(p - q - r)(p^2 + q^2 + r^2 + pq + pr - qr) > 0.$$

The second factor is positive, since $q^2 - qr + r^2 > 0$. Therefore $p - q - r > 0$, that is, $p > q + r$.

On the other hand, from $p \equiv q \pmod{r}$ and $p - q > r$ it follows that $p - q = nr$ for some integer $n > 1$, that is, $p \geq q + 2r$. Now we use the same identity again, this time with $(a, b, c) = (p, -q, -2r)$. Then

$$p^3 - q^3 - 8r^3 - 6pqr = (p - q - 2r)(p^2 + q^2 + 4r^2 + pq + 2pr - 2qr).$$

The first factor is nonnegative because $p \geq q + 2r$. The second factor is positive, since $q^2 - 2qr + 4r^2 > 0$. Thus,

$$p^3 - q^3 - 8r^3 - 6pqr \geq 0 \implies \frac{p^3 - q^3}{2} \geq 3pqr + 4r^3,$$

and the proof is complete.

Solution 3 by Theo Koupelis, Clark College, Washington, USA

Let $p = ar + 1$ and $q = br + 1$, where a, b are integers. It is given that $p > q$ and thus $a > b > 0$ because p, q are primes. Substituting we get

$$p^3 - q^3 = r^3(a^3 - b^3) + 3r^2(a^2 - b^2) + 3r(a - b)$$

and

$$3pqr + r^3 = r^3(3ab + 1) + 3(a + b)r^2 + 3r.$$

It is given that $p^3 - q^3 > 3pqr + r^3$, and thus

$$r^3(a^3 - b^3 - 3ab - 1) + 3(a^2 - b^2 - a - b)r^2 + 3(a - b - 1)r > 0,$$

or

$$r(a - b - 1) [r^2(a^2 + a(b + 1) + (b^2 - b + 1)) + 3(a + b)r + 3] > 0.$$

Therefore, $a \geq b + 2$. The desired inequality is now equivalent to

$$r^3(a^3 - b^3 - 6ab - 8) + 3(a^2 - b^2 - 2a - 2b)r^2 + 3(a - b - 2)r \geq 0,$$

or

$$r(a - b - 2) [r^2(a^2 + a(b + 2) + (b^2 - 2b + 4)) + 3(a + b)r + 3] \geq 0,$$

which is obvious. Equality occurs when $a = b + 2$. From the above we get that it suffices that p, q, r are positive integers that satisfy the given conditions; equality occurs when $p = q + 2r$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Aaditya Bhau-mik Sahu, AMMOC, India; Ervin Ramirez, Richland Center, Wisconsin, USA; Ganghun Kim, Fort Lee High School, NJ, USA; Shilong Zhuang, California, USA; Vibhu Konduru, AMMOC.

S729. Let ABC be an acute triangle with circumcircle $\Gamma(O, R)$. The perpendicular bisector of side BC meets Γ at X , on the minor arc BC , and let Y be the intersection point of AX and the segment BC . Let D and E be the feet of the perpendiculars from Y and C to line AB , respectively, and let Z be the foot of the perpendicular from C to line AX . Let T be the intersection point of lines DZ and BC , K be the intersection point of lines AT and CE and S be the circumcenter of $\triangle BKC$. Prove that

$$\overrightarrow{AS} = \overrightarrow{AK} + \overrightarrow{AO}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Kousik Sett, India

Since OX is the perpendicular bisector of BC , it follows that AX is the angle bisector of $\angle BAC$. So, $\angle XBC = \angle XAC = \angle BAX = \angle A/2$. Also, $\angle AXC = \angle ABC = \angle B$. Then we get,

$$\frac{AD}{DB} = \frac{AY \cos \frac{A}{2}}{BY \cos B} \quad \text{and} \quad \frac{AZ}{ZX} = \frac{AC \cos \frac{A}{2}}{CX \cos B}.$$

Since $\angle BAY = \angle XAC$ and $\angle ABY = \angle ABC = \angle AXC$, we obtain $\triangle ABY \sim \triangle AXC$. Thus,

$$\frac{AY}{BY} = \frac{AC}{CX} \quad \text{and consequently, from the above, we get} \quad \frac{AD}{DB} = \frac{AZ}{ZX} \Rightarrow DZ \parallel BX.$$

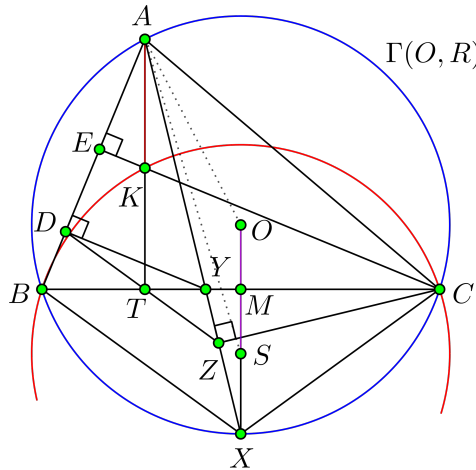
Therefore,

$$\angle ZTC = \angle ZTY = \angle XBY = \angle XBC = \angle A/2.$$

Also,

$$\angle ZAC = \angle XAC = \angle A/2.$$

So, quadrilateral $ZCAT$ is cyclic and $\angle ATC = \angle AZC = 90^\circ$. Thus, AT is perpendicular to BC . Therefore, $K = CE \cap AT$ is the orthocenter of triangle ABC .



It is well-known that the circumradius of $\triangle ABC$ and $\triangle BKC$ are equal. Therefore, $CO = CS$. Again, since $\angle CMO = \angle CMS = 90^\circ$ and CM is the common side, we have $\triangle CMO \cong \triangle CMS$. So, $OM = MS$. It is also well-known that $AK = 2OM$. Thus $AK = 2OM = OM + MS = OS$. Also, OS and AK are perpendicular to BC , so $AK \parallel OS$. Hence, we infer that $AKSO$ is a parallelogram. Thus,

$$\overrightarrow{AS} = \overrightarrow{AO} + \overrightarrow{OS} = \overrightarrow{AO} + \overrightarrow{AK}.$$

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Without loss of generality, let $AB \leq AC$. The quadrilateral $AEZC$ is cyclic because $\angle AEC = \angle AZC = 90^\circ$. Let T' be the point of intersection of the circle $(AEZC)$ and the line BC . Then $AT' \perp BC$. The quadrilateral

$ADT'Y$ is cyclic because $\angle ADY = \angle AT'Y = 90^\circ$, and thus $\angle T'DY = \angle T'AY = \angle BAY - \angle BAT' = \angle A/2 - (90^\circ - \angle B) = (\angle B - \angle C)/2$. Therefore, $\angle T'DB = 90^\circ - (\angle B - \angle C)/2 = \angle A/2 + \angle C$ and $\angle BT'D = 180^\circ - \angle B - (\angle A/2 + \angle C) = \angle A/2 = \angle ZT'C$. Thus, D, T', Z are collinear and therefore $T = T'$ and K is the orthocenter of the triangle. If R' is the radius of the circle (S) then $BC = 2R' \sin \angle(180^\circ - \angle BKC) = 2R' \sin \angle A = 2R \sin \angle A$, and thus $R' = R$. With OS being the perpendicular bisector of BC we get $AE = AK \sin B = b \cos A$ and thus $AK = 2R \cos A = OS$. Thus, $AKSO$ is a parallelogram and therefore $\vec{AS} = \vec{AK} + \vec{AO}$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Andrea Fanchini, Cantù, Italy; William Han, Horace Mann School, Bronx, New York, USA.

S730. Let a, b, c be nonnegative real numbers. Prove that

$$3\sqrt{(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)} + 9abc \geq 2(a + b + c)(ab + bc + ca).$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

Solution 1 by the authors

If one of the numbers a, b, c is 0, say c , the inequality is true because

$$3ab\sqrt{a^2 + b^2} \geq \frac{3ab(a + b)}{\sqrt{2}} \geq 2ab(a + b).$$

We consider that $abc > 0$. The inequality can be rewritten as

$$3\sqrt{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} + a^2b^2c^2 + 9abc \geq 2(a + b)(b + c)(c + a) + 2abc.$$

Dividing by abc , this becomes

$$3\sqrt{(x^2 + 1)(y^2 + 1)(z^2 + 1)} + 1 + 7 \geq 2(x + 1)(y + 1)(z + 1),$$

where $x = \frac{a}{b}$, $y = \frac{b}{c}$, $c = \frac{z}{x}$.

After expanding, given that $xyz = 1$, this becomes

$$3\sqrt{3 + x^2y^2 + y^2z^2 + z^2x^2 + x^2 + y^2 + z^2} \geq 2(xy + yz + zx) + 2(x + y + z) - 3.$$

Setting $u = x + y + z$ and $v = xy + yz + zx$, we get

$$3\sqrt{3 + u^2 - 2v + v^2 - 2u} \geq 2u + 2v - 3.$$

After squaring both sides, the inequality reduces to

$$9(3 + u^2 - 2v + v^2 - 2u) \geq (2u + 2v - 3)^2,$$

or

$$5u^2 + 5v^2 - 8uv - 6u - 6v + 18 \geq 0.$$

Since $\Delta_u = -36(v - 3)^2$ write the inequality as

$$5\left(u - \frac{8v + 6}{10}\right)^2 + \frac{36(v - 3)^2}{20} \geq 0,$$

or

$$\frac{(5u - 4v - 3)^2}{5} + \frac{9(v - 3)^2}{5} \geq 0$$

which is clearly true. The equality holds when $u = v = 3$ which means $x = y = z$ so $a = b = c$.

Solution 2 by Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA

We start with the identity

$$\begin{aligned} &9(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2) - (2(a + b + c)(ab + bc + ca) - 9abc)^2 \\ &= a^2(b - c)^4 + b^2(c - a)^4 + c^2(a - b)^4 + 4(a - b)^2(b - c)^2(c - a)^2. \end{aligned}$$

The right-hand side is clearly nonnegative, hence

$$9(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2) \geq (2(a + b + c)(ab + bc + ca) - 9abc)^2.$$

Also, by AM-GM,

$$(a + b + c)(ab + bc + ca) \geq 3\sqrt[3]{abc} \cdot 3\sqrt[3]{a^2b^2c^2} = 9abc.$$

Therefore

$$2(a + b + c)(ab + bc + ca) - 9abc \geq 0.$$

Taking square roots, we get

$$3\sqrt{(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)} \geq 2(a + b + c)(ab + bc + ca) - 9abc.$$

Thus

$$3\sqrt{(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)} + 9abc \geq 2(a + b + c)(ab + bc + ca),$$

as desired.

Solution 3 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Consider

$$F(a, b, c) = 3\sqrt{(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)} + 9abc - 2(a + b + c)(ab + bc + ca).$$

The inequality is symmetric and homogeneous, so by the *uvw* principle it is enough to check the case

$$b = c = t, \quad a = xt,$$

where $x, t \geq 0$.

If $t = 0$, the inequality is trivial. Hence assume $t > 0$. Substituting $a = xt$, $b = c = t$, we obtain

$$a^2 + b^2 + c^2 = t^2(x^2 + 2), \quad a^2b^2 + b^2c^2 + c^2a^2 = t^4(2x^2 + 1).$$

Thus

$$\sqrt{(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)} = t^3\sqrt{(x^2 + 2)(2x^2 + 1)}.$$

Also, $abc = xt^3$, and

$$(a + b + c)(ab + bc + ca) = t(x + 2) \cdot t^2(2x + 1) = t^3(x + 2)(2x + 1).$$

Therefore the desired inequality becomes

$$3t^3\sqrt{(x^2 + 2)(2x^2 + 1)} + 9xt^3 \geq 2t^3(x + 2)(2x + 1).$$

Dividing by $t^3 > 0$, it remains to prove

$$3\sqrt{(x^2 + 2)(2x^2 + 1)} + 9x \geq 2(x + 2)(2x + 1).$$

Equivalently,

$$3\sqrt{(x^2 + 2)(2x^2 + 1)} \geq 4x^2 + x + 4.$$

Both sides are nonnegative, so we may square. We need to prove

$$9(x^2 + 2)(2x^2 + 1) \geq (4x^2 + x + 4)^2.$$

But

$$9(x^2 + 2)(2x^2 + 1) - (4x^2 + x + 4)^2 = 2(x - 1)^4 \geq 0.$$

Hence

$$3\sqrt{(x^2 + 2)(2x^2 + 1)} \geq 4x^2 + x + 4,$$

and the desired inequality follows.

Equality holds when $x = 1$, that is, when $a = b = c$.

Solution 4 by Prodromos Fotiadis, University of Crete, Greece

Moving $9abc$ to the other side and squaring we get

$$9(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2) \geq \left[2(a + b + c)(ab + bc + ca) - 9abc\right]^2. \quad (1)$$

Using the AM-GM inequality we find

$$2(a + b + c)(ab + bc + ca) \geq 18\sqrt[3]{abc}\sqrt[3]{a^2b^2c^2} = 18abc \geq 9abc$$

so Equation (1) is equivalent to the original statement. Now, we set

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc$$

It is straightforward to rewrite Equation (1) as

$$\begin{aligned} 9(p^2 - 2q)(q^2 - 2pr) &\geq (2pq - 9r)^2 = 4p^2q^2 - 36pqr + 81r^2 \\ \iff 5p^2q^2 + 72pqr &\geq 18p^3r + 18q^3 + 81r^2 \end{aligned}$$

To prove this, we invoke the following well-known result

$$p^2q^2 + 18pqr \geq 4p^3r + 4q^3 + 27r^2, \quad (2)$$

which is obtained by expanding $(a - b)^2(b - c)^2(c - a)^2 \geq 0$. According to Equation (2):

$$p^3r + q^3 \leq \frac{p^2q^2 + 18pqr - 27r^2}{4},$$

Therefore, it suffices to show that

$$\begin{aligned} 5p^2q^2 + 72pqr &\geq 9\frac{p^2q^2 + 18pqr - 27r^2}{2} + 81r^2 \\ \iff p^2q^2 + 81r^2 &\geq 18pqr \end{aligned}$$

But this is obvious by the AM-GM inequality: $p^2q^2 + 81r^2 \geq 2\sqrt{81p^2q^2r^2} = 18pqr$. This completes the proof.

Also solved by Andrea Fanchini, Cantù, Italy; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; José Luis Díaz-Barrero, Barcelona, Spain; Kevin Soto Palacios, National University of Callao (UNAC), Peru; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Ervin Ramirez, Richland Center, Wisconsin, USA; Loukya Ghanta, AMMOC; Parth Andhare, AMMOC; Shaurya Patil, AMMOC; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania.

S731. Let a, b, c, d, e, f be real numbers such that $a + b + c + d + e + f = 0$. Prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \geq 2(ab + bc + cd + de + ef + fa).$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution 1 by the author

Replacing f with $-(a + b + c + d + e)$, the inequality becomes as follows:

$$2a^2 + (b + 2c + 2d + 3e)a + b^2 + c^2 + d^2 + 2e^2 + bd + 2be + 2ce + de \geq 0,$$

$$2 \left(a + \frac{b + 2c + 2d + 3e}{4} \right)^2 + \frac{7(b^2 + e^2) + 10be + 4(c - d)^2 - 4(b - e)(c - d)}{8} \geq 0,$$

It suffices to show that

$$7(b^2 + e^2) + 10be + 4(c - d)^2 - 4(b - e)(c - d) \geq 0.$$

Since

$$7(b^2 + e^2) + 10be = (b - e)^2 + 6(b + e)^2 \geq (b - e)^2,$$

we have

$$7(b^2 + e^2) + 10be + 4(c - d)^2 - 4(b - e)(c - d) \geq (b - e)^2 + 4(c - d)^2 - 4(b - e)(c - d)$$

$$= [(b - e) - 2(c - d)]^2 \geq 0.$$

The equality occurs when

$$a + b + c + d + e + f = 0, \quad 4a + b + 2c + 2d + 3e = 0, \quad b + e = 0, \quad b - e = 2(c - d),$$

i.e. when

$$b, d \in \mathbb{R}, \quad a = -d, \quad c = b + d, \quad e = -b, \quad f = -b - d.$$

Solution 2 by William Han, Horace Mann School, Bronx, New York, USA

Consider the following three quantities:

$$\begin{aligned} X &= (a - b + c - d + e - f)^2 + (a - b + d - e)^2 + 3(c + f)^2, \\ Y &= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2(ab + bc + cd + de + ef + fa), \\ Z &= (c + f)(a + b + c + d + e + f). \end{aligned}$$

A direct calculation gives the identity

$$X = 2Y + 2Z.$$

By the hypothesis, $a + b + c + d + e + f = 0$, so $Z = 0$. Hence $X = 2Y$. But $X \geq 0$, since it is a sum of squares. Therefore $Y \geq 0$, that is,

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \geq 2(ab + bc + cd + de + ef + fa),$$

as desired.

Solution 3 by Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA

Introduce the following six new variables:

$$2u = a + d, \quad 2v = b + e, \quad 2w = c + f,$$

and

$$2x = a - d, \quad 2y = b - e, \quad 2z = c - f.$$

Then

$$a = u + x, \quad b = v + y, \quad c = w + z, \quad d = u - x, \quad e = v - y, \quad f = w - z.$$

From the hypothesis $a + b + c + d + e + f = 0$ we get $u + v + w = 0$. A direct computation gives

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 2(u^2 + v^2 + w^2 + x^2 + y^2 + z^2),$$

and

$$ab + bc + cd + de + ef + fa = 2(uv + vw + wu + xy + yz - zx).$$

Therefore

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2(ab + bc + cd + de + ef + fa) = 2(u^2 + v^2 + w^2 + x^2 + y^2 + z^2 - 2uv - 2vw - 2wu - 2xy - 2yz + 2zx).$$

Now

$$u^2 + v^2 + w^2 + x^2 + y^2 + z^2 - 2uv - 2vw - 2wu - 2xy - 2yz + 2zx = (u - v)^2 + (x - y + z)^2 + w^2 - 2(u + v)w.$$

Since $u + v = -w$, this becomes

$$(u - v)^2 + (x - y + z)^2 + 3w^2.$$

Hence

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2(ab + bc + cd + de + ef + fa) = 2((u - v)^2 + (x - y + z)^2 + 3w^2) \geq 0,$$

and the conclusion follows.

Solution 4 by Kousik Sett, India

Let

$$S = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \quad \text{and} \quad P = 2(ab + bc + cd + de + ef + fa).$$

So, it suffice to prove

$$S \geq P.$$

Consider the following identity:

$$(a - b + c)^2 + (c - d + e)^2 + (e - f + a)^2 = S - P + (a + c + e)^2.$$

After rearranging, we obtain

$$S - P = (a - b + c)^2 + (c - d + e)^2 + (e - f + a)^2 - (a + c + e)^2.$$

Assume

$$x = a - b + c, \quad y = c - d + e, \quad z = e - f + a.$$

Since $a + b + c + d + e + f = 0$, we have

$$x + y + z = 2(a + c + e) - (b + d + e) = 3(a + c + e).$$

The following inequality:

$$3(x^2 + y^2 + z^2) \geq (x + y + z)^2$$

holds for real x, y, z .

Substituting x, y, z , and $x + y + z$ into the above inequality, we obtain

$$3[(a - b + c)^2 + (c - d + e)^2 + (e - f + a)^2] \geq 9(a + c + e)^2.$$

Dividing by 3, we obtain

$$(a - b + c)^2 + (c - d + e)^2 + (e - f + a)^2 \geq 3(a + c + e)^2.$$

Therefore,

$$S - P \geq 3(a + c + e)^2 - (a + c + e)^2 = 2(a + c + e)^2 \geq 0.$$

Thus, we conclude that

$$S - P \geq 0 \Rightarrow S \geq P.$$

This completes the proof.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Hayoon Kwon, Korean Minjok Leadership Academy, South Korea; Kevin Soto Palacios, National University of Callao (UNAC), Peru; Theo Koupelis, Clark College, Washington, USA; Zon Zon, Kweilin, China; Aaditya Bhaumik Sahu, AMMOC, India; Parth Andhare, AMMOC; Shilong Zhuang, California, USA.

S732. Find all positive integers n with the following property: there is a positive integer k and 3-variable polynomials $f_1, \dots, f_k$ with real coefficients such that for all real numbers a, b, c

$$a(a+b)^n + b(b+c)^n + c(c+a)^n = f_1(a, b, c)^2 + \dots + f_k(a, b, c)^2.$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

Solution by the authors

We will prove that the solutions are $n = 1, 3, 5$. Assume first that n is a solution of the problem and let $P_n(X, Y, Z) = X(X+Y)^n + Y(Y+Z)^n + Z(Z+X)^n$. By assumption P takes nonnegative values on $\mathbb{R}^3$. If n is even, then P_n is odd and since P_n is obviously not zero, it cannot take only nonnegative values. Thus n must be odd. Next, taking $a = -1, c = -2$ yields $b(b-2)^n \geq (b-1)^n - 2 \cdot 3^n$ for all b . Choosing $b = 5$ gives $7 \cdot 3^n \geq 4^n$, which forces $n < 7$. Thus we must have $n \in \{1, 3, 5\}$.

Next, we show that $1, 3, 5$ are solutions. We have

$$P_1(a, b, c) = a^2 + b^2 + c^2 + ab + bc + ca = \frac{(a+b)^2 + (b+c)^2 + (c+a)^2}{2}.$$

To treat the cases $n = 3, 5$ it is convenient to make the change of variables $a+b = 2x, b+c = 2y, c+a = 2z$. Then a direct computation shows that

$$P_n(a, b, c) = 2^n [x^{n+1} + y^{n+1} + z^{n+1} + xy^n - yx^n + yz^n - y^n z + zx^n - z^n x].$$

It suffices therefore to show that the polynomial expression $Q_n(x, y, z)$ under the bracket is a sum of squares of polynomials with real coefficients. For this, it suffices to see that $\frac{a^{n+1}+b^{n+1}}{2} + ab^n - a^n b$ is a sum of squares of polynomials. By homogeneity, it suffices to show that $\frac{X^{n+1}+1}{2} + X - X^n$ is a sum of squares. Using that $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$ we obtain

$$\frac{X^4+1}{2} + X - X^3 = \frac{X^4+1}{2} + \frac{(X+1-X^2)^2 - X^2 - (1-X^2)^2}{2} = \frac{X^2 + (X+1-X^2)^2}{2},$$

dealing with the case $n = 3$. As for $n = 5$, we have

$$\begin{aligned} \frac{X^6+1}{2} + X - X^5 &= \frac{(X^2+1)(X^4-X^2+1)}{2} + X(1-X^2)(X^2+1) = \\ &= (X^2+1) \left[\frac{X^4-X^2+1}{2} + X - X^3 \right] = \frac{1}{2}(X^2+1)(X^2-X-1)^2. \end{aligned}$$

Undergraduate problems

U727. Let $\{a_n\}_{n \geq 1}$ be a sequence defined by $a_1 = 2$ and $a_n = a_{n-1}^2 - a_{n-1} + 1$ for $n \geq 2$. Evaluate

$$\lim_{n \rightarrow \infty} \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

From $a_n = a_{n-1}^2 - a_{n-1} + 1$ we get

$$a_n - 1 = a_{n-1}(a_{n-1} - 1).$$

This implies

$$\frac{1}{a_n - 1} = \frac{1}{a_{n-1}(a_{n-1} - 1)} = \frac{1}{a_{n-1} - 1} - \frac{1}{a_{n-1}},$$

or

$$\frac{1}{a_n - 1} - \frac{1}{a_{n-1} - 1} = -\frac{1}{a_{n-1}}.$$

Similarly

$$\begin{aligned} \frac{1}{a_{n-1} - 1} - \frac{1}{a_{n-2} - 1} &= -\frac{1}{a_{n-2}}, \\ &\dots \quad \dots \quad \dots, \\ \frac{1}{a_2 - 1} - \frac{1}{a_1 - 1} &= -\frac{1}{a_1}. \end{aligned}$$

Summing up these relations we obtain

$$\frac{1}{a_n - 1} - \frac{1}{a_1 - 1} = -\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} \right).$$

Consequently

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1 - \frac{1}{a_{n+1} - 1}.$$

It's easy to check that $a_n > 1$ for all n . Furthermore

$$a_n - a_{n-1} = (a_{n-1} - 1)^2 > 0 \quad \forall n.$$

Hence $\{a_n\}$ is a increasing sequence. Now if $\{a_n\}$ is upper bounded, then there exists $\ell = \lim_{n \rightarrow \infty} a_n$ where ℓ is finite. Since $a_n > a_1 = 2$ for all n , then $\ell \geq 2$. From $a_n = a_{n-1}^2 - a_{n-1} + 1$ we take $n \rightarrow \infty$ to get $\ell = \ell^2 - \ell + 1$. This yields $(\ell - 1)^2 = 0$ or $\ell = 1$ which is a contradiction. So, $\lim_{n \rightarrow \infty} a_n = \infty$. From here we obtain

$$\lim_{n \rightarrow \infty} \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{a_{n+1} - 1} \right) = 1.$$

Solution 2 by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let $b_n = a_n - 1$. Then

$$b_n + 1 = a_n = (b_{n-1} + 1)^2 - (b_{n-1} + 1) + 1 \implies b_n = b_{n-1}^2 + b_{n-1}, \quad b_1 = 1.$$

Clearly b_n is strictly increasing $b_n = b_{n-1}^2 + b_{n-1} > b_{n-1}$ and unbounded because $b_1 > 0$. Moreover

$$\frac{1}{b_{n+1}} = \frac{1}{b_n} - \frac{1}{1 + b_n},$$

thus telescoping

$$\sum_{n=1}^N \frac{1}{a_n} = \sum_{n=1}^N \frac{1}{1+b_n} = \sum_{n=1}^N \left(\frac{1}{b_n} - \frac{1}{b_{n+1}} \right) \implies \frac{1}{b_1} - \frac{1}{b_{N+1}} = \sum_{n=1}^N \frac{1}{1+b_n},$$

whence

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{a_n} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{1+b_n} = \lim_{N \rightarrow \infty} \left(\frac{1}{b_1} - \frac{1}{b_{N+1}} \right) = \frac{1}{b_1} = 1.$$

Solution 3 by Prodromos Fotiadis, University of Crete, Greece

It is easy to see that $(a_n)_{n \geq 1}$ is a strictly increasing sequence of positive integers. We define

$$b_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{a_{n+1} - 1}.$$

We now observe that

$$\frac{1}{a_n} + \frac{1}{a_{n+1} - 1} = \frac{1}{a_n} + \frac{1}{a_n^2 - a_n} = \frac{a_n - 1}{a_n(a_n - 1)} + \frac{1}{a_n(a_n - 1)} = \frac{1}{a_n - 1}$$

and thus,

$$b_n = b_{n-1} = b_{n-2} = \dots = b_1 = \frac{1}{a_1} + \frac{1}{a_2 - 1} = \frac{1}{2} + \frac{1}{3 - 1} = 1.$$

It follows that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{a_{n+1} - 1} \right) = 1.$$

Solution 4 by G. C. Greubel, Newport News, VA, USA

The sequence defined by $a_1 = 2$ and $a_n = a_{n-1}^2 - a_{n-1} + 1$ is known as Sylvester's sequence and is sequence A000058 in the Online Encyclopedia of Integer Sequences (OEIS). The series $\sum_{k=1}^n \frac{1}{a_k}$ is the greedy Egyptian fractions representation of 1.

For the purpose of this problem note that

$$a_n \in \{2, 3, 7, 43, 1807, 3263443, 10650056950807, \dots\}_{k \geq 1}$$

and

$$\begin{aligned} a_2 &= a_1 + 1 \\ a_3 &= a_1 a_2 + 1 \\ a_4 &= a_1 a_2 a_3 + 1 \\ &\dots = \dots \\ a_n &= \prod_{j=1}^{n-1} a_j + 1, \end{aligned}$$

and when $S_n = \sum_{k=1}^n \frac{1}{a_k}$,

$$\begin{aligned} S_1 &= \frac{1}{a_1} = \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 1 - \frac{1}{6} = 1 - \frac{1}{a_3 - 1} \\ S_3 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42} = 1 - \frac{1}{a_4 - 1}. \end{aligned}$$

Supposing that

$$S_n = 1 - \frac{1}{a_{n+1} - 1}$$

is true, then

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{a_{n+1}} = 1 - \frac{1}{a_{n+1} - 1} + \frac{1}{a_{n+1}} \\ &= 1 - \frac{a_{n+1} - (a_{n+1} - 1)}{a_{n+1}(a_{n+1} - 1)} = 1 - \frac{1}{a_{n+1}^2 - a_{n+1}} \\ &= 1 - \frac{1}{a_{n+2} - 1}, \end{aligned}$$

which shows that the formula holds for all $n \geq 1$. Since $a_n = \prod_{j=1}^{n-1} a_j + 1$, then

$$S_n = \sum_{k=1}^n \frac{1}{a_k} = 1 - \frac{1}{\prod_{j=1}^{n-1} a_j}.$$

Now, taking the desired we obtain that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{a_k} = 1.$$

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; Hayoon Kwon, Korean Minjok Leadership Academy, South Korea; Kousik Sett, India; Luke Paluso, Blinn College, Brenham, TX, USA; Prakash Pant, The University of Vermont, Bardiya, Nepal; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Aaditya Bhaumik Sahu, AMMOC, India; Diar Gashi, University of Primorska, Slovenia; Ervin Ramirez, Richland Center, Wisconsin, USA; Ganghun Kim, Fort Lee High School, NJ, USA; Shilong Zhuang, California, USA; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

U728. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x)f(-x) = 4$ for all $x \in \mathbb{R}$. Evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(1 + 3 \sin^2 x)(2 + f(x))} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Daniel Pascuas, Barcelona, Spain

We assume that the integral I in the statement is well defined as either a Riemann or Lebesgue integral. The substitution $y = -x$ and the hypothesis $f(x)f(-x) = 4$, for all $x \in \mathbb{R}$, yield that

$$\begin{aligned} I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dy}{(1 + 3 \sin^2 y)(2 + f(y))} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + 3 \sin^2 x)(2 + f(-x))} \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x) dx}{(1 + 3 \sin^2 x)(2f(x) + 4)} = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x) dx}{(1 + 3 \sin^2 x)(2 + f(x))}. \end{aligned}$$

Therefore

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x) dx}{(1 + 3 \sin^2 x)(2 + f(x))},$$

so

$$\begin{aligned} I &= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x) dx}{(1 + 3 \sin^2 x)(2 + f(x))} + \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2 dx}{(1 + 3 \sin^2 x)(2 + f(x))} \\ &= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1 + 3 \sin^2 x} = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{dx}{1 + 3 \sin^2 x}. \end{aligned}$$

Then perform the substitution $t = \tan x$ to get that $dx = \frac{dt}{1 + t^2}$ and $\sin^2 x = \frac{t^2}{1 + t^2}$, so

$$I = \frac{1}{2} \int_0^1 \frac{dt}{1 + 4t^2} = \frac{1}{4} \left[\arctan(2t) \right]_0^1 = \frac{\arctan 2}{4} = 0.27679 \dots$$

Solution 2 by Prodromos Fotiadis, University of Crete, Greece

We have

$$\begin{aligned} I &\equiv \int_{-\pi/4}^{\pi/4} \frac{1}{(1 + 3 \sin^2 x)(2 + f(x))} dx \\ &= \int_0^{\pi/4} \frac{1}{1 + 3 \sin^2 x} \left[\frac{1}{2 + f(-x)} + \frac{1}{2 + f(x)} \right] dx. \end{aligned}$$

Given $f(x)f(-x) = 4$, we have

$$\begin{aligned} \frac{1}{2 + f(-x)} + \frac{1}{2 + f(x)} &= \frac{4 + f(-x) + f(x)}{(2 + f(-x))(2 + f(x))} \\ &= \frac{4 + f(-x) + f(x)}{4 + 2f(x) + 2f(-x) + 4} = \frac{1}{2}. \end{aligned}$$

This yields:

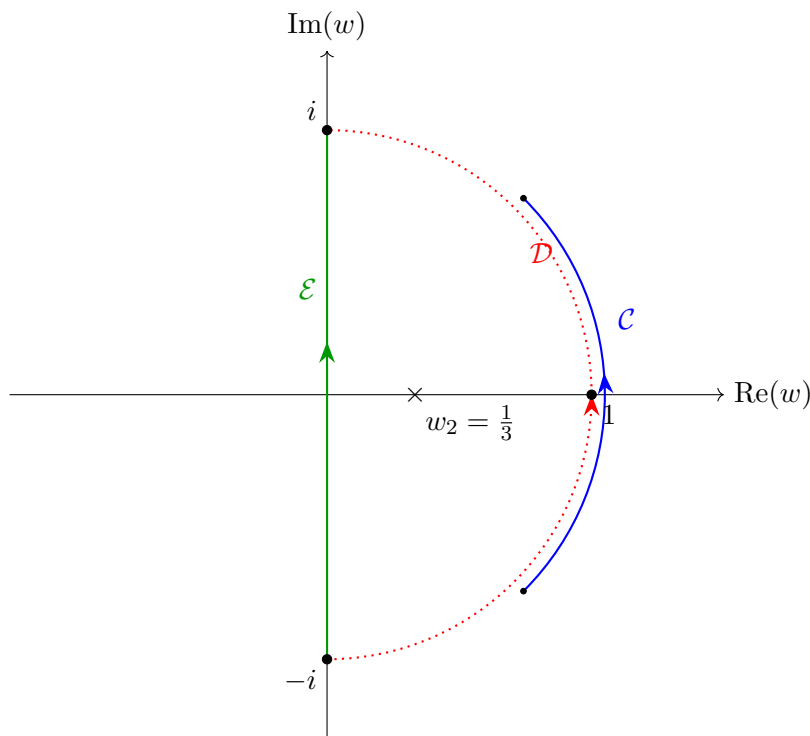
$$I = \frac{1}{2} \int_0^{\pi/4} \frac{1}{1 + 3 \sin^2 x} dx = \frac{1}{4} \int_{-\pi/4}^{\pi/4} \frac{1}{1 + 3 \sin^2 x} dx. \quad (1)$$

Let $z = e^{ix}$, then $\sin x = \frac{1}{2i}(z - z^{-1})$ and $dx = \frac{dz}{iz}$. The integral becomes:

$$I = \frac{1}{4} \int_{\mathcal{C}} \frac{1}{1 - \frac{3}{4} \left(z - \frac{1}{z}\right)^2} \frac{dz}{iz} = \frac{i}{2} \int_{\mathcal{C}} \frac{1}{3z^4 - 10z^2 + 3} d(z^2) \quad (2)$$

$$\implies I = \frac{i}{2} \int_{\mathcal{D}} g(w) dw, \quad g(w) \equiv \frac{1}{3w^2 - 10w + 3} \quad (3)$$

where $\mathcal{C}$ is the arc $z = e^{ix}$ for $x \in [-\pi/4, \pi/4]$ and $\mathcal{D} = \mathcal{C}^2$.



We now observe that

$$3w^2 - 10w + 3 = 3(w - 3) \left(w - \frac{1}{3}\right) \quad (4)$$

hence, $g(w)$ has simple poles at $w_1 = 3$ and $w_2 = \frac{1}{3}$. Using the Cauchy Residue Theorem:

$$\int_{\mathcal{D}} g(w) dw = 2\pi i \operatorname{Res}(g, 1/3) + \int_{\mathcal{E}} g(w) dw \quad (5)$$

We compute

$$\operatorname{Res}(g, 1/3) = \frac{1}{3(w - 3)} \Big|_{w=1/3} = -\frac{1}{8} \quad (6)$$

and

$$\begin{aligned}
\int_{\mathcal{E}} g(w)dw &= \int_{-1}^1 \frac{i}{3(ix-3)(ix-\frac{1}{3})} dx = \frac{1}{8} \int_{-1}^1 \left[\frac{1}{x+3i} - \frac{1}{x+\frac{i}{3}} \right] dx \\
&= \frac{1}{8} \left[\ln(1+3i) - \ln(-1+3i) - \ln\left(1+\frac{i}{3}\right) + \ln\left(-1+\frac{i}{3}\right) \right] \\
&= \frac{1}{8} \left[\ln\sqrt{10} + i \arctan(3) - \ln\sqrt{10} - i(\pi - \arctan(3)) \right. \\
&\quad \left. - \ln\frac{\sqrt{10}}{3} - i \arctan\left(\frac{1}{3}\right) + \ln\frac{\sqrt{10}}{3} + i\left(\pi - \arctan\left(\frac{1}{3}\right)\right) \right] \\
&= \frac{1}{8} \left[i \arctan(3) - i\pi + i \arctan(3) - i\left(\frac{\pi}{2} - \arctan(3)\right) + i\pi - i\left(\frac{\pi}{2} - \arctan(3)\right) \right] \\
&= \frac{i}{8} [4 \arctan(3) - \pi]
\end{aligned}$$

It follows from (3) and (5) that

$$I = \frac{i}{2} \left(-\frac{1}{4}\pi i + \frac{i}{8} [4 \arctan(3) - \pi] \right) \implies I = \frac{1}{4} \left(\frac{3\pi}{4} - \arctan(3) \right) = \frac{1}{4} \arctan(2). \quad (7)$$

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Anita Tufisi, Romania; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, New York, USA; Kousik Sett, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prakash Pant, The University of Vermont, Bardiya, Nepal; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Sundaresh H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Diar Gashi, University of Primorska, Slovenia; Ervin Ramirez, Richland Center, Wisconsin, USA; Hoang-Bao Le, Dublin, Ireland; Parth Andhare, AMMOC; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania; William Han, Horace Mann School, Bronx, New York, USA.

U729. Determine all natural numbers $n \geq 2$ for which the following statement holds:

For every $A, B \in M_n(\mathbb{C})$ such that $A^n = O_n$ and $\det(A + B) = \det B = 0$, it follows that $BAB = O_n$.

Proposed by Florin Stănescu, Găești, Romania

Solution 1 by the author

We claim that the only such natural number is $n = 2$.

First, let $n = 2$. Suppose that $A, B \in M_2(\mathbb{C})$ satisfy $A^2 = O_2$ and $\det(A + B) = \det B = 0$.

Since $A^2 = O_2$, we have $\text{tr} A = \det A = 0$. By the Cayley-Hamilton theorem applied to $A + B$, we get

$$(A + B)^2 - \text{tr}(A + B)(A + B) + \det(A + B)I_2 = O_2.$$

Since $\det(A + B) = 0$ and $\text{tr}(A + B) = \text{tr} B$, this becomes

$$(A + B)^2 - \text{tr}(B)(A + B) = O_2.$$

Expanding,

$$A^2 + AB + BA + B^2 - \text{tr}(B)A - \text{tr}(B)B = O_2.$$

Also, by Cayley-Hamilton applied to B , and using $\det B = 0$, we have

$$B^2 - \text{tr}(B)B = O_2.$$

Since $A^2 = O_2$, it follows that

$$AB + BA = \text{tr}(B)A. \quad (1)$$

Multiplying (1) on the left by A , we obtain

$$AAB + ABA = \text{tr}(B)A^2,$$

hence

$$ABA = O_2. \quad (2)$$

Now we use Frobenius' rank inequality in the form

$$\text{rank}(XY) + \text{rank}(YZ) \leq \text{rank}(Y) + \text{rank}(XYZ).$$

Taking $X = A$, $Y = B$, and $Z = A$, we get

$$\text{rank}(AB) + \text{rank}(BA) \leq \text{rank}(B) + \text{rank}(ABA).$$

Since $\det B = 0$, we have $\text{rank}(B) \leq 1$, and by (2), $\text{rank}(ABA) = 0$. Therefore

$$\text{rank}(AB) + \text{rank}(BA) \leq 1.$$

Thus at least one of AB and BA is the zero matrix. If $AB = O_2$, then $BAB = B(AB) = O_2$. If $BA = O_2$, then $BAB = (BA)B = O_2$. Hence the statement is true for $n = 2$.

We now show that the statement is false for every $n \geq 3$. Let

$$A = E_{1n}, \quad B = E_{11} + E_{nn},$$

where E_{ij} denotes the $n \times n$ matrix with 1 in the (i, j) -entry and 0 elsewhere. Since $A^2 = O_n$, we have $A^n = O_n$. Also, $\text{rank}(B) = 2 < n$, so $\det B = 0$. Moreover, $\text{rank}(A + B) \leq 2 < n$, hence $\det(A + B) = 0$. However,

$$BAB = (E_{11} + E_{nn})E_{1n}(E_{11} + E_{nn}) = E_{1n} = A \neq O_n.$$

Thus the implication fails for every $n \geq 3$.

Therefore, the only natural number $n \geq 2$ for which the statement holds is $n = 2$.

Solution 2 by Diar Gashi, University of Primorska, Slovenia

The only such n is $n = 2$.

Firstly, we claim that the statement is false for every $n \geq 3$.

We can take $A = E_{1,2}$ and $B = E_{2,1}$, where $E_{i,j}$ denotes the matrix with 1 in the (i, j) -entry and 0 elsewhere. Then, it is easy to see that:

- $A^2 = O_n$, so $A^n = O_n$ for all $n \geq 3$.
- $\det B = 0$ because B has rank 1 and $n \geq 3$.
- For $n \geq 3$, the matrix $A + B$ has at most two non-zero entries and at least one zero row, hence $\det(A + B) = 0$.
- $BAB = E_{2,1}E_{1,2}E_{2,1} = E_{2,2}E_{2,1} = E_{2,1} \neq O_n$.

Therefore the conclusion $BAB = O_n$ fails. Hence the statement is false for every $n \geq 3$.

Now, let $A, B \in M_2(\mathbb{C})$ satisfy $A^2 = O_2$, $\det(A + B) = \det B = 0$. If $A = O_2$, then $BAB = O_2$ trivially. Otherwise, A is a non-zero nilpotent 2×2 matrix, hence similar to the Jordan block

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence there exists an invertible matrix P such that $P^{-1}AP = J$. Let's define $B' = P^{-1}BP$. Thus,

$$A^2 = O_2 \iff O_2 = (P^{-1}AP)^2 = J^2,$$

$$0 = \det(A + B) = \det(P^{-1}(A + B)P) = \det(J + B'),$$

$$\det B = 0 = \det B'.$$

The desired conclusion $BAB = O_2$ is equivalent to

$$O_2 = P^{-1}(BAB)P = (P^{-1}BP)(P^{-1}AP)(P^{-1}BP) = B'JB'.$$

Thus, the statement holds for the original pair (A, B) if and only if it holds for (J, B') . Write

$$B' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The hypotheses give

$$ad - bc = \det B' = 0, \quad \det(J + B') = \det \begin{pmatrix} a & b+1 \\ c & d \end{pmatrix} = ad - c(b+1) = (ad - bc) - c = -c = 0.$$

Hence $c = 0$, and then $ad = 0$. Now we see that:

$$B'JB' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ad \\ 0 & 0 \end{pmatrix} = O_2.$$

Thus, the statement holds for $n = 2$.

Also solved by Prakash Pant, The University of Vermont, Bardiya, Nepal; The Ring Lords, Baden-Württemberg, Germany.

U730. Prove that 2 is the least value of the exponent k such that

$$\frac{a^4 + b^4 + c^4}{3} \leq \left(\frac{a^3 + b^3 + c^3}{3} \right)^k$$

for any nonnegative real numbers a, b, c with at most one of them less than 1 and $a + b + c = 3$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution by the author

For $b = 1$, $a = 1 + \sqrt{t}$ and $c = 1 - \sqrt{t}$, where $t \in [0, 1]$, the constraints are satisfied and the inequality becomes $f(t) \leq 0$, where

$$f(t) = \frac{3 + 12t + 2t^2}{3} - (1 + 2t)^k.$$

We have

$$f(0) = 0, \quad f'(t) = \frac{12 + 4t}{3} - 2k(1 + 2t)^{k-1}, \quad f'(0) = 4 - 2k.$$

Since $f(0) = 0$, the condition $f'(0) \leq 0$ is necessary to have $f(t) \leq 0$ for $t \in [0, 1]$. So, we get $k \geq 2$. To show that 2 is the least value of k we need to prove the inequality $F \geq 0$, where

$$F = (a^3 + b^3 + c^3)^2 - 3(a^4 + b^4 + c^4).$$

Assume that $a \geq b \geq 1 \geq c$. In addition, for fixed a , assume that c and F are functions of b . We have $c' = -1$ and

$$\frac{F'(b)}{6} = b^2(a^3 + b^3 + c^3) - 2b^3 + [c^2(a^3 + b^3 + c^3) - 2c^3]c' = (b^2 - c^2)(a^3 + b^3 + c^3) - 2(b^3 - c^3).$$

We will show that $F'(b) \geq 0$. This is true if

$$(b + c)(a^3 + b^3 + c^3) \geq 2(b^2 + bc + c^2),$$

which is equivalent to the homogeneous inequality

$$9(b + c)(a^3 + b^3 + c^3) \geq 2(b^2 + bc + c^2)(a + b + c)^2.$$

Since

$$9(a^3 + b^3 + c^3) \geq (a + b + c)^3,$$

it suffices to show that $(b + c)(a + b + c) \geq 2(b^2 + bc + c^2)$. Indeed, we have

$$(b + c)(a + b + c) - 2(b^2 + bc + c^2) \geq (b + c)(2b + c) - 2(b^2 + bc + c^2) = bc - c^2 \geq 0.$$

From $F'(b) \geq 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. So, it suffices to prove the required inequality for $b = 1$. We need to show that $a + c = 2$ implies $F \geq 0$, where

$$F = (a^3 + c^3 + 1)^2 - 3(a^4 + c^4 + 1).$$

Let $x = ac$, $x \leq 1$. Since

$$\begin{aligned} a^3 + c^3 &= (a + c)^3 - 3ac(a + c) = 8 - 6x, \\ a^4 + c^4 &= (a^2 + c^2)^2 - 2a^2c^2 = 4(2 - x)^2 - 2x^2 = 2x^2 - 16x + 16, \end{aligned}$$

we have

$$F = (9 - 6x)^2 - 3(2x^2 - 16x + 17) = 30(x - 1)^2 \geq 0.$$

For $k = 2$, the equality occurs for $a = b = c = 1$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Theo Koupelis, Clark College, Washington, USA; Zon Zon, Kweilin, China; Parth Andhare, AMMOC.

U731. Prove

$$\int_0^1 \frac{\arcsin x}{4-x^2} dx = \frac{\pi}{12} \log(9+6\sqrt{3}) - \frac{2}{3}G.$$

Here $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant.

Proposed by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

Solution 1 by Theo Koupelis, Clark College, Washington, USA

Using the substitution $y = \arcsin x$ we get

$$\begin{aligned} I &:= \int_0^1 \frac{\arcsin x}{4-x^2} dx = \frac{1}{4} \int_0^{\pi/2} \frac{y}{2-\sin(y)} d(\sin y) + \frac{1}{4} \int_0^{\pi/2} \frac{y}{2+\sin(y)} d(\sin y) \\ &= -\frac{1}{4} [y \cdot \ln(2-\sin y)] \Big|_0^{\pi/2} + \frac{1}{4} \int_0^{\pi/2} \ln(2-\sin y) dy \\ &\quad + \frac{1}{4} [y \cdot \ln(2+\sin y)] \Big|_0^{\pi/2} - \frac{1}{4} \int_0^{\pi/2} \ln(2+\sin y) dy \\ &= \frac{\pi}{8} \ln 3 + \frac{1}{4} \int_0^{\pi/2} \ln(2-\sin y) dy - \frac{1}{4} \int_0^{\pi/2} \ln(2+\sin y) dy. \quad (1) \end{aligned}$$

Let $f(\theta) = \int_0^{\pi/2} \ln(1 + \sin \theta \cdot \sin x) dx$, where $\theta \in (-\pi/2, \pi/2)$. Then

$$\begin{aligned} f'(\theta) &= \int_0^{\pi/2} \frac{\cos \theta \cdot \sin x}{1 + \sin \theta \cdot \sin x} dx = \frac{1}{\tan \theta} \int_0^{\pi/2} \frac{\sin \theta \sin x}{1 + \sin \theta \sin x} dx \\ &= \frac{\pi}{2 \tan \theta} - \frac{1}{\tan \theta} \int_0^{\pi/2} \frac{1}{1 + \sin \theta \sin x} dx \\ &= \frac{\pi}{2 \tan \theta} - \frac{2}{\tan \theta} \int_0^{\pi/2} \frac{d \tan \frac{x}{2}}{(1 - \sin \theta) \tan^2 \frac{x}{2} + (1 + \sin \theta)} \\ &= \frac{\pi}{2 \tan \theta} - \frac{2}{\tan \theta} \cdot \frac{1}{\cos \theta} \cdot \arctan \left(\sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} \cdot \tan \frac{x}{2} \right) \Big|_0^{\pi/2} \\ &= \frac{\pi}{2 \tan \theta} - \frac{2}{\sin \theta} \cdot \arctan \left(\tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right) = \frac{\theta}{\sin \theta} - \frac{\pi}{2} \tan \frac{\theta}{2}. \end{aligned}$$

Therefore, using the substitution $\theta = 2\phi$ and integrating by parts we get

$$\begin{aligned} A &:= \int_0^{\pi/2} \ln(2 + \sin x) dx = \frac{\pi \ln 2}{2} + f\left(\frac{\pi}{6}\right) = \frac{\pi \ln 2}{2} + \int_0^{\pi/6} f'(\theta) d\theta \\ &= \frac{\pi \ln 2}{2} + 4 \int_0^{\pi/12} \frac{\phi}{\sin(2\phi)} d\phi - \pi \int_0^{\pi/12} \tan \phi d\phi \\ &= \frac{\pi \ln 2}{2} - 2\phi \ln(\cot \phi) \Big|_0^{\pi/12} + 2 \int_0^{\pi/12} \ln(\cot \phi) d\phi + \pi \ln(\cos \phi) \Big|_0^{\pi/12} \\ &= \pi \cdot \frac{\ln(2 + \sqrt{3})}{3} - \frac{\pi \ln 2}{2} + \frac{4}{3}G, \quad (2) \end{aligned}$$

where we used the lemma that $\int_0^{\pi/12} \ln(\cot t) dt = \frac{2}{3}G$.

[Proof of lemma: Using the identity $\tan(3x) = \tan(x) \tan\left(\frac{\pi}{3} + x\right) \tan\left(\frac{\pi}{3} - x\right)$, we get

$$\begin{aligned}
 S &:= \int_0^{\pi/12} \ln(\tan(3x)) \, dx = \int_0^{\pi/12} \ln(\tan x) \, dx + \int_0^{\pi/12} \ln\left(\tan\left(\frac{\pi}{3} + x\right)\right) \, dx \\
 &\quad + \int_0^{\pi/12} \ln\left(\tan\left(\frac{\pi}{3} - x\right)\right) \, dx = \int_0^{\pi/12} \ln(\tan x) \, dx + \int_{\pi/3}^{5\pi/12} \ln(\tan x) \, dx \\
 &\quad + \int_{\pi/4}^{\pi/3} \ln(\tan x) \, dx = \int_0^{\pi/12} \ln(\tan x) \, dx + \int_{\pi/4}^{5\pi/12} \ln(\tan x) \, dx \\
 &= \int_0^{\pi/12} \ln(\tan x) \, dx - \int_{\pi/4}^{\pi/12} \ln\left(\tan\left(\frac{\pi}{2} - t\right)\right) \, dt = \int_0^{\pi/12} \ln(\tan x) \, dx \\
 &\quad + \int_{\pi/4}^{\pi/12} \ln(\tan x) \, dx = 2 \int_0^{\pi/12} \ln(\tan x) \, dx - \int_0^{\pi/4} \ln(\tan x) \, dx \\
 &= 2 \int_0^{\pi/12} \ln(\tan x) \, dx - 3 \int_0^{\pi/12} \ln(\tan(3x)) \, dx = 2 \int_0^{\pi/12} \ln(\tan x) \, dx - 3S.
 \end{aligned}$$

Thus, $S = \frac{1}{2} \int_0^{\pi/12} \ln(\tan(x)) \, dx$, and using the change in variable $t = 3x$ we get

$$\int_0^{\pi/12} \ln(\cot x) \, dx = \frac{2}{3} \int_0^{\pi/4} \ln(\cot t) \, dt = \frac{2}{3} G,$$

where we used the well-known expression $G = \int_0^{\pi/4} \ln(\cot t) \, dt$.]

Working similarly as above we get

$$B := \int_0^{\pi/2} \ln(2 - \sin x) \, dx = \frac{\pi \ln 2}{2} + f\left(-\frac{\pi}{6}\right) = 2\pi \cdot \frac{\ln(2 + \sqrt{3})}{3} - \frac{\pi \ln 2}{2} - \frac{4}{3} G. \quad (3)$$

From (1), (2), (3) we get the desired result.

Solution 2 by G. C. Greubel, Newport News, VA, USA

The integral can be evaluated by using an expansion, the hypergeometric function (${}_2F_1$), and a known series value, which are

$$\begin{aligned}
 \frac{1}{1-x^2} &= \sum_{n=0}^{\infty} x^{2n} \\
 {}_2F_1(a, b; c; 1) &= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \\
 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \binom{2n}{n}} &= \frac{8G}{3} - \frac{\pi}{3} \ln(2 + \sqrt{3}).
 \end{aligned}$$

Let

$$f(x) = 2(n+1)x^{2n+1} \sin^{-1}(x) - x^{2n+2} {}_2F_1\left(\frac{1}{2}, n+1; n+2; x^2\right)$$

and note that $f(0) = 0$, and

$$\begin{aligned}
 f(1) &= 2(n+1) \sin^{-1}(1) - {}_2F_1\left(\frac{1}{2}, n+1; n+2; 1\right) \\
 &= (n+1)\pi - \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \\
 &= (n+1)\pi - \frac{2(n+1)}{2n+1} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \\
 f(1) &= (n+1)\pi - \frac{2(n+1)}{2n+1} \frac{4^n}{\binom{2n}{n}}.
 \end{aligned}$$

With this, then

$$\begin{aligned}
I &= \int_0^1 \frac{\sin^{-1}(x)}{4-x^2} dx = \frac{1}{4} \int_0^1 \frac{\sin^{-1}(x)}{1-\left(\frac{x}{2}\right)^2} dx \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{4^n} \int_0^1 x^{2n} \sin^{-1}(x) dx \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{4^n (2n+1)(2n+2)} [f(x)]_0^1 \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \frac{f(1) - f(0)}{4^n (2n+1)(2n+2)} \\
&= \frac{\pi}{8} \sum_{n=0}^{\infty} \frac{1}{4^n (2n+1)} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \binom{2n}{n}} \\
&= \frac{\pi}{8} \ln(3) - \frac{1}{4} \left(\frac{8G}{3} - \frac{\pi}{3} \ln(2+\sqrt{3}) \right) \\
&= \frac{\pi}{12} \ln(9+6\sqrt{3}) - \frac{2G}{3}.
\end{aligned}$$

This can be stated as

$$\int_0^1 \frac{\sin^{-1}(x)}{4-x^2} dx = \frac{\pi}{12} \log(9+6\sqrt{3}) - \frac{2}{3} G.$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prakash Pant, The University of Vermont, Bardiya, Nepal; The Ring Lords, Baden-Württemberg, Germany.

- U732. (a) Find all finite groups G having exactly 5 distinct subgroups $H_1, H_2, H_3, H_4, H_5 \leq G$ of the same order $m \geq 2$ such that $H_i \cap H_j = \{1\}$ for all $i \neq j$ and $G = \bigcup_{i=1}^5 H_i$.
- (b) Find all finite groups G having exactly 6 distinct subgroups $H_1, H_2, H_3, H_4, H_5, H_6 \leq G$ of the same order $m \geq 2$ such that $H_i \cap H_j = \{1\}$ for all $i \neq j$ and $G = \bigcup_{i=1}^6 H_i$.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

- (a) We will prove that there is no finite group satisfying the given properties. Because the subgroups intersect only in $\{1\}$, their union is disjoint outside $\{1\}$. Hence

$$|G| = 1 + \sum_{i=1}^5 (|H_i| - 1) = 1 + 5(m - 1) = 5m - 4.$$

For any $i \neq j$ we have $|H_i H_j| = \frac{|H_i| \cdot |H_j|}{|H_i \cap H_j|} = m^2$, so

$$m^2 \leq |G| = 5m - 4 \implies (m - 1)(m - 4) \leq 0,$$

so $m \in \{2, 3, 4\}$.

If $m = 2$, then $|G| = 6$. The only groups of order 6 are $\mathbb{Z}_6$ (one subgroup of order 2) and S_3 (three subgroups of order 2). In both cases we have no five subgroups of order 2.

If $m = 3$, then $|G| = 11$. So, $G \cong \mathbb{Z}_{11}$, which has no subgroup of order 3.

If $m = 4$, then $|G| = 16$. Thus, G is a 2-group. Since every finite 2-group has nontrivial center, $Z(G)$ is a nontrivial 2-group. Hence, by Cauchy's theorem, $Z(G)$ contains an element z of order 2. Since $G = \bigcup_{i=1}^5 H_i$, the element z belongs to some H_i . Relabeling if necessary, assume $z \in H_1$. Because $H_1 \cap H_j = \{1\}$ for every $j \neq 1$, we have $z \notin H_j$, $j = 2, 3, 4, 5$. Now take H_2 . Since $|H_2| = 4$, the subgroup H_2 contains a nonidentity element u of order 2. Clearly $u \neq z$, since $z \notin H_2$. Because z is central and both z and u have order 2, the set $K = \langle z, u \rangle$ is a subgroup of G isomorphic to V_4 . In particular, $|K| = 4$. But $K \neq H_1$, since $u \in K$ and $u \notin H_1$. Also, for every $j \neq 1$, we cannot have $K = H_j$, because $z \in K \cap H_1$ while each H_j with $j \neq 1$ intersects H_1 only in $\{1\}$.

Hence, K is a subgroup of G of order 4 different from H_1, H_2, H_3, H_4, H_5 . This contradicts the assumption that G has exactly five subgroups of order 4.

Therefore, the case $m = 4$ is impossible.

- (b) Because the subgroups intersect only in $\{1\}$, their union is disjoint outside $\{1\}$. Hence

$$|G| = 1 + \sum_{i=1}^6 (|H_i| - 1) = 1 + 6(m - 1) = 6m - 5.$$

For any $i \neq j$ we have $|H_i H_j| = \frac{|H_i| \cdot |H_j|}{|H_i \cap H_j|} = m^2$, so

$$m^2 \leq |G| = 6m - 5 \implies (m - 1)(m - 5) \leq 0,$$

so $m \in \{2, 3, 4, 5\}$.

If $m = 2, 3, 4$, we get $|G| = 7, 13, 19$, respectively. Each is prime, so G would be cyclic of prime order and has no subgroup of order 2, 3, or 4, contradiction.

So, $m = 5$ and $|G| = 25$. A group of order p^2 , where p is prime, is abelian. So $G \cong \mathbb{Z}_{25}$ or $G \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. In $\mathbb{Z}_{25}$ there is a unique subgroup of order 5. In $\mathbb{Z}_5 \times \mathbb{Z}_5$ the subgroups of order 5 are exactly the 1-dimensional subspaces of the 2-dimensional $\mathbb{F}_5$ -vector space. Their number is $\frac{5^2 - 1}{5 - 1} = 6$. Any two distinct such subgroups intersect trivially and their union is all of $\mathbb{Z}_5 \times \mathbb{Z}_5$.

So, the unique solution (up to isomorphism) is $G \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

Remark

The results that we have proved (finite groups that admit a partition by subgroups all of the same order) are part of a more general result. Indeed, the groups we have presented were classified by I. M. Isaacs in his article *Equally partitioned groups*, Pacific J. Math. **49** (1973), 109-116. Isaacs proved that a finite group can be partitioned by subgroups of equal order if and only if it is a p -group of exponent p . In particular $m = p$ and the number of parts is $t = (|G| - 1)/(p - 1)$. This sits inside the broader “groups with partitions” line of work initiated by Miller and developed/classified by Baer-Kegel-Suzuki (for general partitions, not necessarily equal size). Good overviews include:

- Baer, R., *Partitioner endlicher Gruppen*, Math. Z. **75** (1960/1961), 333-372.
- Suzuki, M. *On a finite group with a partition*, Arch. Math. **12** (1961), 241-274.

Olympiad problems

O727. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{b+c} + \frac{b}{c+a}\right) \left(\frac{b}{c+a} + \frac{c}{a+b}\right) \left(\frac{c}{a+b} + \frac{a}{b+c}\right) + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution 1 by Theo Koupelis, Clark College, Washington, USA

Let $x = \frac{a}{b+c}$, $y = \frac{b}{c+a}$, $z = \frac{c}{a+b}$. Then the desired inequality becomes

$$(x+y)(y+z)(z+x) + 8xyz \geq 2.$$

We first note that $xy + yz + zx + 2xyz = 1$. Indeed,

$$\begin{aligned} xy + yz + zx + 2xyz &= \frac{ab}{(b+c)(c+a)} + \frac{bc}{(c+a)(a+b)} + \frac{ca}{(a+b)(b+c)} + \frac{2abc}{(a+b)(b+c)(c+a)} \\ &= \frac{ab(a+b) + bc(b+c) + ca(c+a) + 2abc}{(a+b)(b+c)(c+a)} \\ &= 1. \end{aligned}$$

Denote

$$p = x + y + z, \quad q = xy + yz + zx, \quad r = xyz.$$

Then $q + 2r = 1$. Also,

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz = pq - r.$$

Hence the desired inequality is equivalent to $pq - r + 8r \geq 2$, that is,

$$pq + 7r \geq 2. \tag{1}$$

Since $q + 2r = 1$, we have $r = (1 - q)/2$. Thus (1) is equivalent to $pq + \frac{7(1-q)}{2} \geq 2$, or

$$q(7 - 2p) \leq 3. \tag{2}$$

By Nesbitt's inequality, $p = x + y + z \geq \frac{3}{2}$.

If $p \geq \frac{7}{2}$, then $7 - 2p \leq 0$, and (2) is immediate.

If $2 \leq p < \frac{7}{2}$, then $0 < 7 - 2p \leq 3$, while $q < 1$, so again $q(7 - 2p) < 3$.

It remains to consider the case $\frac{3}{2} \leq p < 2$.

For positive real numbers x, y, z , we have that the inequality $(x-y)^2(y-z)^2(z-x)^2 \geq 0$ can be written as

$$p^2q^2 - 4q^3 + 18pqr - 4p^3r - 27r^2 \geq 0.$$

Substituting $r = \frac{1-q}{2}$, we get

$$(2p + q + 3) [-16q^2 + q(4p^2 - 4p + 21) - 4p^2 + 6p - 9] \geq 0.$$

Since $2p + q + 3 > 0$, it follows that

$$16q^2 - q(4p^2 - 4p + 21) + 4p^2 - 6p + 9 \leq 0. \tag{3}$$

The discriminant of this quadratic in q is $\Delta = (2p - 3)^3(2p + 5)$. Thus, from (3),

$$q \leq \frac{4p^2 - 4p + 21 + \sqrt{(2p - 3)^3(2p + 5)}}{32}.$$

Therefore, it is enough to prove that

$$\frac{4p^2 - 4p + 21 + \sqrt{(2p - 3)^3(2p + 5)}}{32} \leq \frac{3}{7 - 2p}.$$

This is equivalent to

$$(7 - 2p)\sqrt{(2p - 3)^3(2p + 5)} \leq (2p - 3)(4p^2 - 12p + 17).$$

Both sides are nonnegative for $\frac{3}{2} \leq p < 2$. Squaring, this reduces to

$$(2p - 3)^2(4p^2 - 12p + 17)^2 - (7 - 2p)^2(2p - 3)^3(2p + 5) \geq 0,$$

or

$$256(p - 2)^2(2p - 3)^2 \geq 0,$$

which is true.

Hence (2) holds, and therefore $pq + 7r \geq 2$. Thus

$$(x + y)(y + z)(z + x) + 8xyz \geq 2.$$

Equality holds when $x = y = z = \frac{1}{2}$, which corresponds to $a = b = c$.

If we were to allow that at most one of the variables a, b, c were zero, then equality would also occur when $(p, q, r) = (2, 1, 0)$ or $(a, b, c) = (t, t, 0)$ and cyclic permutations, where t is a positive real number.

Solution 2 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Set $x = \frac{a}{b+c}$, $y = \frac{b}{c+a}$, $z = \frac{c}{a+b}$. Then the left-hand side becomes

$$E = (x + y)(y + z)(z + x) + 8xyz.$$

Also, $xy + yz + zx + 2xyz = 1$. Indeed,

$$xy + yz + zx + 2xyz = \frac{ab(a + b) + bc(b + c) + ca(c + a) + 2abc}{(a + b)(b + c)(c + a)} = 1.$$

Thus it remains to prove that

$$(x + y)(y + z)(z + x) + 8xyz \geq 2$$

for positive real numbers x, y, z satisfying $xy + yz + zx + 2xyz = 1$.

The inequality and the condition are both symmetric in x, y, z . Hence, by the standard uvw principle, it is enough to consider the case $y = z = t$, $x = s$. Then the condition gives

$$2st + t^2 + 2st^2 = 1.$$

Hence

$$2st(1 + t) = 1 - t^2 \implies s = \frac{1 - t}{2t}.$$

Since $s, t > 0$, we have $0 < t < 1$. Now

$$E = (s + t)(t + t)(t + s) + 8st^2 = 2t(s + t)^2 + 8st^2.$$

Substituting $s = \frac{1-t}{2t}$, we obtain

$$E = 2 + \frac{(1 - t)^2(2t - 1)^2}{2t} \geq 2.$$

Therefore,

$$(x + y)(y + z)(z + x) + 8xyz \geq 2.$$

Equality holds when $t = s = 1/2$, i.e. $x = y = z = \frac{1}{2}$, which corresponds to $a = b = c$.

Also solved by Arkady Alt, San Jose, California, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; G. C. Greubel, Newport News, VA, USA; The Ring Lords, Baden-Württemberg, Germany; Aaditya Bhaumik Sahu, AMMOC, India; Hoang-Bao Le, Dublin, Ireland; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania.

O728. Find all positive integers n such that $1 + 2 + 2^2 + \cdots + 2^n$ divides $(1 + 2)(1 + 2^2) \cdots (1 + 2^n) - 1$.

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

Solution 1 by the authors

Clearly $n = 1$ is not a solution, so from now on we assume that $n > 1$. We will prove that n is a solution of the problem if and only if $n + 1$ is a prime.

Let μ_n^* be the set of primitive n th roots of unity and let

$$\Phi_n(X) = \prod_{z \in \mu_n^*} (X - z).$$

We have $\prod_{d|n} \Phi_d(X) = X^n - 1$, which easily implies (by strong induction) that Φ_n has integer coefficients. We claim that for any even positive integer n we have

$$\Phi_{n+1}(X) \mid (1 + X)(1 + X^2) \cdots (1 + X^n) - 1$$

in $\mathbf{Z}[X]$. Since both polynomials are monic, with integer coefficients, and since $\Phi_{n+1}(X)$ has no multiple roots, it suffices to check that $(1 + X)(1 + X^2) \cdots (1 + X^n) - 1$ vanishes at all $z \in \mu_{n+1}^*$. This is equivalent to $\prod_{i=1}^n (1 - z^i) = \prod_{i=1}^n (1 - z^{2^i})$ and it is clear since $n + 1$ is odd, thus the remainders mod $n + 1$ of the numbers $2, 4, \dots, 2n$ are a permutation of $1, 2, \dots, n$.

Coming back to the problem, suppose first that $n + 1 = p$ is a prime. Then $\Phi_{n+1}(2) \mid (1 + 2)(1 + 2^2) \cdots (1 + 2^n) - 1$ and it suffices to observe that $\Phi_{n+1}(2) = 2^{n+1} - 1 = 1 + 2 + \cdots + 2^n$ since $\Phi_p(X) = \frac{X^p - 1}{X - 1}$ for any prime p .

For the harder implication, suppose that $2^{n+1} - 1$ divides $(1 + 2)(1 + 2^2) \cdots (1 + 2^n) - 1$. Clearly $2^{n+1} - 1$ is not a multiple of 3, thus n must be even. Let p be the smallest prime factor of $n + 1$ and write $n + 1 = kp$. We have

$$\Phi_k(2) \mid 2^k - 1 \mid 2^{n+1} - 1 \mid (1 + 2)(1 + 2^2) \cdots (1 + 2^n) - 1 = (1 + 2)(1 + 2^2) \cdots (1 + 2^{kp-1}) - 1.$$

On the other hand $\Phi_k(2) \mid 2^k - 1$, thus (using that $(1 + 2)(1 + 2^2) \cdots (1 + 2^{k-1}) \equiv 1 \pmod{\Phi_k(2)}$), which follows from our previous discussion)

$$(1 + 2)(1 + 2^2) \cdots (1 + 2^{kp-1}) \equiv [(1 + 2)(1 + 2^2) \cdots (1 + 2^{k-1})]^p 2^{p-1} \equiv 2^{p-1} \pmod{\Phi_k(2)}.$$

Thus $\Phi_k(2)$ divides both $2^{p-1} - 1$ and $2^{n+1} - 1$, thus it divides $2^{\gcd(n+1, p-1)} - 1 = 1$, the last equality being a consequence of the choice of p . Thus $|\Phi_k(2)| = 1$ and so $\prod_{z \in \mu_k^*} |2 - z| = 1$. But $|2 - z| \geq 1$ for all $z \in \mu_k^*$, thus we must have $|2 - z| = 1$ and then $z = 1$ for all $z \in \mu_k^*$. This forces $k = 1$ and shows that $n + 1$ is a prime, finishing the proof.

Solution 2 by William Han, Horace Mann School, Bronx, New York, USA

We will show that this property holds if and only if $n = p - 1$, where p is an odd prime. Assume first that $n + 1 = p$, with p an odd prime. Let $F_p(x) = \prod_{k=1}^{p-1} (1 + x^k) - 1$ and let ω be a primitive p -th root of unity. Then $\omega, \omega^2, \dots, \omega^{p-1}$ are the nontrivial p -th roots of unity and

$$\prod_{k=1}^{p-1} (x - \omega^k) = \frac{x^p - 1}{x - 1}.$$

Substituting $x = -1$, and since p is odd, we get $\prod_{k=1}^{p-1} (1 + \omega^k) = 1$.

It follows that $F_p(\omega^k) = 0$ for every $1 \leq k \leq p - 1$, and therefore the cyclotomic polynomial $\Phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$ divides $F_p(x)$ in $\mathbf{Z}[x]$. In particular, $\Phi_p(2) = 1 + 2 + 2^2 + \cdots + 2^{p-1}$ divides $F_p(2) = (1 + 2)(1 + 2^2) \cdots (1 + 2^{p-1}) - 1$, as desired. This proves the direct implication.

For the converse, let $m = n + 1$, where m is composite, and write

$$P_m = \prod_{k=1}^{m-1} (1 + 2^k).$$

We have two cases.

(i) $m = 2m'$. Then $3 = 1 + 2$, thus

$$P_m - 1 = (1 + 2)(1 + 2^2) \cdots (1 + 2^{m-1}) - 1 \equiv -1 \pmod{3},$$

while

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 = 2^m - 1 = 4^{m'} - 1 \equiv 0 \pmod{3}.$$

So, in this case, the property is not satisfied.

(ii) m is odd composite. Let p be the smallest prime divisor of m , and write $m = pt$, where $t > 1$ is odd. By Zsigmondy's theorem, there exists a prime $r \mid (2^t - 1)$ such that the order of 2 modulo r is exactly t .

Then the elements $1, 2, 2^2, \dots, 2^{t-1}$ are precisely the nonzero t -th roots of unity modulo r , so

$$\prod_{k=1}^{t-1} (1 + 2^k) \equiv 1 \pmod{r}.$$

Now group the product:

$$P_m = \left(\prod_{k=1}^t (1 + 2^k) \right)^{p-1} \prod_{k=1}^{t-1} (1 + 2^k).$$

Since $2^t \equiv 1 \pmod{r}$, we have

$$\prod_{k=1}^t (1 + 2^k) = (1 + 2^t) \prod_{k=1}^{t-1} (1 + 2^k) \equiv 2 \cdot 1 = 2 \pmod{r}.$$

Hence $P_m \equiv 2^{p-1} \pmod{r}$. Because the order of 2 modulo r is t , and $p - 1 < t$, we have $2^{p-1} \not\equiv 1 \pmod{r}$. Thus $P_m \not\equiv 1 \pmod{r}$. Since $r \mid (2^t - 1)$ and $t \mid m$, it follows that $r \mid (2^m - 1)$. Therefore

$$2^m - 1 \nmid P_m - 1.$$

Thus, for $m > 2$,

$$2^m - 1 \mid \prod_{k=1}^{m-1} (1 + 2^k) - 1 \iff m \text{ is prime}.$$

O729. Find all prime numbers p such that $\frac{2^{p-1} - 1}{p}$ is a perfect power.

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by Daniel Pascuas, Barcelona, Spain

Let p be a prime number such that $(2^{p-1} - 1)/p = 1$ is a perfect power. Since $(2^{2-1} - 1)/2 = 1/2$ is not a perfect power, p must be an odd prime number. Thus we write $p = 2q + 1$, where q is a positive integer. Then

$$2^{p-1} - 1 = 2^{2q} - 1 = (2^q + 1)(2^q - 1).$$

Since the above two factors are consecutive odd integers, they are coprime, that is, they have no common prime divisor. Therefore only one of them is divisible by p and the other one must be a perfect power. Hence we only have the following two cases:

Case 1. $2^q + 1$ is a perfect power. In this case, there are integers $x, a > 1$ such that $2^q + 1 = x^a$. Then x is odd and 2 is the only prime divisor of $x^a - 1$. Thus we may apply Zsigmondy's theorem (see the article *Zsigmondy's Theorem and its Applications* by Sunaina Pati, published in *Mathematical Reflections* 2 (2025)) to get that $a = 2$, so $2^q = x^2 - 1 = (x - 1)(x + 1)$. Therefore $x - 1 = 2^b$ and $x + 1 = 2^c$, where $0 \leq b < c$ are integers. It follows that $2 = 2^c - 2^b = 2^b(2^{c-b} - 1)$, so $b = 1$, $x = 3$, and $q = 3$. Hence $p = 2q + 1 = 7$, which satisfies that

$$\frac{2^{p-1} - 1}{p} = \frac{2^6 - 1}{7} = \frac{63}{7} = 9 = 3^2 \quad \text{is a perfect power.}$$

Case 2. $2^q - 1$ is a perfect power. In this case, there are integers $x > 0$ and $a > 1$ such that $2^q - 1 = x^a$. Then x is odd and 2 is the only prime divisor of $x^a + 1$. By the sum version of Zsigmondy's theorem (see Theorem 7 on page 7 of the cited article), $x = 1$, so $q = 1$ and $p = 2q + 1 = 3$, which satisfies that

$$\frac{2^{p-1} - 1}{p} = \frac{2^2 - 1}{3} = 1 \quad \text{is a perfect power.}$$

Summarizing, we have just shown that 3 and 7 are the only prime numbers p such that $(2^{p-1} - 1)/p$ is a perfect power.

Solution 2 by Mohammed Aassila, Strasbourg, France

Let

$$N_p = \frac{2^{p-1} - 1}{p}.$$

We seek all primes p for which N_p is a perfect power. First check small cases.

For $p = 3$, $N_3 = \frac{2^2 - 1}{3} = \frac{3}{3} = 1$, which is a perfect power.

For $p = 7$, $N_7 = \frac{2^6 - 1}{7} = \frac{63}{7} = 9 = 3^2$, which is also a perfect power.

Now suppose $p \geq 5$ and that N_p is a perfect power. Then $N_p = a^m$ for some integers $a \geq 1$ and $m > 1$. Let q be a prime divisor of m . Then N_p is also a q -th power: $N_p = b^q$ for some integer b . Hence $\frac{2^{p-1} - 1}{p} = b^q$, so

$$2^{p-1} - pb^q = 1.$$

Thus we obtain an equation of the form

$$2^n - Dy^q = 1,$$

where $D = p$ is prime and $q \geq 2$. A classical theorem of Ljunggren on generalized Ramanujan-Nagell equations states that the only solution of $2^n - Dy^q = 1$ with D odd, $q \geq 2$, and $y > 1$ is $2^6 - 7 \cdot 3^2 = 1$. Therefore,

$$p = 7, \quad b = 3, \quad q = 2.$$

Combining this with the earlier check $p = 3$, we conclude that the only primes satisfying the condition are $p = 3$ and $p = 7$.

Solution 3 by Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA

If $p = 2$, then $\frac{2^{p-1}-1}{p} = \frac{1}{2}$, which is not an integer. So, we can assume that $p = 2m + 1$ is an odd prime. If $m = 1$, then $p = 3$, and $(2^{p-1} - 1)/p = 1$, which may be regarded as a perfect power.

In the sequel, consider $m \geq 2$, that is, $p \geq 5$. We have

$$2^{p-1} - 1 = 2^{2m} - 1 = (2^m - 1)(2^m + 1).$$

We distinguish two cases.

(i) $p \mid 2^m - 1$. Then

$$\frac{2^{p-1} - 1}{p} = \frac{2^m - 1}{p} \cdot (2^m + 1),$$

and the two factors on the right are coprime.

Since their product is a perfect power, each one of these factors must be a perfect power. In particular,

$$2^m + 1 = y^n$$

for some integers $y \geq 2$, $n \geq 2$. But by Mihăilescu's Theorem (Catalan's Conjecture), the equation $y^n - 2^m = 1$ has only one solution when y, n, m are all at least 2, and that solution is $3^2 - 2^3 = 1$. This implies $m = 3$, from which $p = 7$.

(i) $p \mid 2^m + 1$. Then

$$\frac{2^{p-1} - 1}{p} = \frac{2^m + 1}{p} \cdot (2^m - 1),$$

and the two factors on the right are coprime.

Since their product is a perfect power, each one of these factors must be a perfect power. In particular,

$$2^m - 1 = y^n$$

for some integers $y \geq 2$, $n \geq 2$. Using Mihăilescu's Theorem again, the equation $2^m - y^n = 1$ has no integer solutions with $y, n, m \geq 2$. Hence, we have no solutions in this case.

In conclusion, the only primes having the desired property are $p = 3$ and $p = 7$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Enes Hoxha, New York School of Science in Kosova, Prishtina, Kosovo; G. C. Greubel, Newport News, VA, USA; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Shilong Zhuang, California, USA; William Han, Horace Mann School, Bronx, New York, USA.

O730. Let $f(x) = \frac{x^3}{3} + x$ and let a, b, c be distinct real numbers such that $abc = 1$ and

$$f\left(a - \frac{1}{b}\right) + f\left(b - \frac{1}{c}\right) + f\left(c - \frac{1}{a}\right) = 0.$$

Prove that

$$f\left(a - \frac{1}{a}\right) + f\left(b - \frac{1}{b}\right) + f\left(c - \frac{1}{c}\right) = \left(a^2 - \frac{1}{a^2}\right) + \left(b^2 - \frac{1}{b^2}\right) + \left(c^2 - \frac{1}{c^2}\right).$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

Solution by the authors

We have

$$\left(a - \frac{1}{b}\right)^3 + \left(b - \frac{1}{c}\right)^3 + \left(c - \frac{1}{a}\right)^3 + 3\left(a - \frac{1}{b}\right) + 3\left(b - \frac{1}{c}\right) + 3\left(c - \frac{1}{a}\right) = 0,$$

which, taking into account that $abc = 1$, is equivalent to

$$\left(a - \frac{1}{b}\right)^3 + \left(b - \frac{1}{c}\right)^3 + \left(c - \frac{1}{a}\right)^3 - 3\left(a - \frac{1}{b}\right)\left(b - \frac{1}{c}\right)\left(c - \frac{1}{a}\right) = 0.$$

If $a - \frac{1}{b} = b - \frac{1}{c} = c - \frac{1}{a}$, then

$$a - b = -\frac{b - c}{bc}, \quad b - c = -\frac{c - a}{ca}, \quad c - a = -\frac{a - b}{ab}.$$

Multiplying the last three relations and dividing by $(a - b)(b - c)(c - a) \neq 0$ yields

$$1 = -\frac{1}{(abc)^2},$$

which is impossible. But if u, v, w are real numbers, not all equal, $u^3 + v^3 + w^3 - 3uvw = 0$ if and only if $u + v + w = 0$. Hence

$$\left(a - \frac{1}{b}\right) + \left(b - \frac{1}{c}\right) + \left(c - \frac{1}{a}\right) = 0,$$

so

$$\left(a - \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) + \left(c - \frac{1}{c}\right) = 0,$$

and, by the same property,

$$\left(a - \frac{1}{a}\right)^3 + \left(b - \frac{1}{b}\right)^3 + \left(c - \frac{1}{c}\right)^3 - 3\left(a - \frac{1}{a}\right)\left(b - \frac{1}{b}\right)\left(c - \frac{1}{c}\right) = 0.$$

It follows that

$$\begin{aligned} & \frac{1}{3} \left(\left(a - \frac{1}{a}\right)^3 + \left(b - \frac{1}{b}\right)^3 + \left(c - \frac{1}{c}\right)^3 \right) + \left(a - \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) + \left(c - \frac{1}{c}\right) \\ &= \left(a - \frac{1}{a}\right)\left(b - \frac{1}{b}\right)\left(c - \frac{1}{c}\right) \\ &= \left(abc - \frac{1}{abc}\right) + \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) - \left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right) \\ &= (a^2 + b^2 + c^2) - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right), \end{aligned}$$

where we used again that $abc = 1$, and the conclusion follows.

There are triples (a, b, c) which satisfy the conditions in the problem, for example $(2, 1/2, 1)$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Theo Koupelis, Clark College, Washington, USA; Hoang-Bao Le, Dublin, Ireland; Shilong Zhuang, California, USA.

O731. Let $P(x)$ be a polynomial of degree d .

- (a) Let c be an integer such that for all integers $c \leq x \leq c + d$, $P(x)$ is an integer. Prove that for all integers n , $P(n)$ is an integer.
- (b) Prove that for each positive integer d , there is a polynomial $P(x)$ of degree d for which there is an integer c such that for all integers $c \leq x \leq c + d - 1$, $P(x)$ is an integer, but for any other integer n , $P(n)$ is irrational.

Proposed by Mohammadreza Ghelichkhani, Shahed University, Tehran, Iran

Solution by the author

- (a) If $d = 1$, let $P(x) = ax + b$. Since $P(c)$ and $P(c + 1)$ are both integers, we have $P(c + 1) - P(c) = a$, which is also an integer. Therefore, a must be an integer, and since $P(c) = ac + b$ is an integer, it follows that b is an integer as well. Thus, $P(x) \in \mathbb{Z}[x]$, completing the proof for this case.

Now, assume the statement is true for all positive integers 1 through $d - 1$. We will prove it for d .

We know that $P(c), P(c + 1), \dots, P(c + d)$ are all integers. Let $q(x) = P(x + 1) - P(x)$. It is obvious that $\deg(q) < d$ and that for all $c \leq x \leq c + d - 1$, $q(x)$ is an integer. Therefore, for all integers n , $q(n)$ is an integer. This implies that for all integers n , either $P(n + 1)$ and $P(n)$ are both integers or neither is. Since $P(c)$ is an integer, the proof is completed.

- (b) Consider the polynomial $P(x) = \sqrt{2}x(x - 1) \cdots (x - d + 1)$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo.

O732. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^2bc \geq 1$, and let

$$F(a, b, c) = \frac{1}{\sqrt[3]{abc}} - \sqrt{\frac{3}{ab + bc + ca}}.$$

Prove that

$$F(a, b, c) \leq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

Proposed by Vasile Cîrtoaje, Romania and Vasile Mircea Popa, Romania

Solution by the author

Since $F(a, b, c) \geq 0$ and $F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \geq 0$, it suffices to prove the homogeneous inequality

$$\sqrt{a^2bc} \cdot F(a, b, c) \leq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

for $a = \min\{a, b, c\}$. Due to homogeneity, we may set $a = 1$, hence $b, c \geq 1$. Thus, we need to show that

$$\sqrt{bc} \cdot F(a, b, 1) \geq F\left(\frac{1}{a}, \frac{1}{b}, 1\right),$$

which is equivalent to

$$\sqrt{bc} \left(\frac{1}{\sqrt[3]{bc}} - \sqrt{\frac{3}{b + c + bc}} \right) \leq \sqrt[3]{bc} - \sqrt{\frac{3bc}{1 + b + c}}.$$

Denote

$$s = \frac{b + c}{2}, \quad p = \sqrt{bc},$$

with $s \geq p \geq 1$. For fixed p , the desired inequality is equivalent to $F(s) \geq 0$, where

$$F(s) = \sqrt[3]{p^2} - p\sqrt{\frac{3}{2s + 1}} - p \left(\frac{1}{\sqrt[3]{p^2}} - \sqrt{\frac{3}{2s + p^2}} \right).$$

We have

$$F'(s) = p\sqrt{3} \left[\frac{1}{(2s + 1)^{3/2}} - \frac{1}{(2s + p^2)^{3/2}} \right] \geq 0.$$

Thus, $F(s)$ is increasing and $F(s) \geq F(p)$. To end the proof, we need to show that $F(p) \geq 0$, which is equivalent to

$$p^{1/3} - p^{2/3} \sqrt{\frac{3}{2p + 1}} \geq 1 - \sqrt{\frac{3p^{1/3}}{p + 2}}.$$

Denoting $p = x^3$, where $x \geq 1$, we need to show that

$$x - x^2 \sqrt{\frac{3}{2x^3 + 1}} \geq 1 - \sqrt{\frac{3x}{x^3 + 2}},$$

i.e.

$$\begin{aligned} \frac{x \left(\sqrt{2x^3 + 1} - x\sqrt{3} \right)}{\sqrt{2x^3 + 1}} &\geq \frac{\sqrt{x^3 + 2} - \sqrt{3x}}{\sqrt{x^3 + 2}}, \\ \frac{x(2x + 1)(x - 1)^2}{\sqrt{2x^3 + 1} \left(\sqrt{2x^3 + 1} + x\sqrt{3} \right)} &\geq \frac{(x + 2)(x - 1)^2}{\sqrt{x^3 + 2} \left(\sqrt{x^3 + 2} + \sqrt{3x} \right)}. \end{aligned}$$

It is true if

$$\frac{x(2x+1)}{2x^3+1+x\sqrt{3(2x^3+1)}} \geq \frac{x+2}{x^3+2+\sqrt{3x(x^3+2)}},$$

which is equivalent to $f(x) + \sqrt{3}xg(x) \geq 0$, where

$$f(x) = 2x^5 - x^4 - 4x^3 + 4x^2 + x - 2 = (x-1)(2x^4 + x^3 - 3x^2 + x + 2) \geq 0$$

and

$$\begin{aligned} g(x) &= (2x+1)\sqrt{x(x^3+2)} - (x+2)\sqrt{2x^3+1} \geq (x+2) \left[\sqrt{x(x^3+2)} - \sqrt{2x^3+1} \right] \\ &= \frac{(x+2)(x^4 - 2x^3 + 2x - 1)}{\sqrt{x(x^3+2)} + \sqrt{2x^3+1}} = \frac{(x+2)(x+1)(x-1)^3}{\sqrt{x(x^3+2)} + \sqrt{2x^3+1}} \geq 0. \end{aligned}$$

The proof is completed. The equality occurs for $a = b = c \geq 1$.

Remark. The inequality is true in the particular case $a, b, c \geq 1$ (which involves $a^2bc \geq 1$).

Also solved by Theo Koupelis, Clark College, Washington, USA.