

Junior Problems

J733. Solve in real numbers the system of equations

$$\begin{aligned}x(x^2 + 1) + y &= 5 \\ y^3 + 3xy(x + y) + z &= 60 \\ z^3 + 3z(x + y + z)(x + y) &= -63.\end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

J734. Let x and y be real numbers such that

$$4^{x+y} - 16^x + 4^{x-y} = 1.$$

Prove that $|x| = |y|$.

Proposed by Adrian Andreescu, Dallas, USA

J735. Let $a \geq b \geq 1 \geq c > 0$ such that $a + b + c = 3$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

Proposed by Mircea Lascu, Zalău, Romania

J736. Let a, b, c be real numbers such that $b + ca + 3c^2 = 0$. Prove that there are two roots u and v of the polynomial $p(x) = x^3 + ax^2 + bx + c^3$ such that $(c - u)v = c^2$.

Proposed by Li Zhou, Polk State College, USA

J737. Let ABC be an acute triangle with the circumcircle $\Gamma(O, R)$. The points X, Y, Z represent the intersections of the lines AO, BO, CO with the circumscribed circles of the triangles BOC, AOC , and AOB , respectively. If $\max \{AX, BY, CZ\} \leq 3R$, show that $\triangle ABC$ is an equilateral triangle.

Proposed by Mihaela Berindeanu, Bucharest, Romania

J738. Let J be the A -excenter of $\triangle ABC$ and K be the center of the circumcircle Ω of $\triangle JBC$. Suppose that the incircle ω of $\triangle ABC$ intersects Ω at P, Q and is tangent to BC, CA at D, E , respectively. Let L be the point where DE intersects PQ . Prove that DL, KJ , and KL form the sides of a right triangle.

Proposed by Li Zhou, Polk State College, USA

Senior Problems

S733. Solve in integers the equation

$$\frac{x^2 + y^2}{2} + \left| \frac{x + y}{2} + 1 \right| = 2026.$$

Proposed by Titu Andreescu, Dallas, USA and Alessandro Ventullo, Milan, Italy

S734. Prove that for any positive integer n ,

$$\sum_{k=1}^n \gcd(k, n) = n \cdot \sum_{d|n} \frac{\varphi(d)}{d},$$

where φ is Euler's totient function.

Proposed by Alessandro Ventullo, Milan, Italy

S735. Let $n \geq 3$ be an integer and $A_0 A_1 \dots A_{n-1}$ a regular polygon of center O . On the ray OA_i , a point B is taken such that $OB = A_i A_{i+1}$. It is known that the triangle $A_{i-1} B A_{i+1}$ is equilateral. Find n .

Proposed by Mircea Becheanu, Montreal, Canada

S736. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 3 \geq \frac{27}{100} (a^3 + b^3 + c^3 - 3).$$

Proposed by Marius Stănean, Zalău, Romania

S737. What is the smallest positive exponent k such that

$$a^k + b^k + c^k \geq 3$$

for any nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq d$ and $ab + bc + cd + da = 4$?

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

S738. Let ABC be a triangle with an arbitrary point $D \in BC$. Triangles ABD , ADC , and ABC have circumradii R_1 , R_2 , and R . If $\max\{R_1, R_2\} = R \cos \frac{A}{2}$, prove that

$$(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{BC} = 0.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Undergraduate Problems

U733. Evaluate

$$\int \frac{x-2}{x^4+16x-12} dx.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U734. For $x \in \left(0, \frac{\pi}{2}\right)$ evaluate

$$\int \frac{1 - \sin x}{3 \sin x + 5(1 + \cos x)e^x} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

U735. Prove that $\frac{7}{3}$ is the smallest positive exponent k which does not depend on n such that the inequality

$$\frac{a_1^4 + a_2^4 + \cdots + a_n^4}{n} \leq \left(\frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n} \right)^k$$

holds for any $n \geq 2$ and nonnegative real numbers a_i with at most one $a_i < 1$ and $a_1 + a_2 + \cdots + a_n = n$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

U736. Given $a, b, c \in \mathbb{R}_{\geq 0}$, prove that the following statements are true.

(i) $(a-b)^2 + (b-c)^2 + (c-a)^2 \neq 0$ is a necessary and sufficient condition for the existence of an infinite, non-constant sequence x_n such that

$$\frac{3}{2} \leq x_n \leq \sum_{\text{cyc}} \frac{a}{b+c}, \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n = \frac{3}{2}.$$

(ii) For all $t \geq 0$, the following inequality holds:

$$\sum_{\text{cyc}} \frac{ab}{(b+c)(c+a)} \geq \sum_{\text{cyc}} \frac{(a+t)(b+t)}{(b+c+2t)(c+a+2t)}.$$

Determine the conditions under which equality occurs.

Proposed by Shilong Zhuang, California, USA

U737. Calculate

$$\lim_{n \rightarrow \infty} n \sqrt[n]{\int_0^1 \left| \arctan x - x + \frac{x^3}{3} - \cdots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right|^n dx}.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

U738. Let $n \geq 3$ and let

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

be the Dihedral group of order $2n$. Fix a divisor $d \mid n$ and let $H = \langle r^d \rangle \leq \langle r \rangle$. Determine the largest possible size of a set

$$\mathcal{R} \subseteq \{sr^0, sr^1, \dots, sr^{n-1}\}$$

of reflections with the property that for every distinct $x, y \in \mathcal{R}$, then $xy \in H$. What are the families $\mathcal{R}$ achieving this maximum?

Proposed by Alessandro Ventullo, Milan, Italy

Olympiad Problems

O733. At a party with $n \geq 3$ people, it is known that among any three people, there are at least two pairs of them who know each other. Prove that:

- (a) you can always line them up in a row so that each person is next to only acquaintances;
- (b) if $n \geq 4$ you can seat them around a round table so that everyone has acquaintances on both sides.

Proposed by Alessandro Ventullo, Milan, Italy

O734. Let $p > 3$ be a prime, and let

$$k = \left\lfloor \frac{3p^2 - 6p - 1}{4p} \right\rfloor$$

(where $\lfloor u \rfloor$ denotes the integer part of the real number u). Show that among the numbers $1 + p, 1 + 2p, \dots, 1 + kp$ there exists at least one that can be written as the sum of three nonzero squares.

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

O735. Prove that in all acute triangles ABC and for any $\lambda > 0$,

$$\sum_{cyc} \frac{\cos A}{\sqrt{\lambda + \cos^2 A}} \leq \frac{3(R + r)}{\sqrt{(9\lambda + 1)R^2 + 2Rr + r^2}}.$$

Proposed by Mihály Bencze, Braşov, Romania

O736. For every positive integer x , let $\alpha(x)$ denote the largest positive divisor z of 2025 for which there is an integer y such that $x = y^z$. Let $n \geq 2$ be a positive integer and let $x_1, x_2, \dots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \dots + x_n = 2$. Determine the largest possible value of

$$\sum_{1 \leq i < j \leq n} (\alpha(j - i) - 1)x_i x_j.$$

Proposed by Andrei-Giovani Chirita, University of Cambridge, UK

O737. Let $x, y, z \in \left[0, \frac{7}{4}\right]$ be real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$9(x + y + z) \geq 23 + 4xyz.$$

Proposed by Marius Stănean, Zalău, Romania

O738. Let $a_1, a_2, \dots, a_n$ be positive real numbers such that at most one of them is less than 1 and $a_1^3 + a_2^3 + \dots + a_n^3 = n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1^4 + a_2^4 + \dots + a_n^4.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania