

APPLICATIONS OF THE FUNDAMENTAL SYMMETRIC SUMS IN THE STUDY OF SOME ALGEBRAIC INEQUALITIES

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Abstract. We will prove a general inequality from which we will obtain known and new inequalities.

Let us consider the real numbers a, b , and c , and their fundamental symmetric sums $\sigma_1 = a+b+c$, $\sigma_2 = ab + bc + ca$, $\sigma_3 = abc$, with the help of which we define

$$s_1 = \frac{1}{27} \left(-2\sigma_1^3 + 9\sigma_1\sigma_2 - 2\sqrt{(\sigma_1^2 - 3\sigma_2)^3} \right),$$
$$s_2 = \frac{1}{27} \left(-2\sigma_1^3 + 9\sigma_1\sigma_2 + 2\sqrt{(\sigma_1^2 - 3\sigma_2)^3} \right).$$

We will keep these notations throughout this paper.

Theorem 1. *Let a, b, c be real numbers. Then the inequalities*

$$s_1 \leq \sigma_3 \leq s_2, \tag{1}$$

hold, with equality if and only if $(a-b)(b-c)(c-a) = 0$, that is if and only if $b = c$, or $c = a$, or $a = b$. Moreover, when $b = c$ we have

- (i) if $a < b$, then $s_1 = \sigma_3 < s_2$;*
- (ii) if $a > b$, then $s_1 < \sigma_3 = s_2$;*
- (iii) if $a = b$, then $s_1 = \sigma_3 = s_2$,*

and similar conclusions can be drawn in the cases $c = a$ or $a = b$.

Proof. The inequality $(a-b)^2(b-c)^2(c-a)^2 \geq 0$ holds for all $a, b, c \in \mathbb{R}$, with equality if and only if $(a-b)(b-c)(c-a) = 0$. After performing some calculations, the above inequality is equivalent to

$$-27\sigma_3^2 + (18\sigma_1\sigma_2 - 4\sigma_1^3)\sigma_3 + \sigma_1^2\sigma_2^2 - 4\sigma_2^3 \geq 0, \tag{2}$$

with equality if and only if $(a-b)(b-c)(c-a) = 0$. The discriminant of the quadratic trinomial in σ_3 from (2) is $\Delta = 16(\sigma_1^2 - 3\sigma_2)^3$ and it is nonnegative since

$$\sigma_1^2 \geq 3\sigma_2 \Leftrightarrow (b-c)^2 + (c-a)^2 + (a-b)^2 \geq 0$$

(with equality if and only if $a = b = c$), so the (real) roots of the trinomial are s_1 and s_2 , with $s_1 \leq s_2$. Now it is clear that the inequality from (2) is true if and only if $\sigma_3 \in [s_1, s_2]$, which is equivalent to the inequalities (1). We have equality in (2) if and only if $(a-b)(b-c)(c-a) = 0$, meaning that $b = c$, or $c = a$, or $a = b$.

Consider now that $b = c$; then we have $\sigma_1 = a + 2b$, $\sigma_2 = 2ab + b^2$, $\sigma_3 = ab^2$ and, by replacing these in s_1 and s_2 , we obtain

$$\begin{aligned} s_1 &= \frac{1}{27} (-2a^3 + 6a^2b + 21ab^2 + 2b^3 - 2|a - b|^3), \\ s_2 &= \frac{1}{27} (-2a^3 + 6a^2b + 21ab^2 + 2b^3 + 2|a - b|^3). \end{aligned}$$

If $a < b$ then $|a - b| = -a + b$ and we have

$$\begin{aligned} s_1 &= ab^2 = \sigma_3, \\ s_2 &= \frac{1}{27} (-4a^3 + 12a^2b + 15ab^2 + 4b^3). \end{aligned}$$

The inequality $s_2 > \sigma_3 = ab^2$ is equivalent to $4(b - a)^3 > 0$, which is clearly true, thus we get that $s_1 = \sigma_3 < s_2$, which represents (i). Similarly we obtain (ii). If $a = b = c$ it follows immediately that $s_1 = \sigma_3 = s_2 = a^3$, that is, (iii). $\square$

Corollary 1. *Let a, b, c be real numbers. Then we have the inequality*

$$|27\sigma_3 + 2\sigma_1^3 - 9\sigma_1\sigma_2| \leq 2\sqrt{(\sigma_1^2 - 3\sigma_2)^3}, \quad (3)$$

with equality if and only if $b = c$, or $c = a$, or $a = b$.

The proof of the above follows from (1). We solve some other applications using the above results.

Application 1. Let a and b be real numbers. Prove that

$$|(a + b)(2a^2 - 5ab + 2b^2)| \leq 2\sqrt{(a^2 - ab + b^2)^3}. \quad (4)$$

(Ovidiu T. Pop and Marius Drăgan)

Solution. The inequality (4) immediately follows by taking $c = 0$ in (3). Equality holds if and only if $a = b$, or $a = 0$, or $b = 0$.

Application 2. Let a, b, c be positive real numbers. Prove that

$$\frac{4(a + b + c)^2}{a^2 + b^2 + c^2} + \frac{a^3 + b^3 + c^3}{abc} - \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 14. \quad (5)$$

(Marius Drăgan and Ovidiu T. Pop)

Solution. Inequality (5) may be written in the form

$$\frac{4\sigma_1^2}{\sigma_1^2 - 2\sigma_2} + \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3}{\sigma_3} - \frac{\sigma_1^2 - 2\sigma_2}{\sigma_2} \geq 14.$$

Due to homogeneity we may assume, without loss of generality, that $\sigma_1 = a + b + c = 1$. Thus we have to prove

$$(1 - 3\sigma_2) \left(\frac{1}{\sigma_3} - \frac{6\sigma_2 + 1}{\sigma_2(1 - 2\sigma_2)} \right) \geq 0, \quad (6)$$

where we have (because $(a + b + c)^2 \geq 3(ab + bc + ca)$, for any $a, b, c \in \mathbb{R}$)

$$\sigma_2 \leq \frac{1}{3}, \quad (7)$$

with equality if and only if $a = b = c$. According to (7) we still need to show that

$$\frac{1}{\sigma_3} - \frac{6\sigma_2 + 1}{\sigma_2(1 - 2\sigma_2)} \geq 0, \quad (8)$$

or (note that all numerators and denominators are positive)

$$\sigma_3 \leq \frac{\sigma_2(1 - 2\sigma_2)}{6\sigma_2 + 1}, \quad (9)$$

where $\sigma_2 \in (0, \frac{1}{3}]$.

By using the right inequality from (1) it will be enough to prove that

$$s_2 \leq \frac{\sigma_2(1 - 2\sigma_2)}{6\sigma_2 + 1},$$

or

$$\frac{1}{27} \left(-2 + 9\sigma_2 + 2\sqrt{(1 - 3\sigma_2)^3} \right) \leq \frac{\sigma_2(1 - 2\sigma_2)}{6\sigma_2 + 1}, \quad (10)$$

or, finally, after performing some calculations,

$$(1 - 3\sigma_2) \left((6\sigma_2 + 1)\sqrt{1 - 3\sigma_2} - 18\sigma_2 - 1 \right) \leq 0, \quad (11)$$

where $\sigma_2 \in (0, \frac{1}{3}]$.

But this is true, since $1 - 3\sigma_2 \geq 0$, and $(6\sigma_2 + 1)\sqrt{1 - 3\sigma_2} < 6\sigma_2 + 1 < 18\sigma_2 + 1$. Equality holds if and only if $a = b = c$.

Application 3. Let a, b, c be real numbers. Then we have

$$\begin{aligned} & (a^2 + b^2 + c^2)(a^2 + b^2 + c^2 - ab - bc - ca) \geq \\ & \geq (\sqrt{6} - 2)(a + b + c)(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 - 6abc). \end{aligned} \quad (12)$$

(Marius Drăgan and Ovidiu T. Pop)

Solution. The inequality (12) can be expressed as

$$(\sigma_1^2 - 2\sigma_2)(\sigma_1^2 - 3\sigma_2) \geq (\sqrt{6} - 2)\sigma_1(\sigma_1\sigma_2 - 9\sigma_3).$$

Without loss of generality we consider $\sigma_1 = 1$, since the inequality is evidently true for $a + b + c = 0$, while for a, b, c with $a + b + c \neq 0$ it can be obtained by applying the already proved case (when a, b, c sum to 1) to the numbers

$$\frac{a}{a + b + c}, \frac{b}{a + b + c}, \frac{c}{a + b + c},$$

and by multiplying with $(a + b + c)^4 > 0$. Thus, we need prove

$$\sigma_3 \geq \frac{1}{9} \left(\sigma_2 - \frac{(1 - 2\sigma_2)(1 - 3\sigma_2)}{\sqrt{6} - 2} \right). \quad (13)$$

Due to (1) we have $\sigma_3 \geq s_1$, so it would be enough to show that

$$\frac{1}{27} \left(-2 + 9\sigma_2 - 2\sqrt{(1 - 3\sigma_2)^3} \right) \geq \frac{1}{9} \left(\sigma_2 - \frac{(1 - 2\sigma_2)(1 - 3\sigma_2)}{\sqrt{6} - 2} \right)$$

or

$$(1 - 3\sigma_2) \left((2 + \sqrt{6}) (6\sigma_2 + 2\sqrt{6} - 7) + 4\sqrt{1 - 3\sigma_2} \right) \leq 0. \quad (14)$$

Since $\sigma_2 \in (0, \frac{1}{3}]$, we still have to prove that

$$(2 + \sqrt{6}) (6\sigma_2 + 2\sqrt{6} - 7) + 4\sqrt{1 - 3\sigma_2} \leq 0, \quad (15)$$

for any $\sigma_2 \in (0, \frac{1}{3}]$.

In order to prove (15), we write it as

$$2(\sqrt{6} - 2)\sqrt{1 - 3\sigma_2} \leq 7 - 2\sqrt{6} - 6\sigma_2.$$

Note that $7 - 2\sqrt{6} - 6\sigma_2 \geq 5 - 2\sqrt{6} > 0$, so, after squaring and performing some calculations, the inequality to prove becomes

$$(6\sigma_2 - 2\sqrt{6} + 3)^2 \geq 0,$$

which is definitely true.

The equality holds in (13) if and only if $\sigma_3 = s_1$ and we also have equality in (14), that is, whenever we have either $\sigma_2 = \frac{1}{3}$, or $\sigma_2 = \frac{2\sqrt{6} - 3}{6}$ (besides $\sigma_3 = s_1$).

The first case leads to equality in (12) in the obvious situation $a = b = c$.

Because in the second case we still have $\sigma_3 = s_1$, it follows by Theorem 1 that two of the numbers are equal. Assuming $b = c$, we must have (also by Theorem 1) $b = c \geq a$. Thus we now get $2ab + b^2 = \frac{2\sqrt{6} - 3}{6}$ and $a + 2b = 1$, yielding

$$a = \frac{3 - \sqrt{6}}{3}, \quad b = c = \frac{1}{\sqrt{6}}, \quad \text{or} \quad a = \frac{\sqrt{6} - 1}{3}, \quad b = c = \frac{4 - \sqrt{6}}{6},$$

but the second triple does not verify the condition $a \leq b$. It remains that equality in (12) also holds for triples of one of the forms $a = \frac{k(3 - \sqrt{6})}{3}$, $b = c = \frac{k}{\sqrt{6}}$, or $b = \frac{k(3 - \sqrt{6})}{3}$, $c = a = \frac{k}{\sqrt{6}}$, or $c = \frac{k(3 - \sqrt{6})}{3}$, $a = b = \frac{k}{\sqrt{6}}$, for some real number k .

Remark 1. Inequality (12) represents a refinement of the inequality

$$\begin{aligned} & (a^2 + b^2 + c^2)(a^2 + b^2 + c^2 - ab - bc - ca) \geq \\ & \geq \frac{7}{24}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 - 6abc) \end{aligned} \quad (16)$$

from [3].

Application 4. Let x, y, z be real numbers such that $x > 1$, $y > 1$, $z > 1$, and

$$\frac{1}{x-1} + \frac{1}{y-1} + \frac{1}{z-1} = 1.$$

Prove that

$$xyz + 44 \geq 9(x + y + z). \quad (17)$$

(Marius Stănean, see [2])

Proof. We denote $a = \frac{1}{x-1}$, $b = \frac{1}{y-1}$, $c = \frac{1}{z-1}$, from which $a, b, c > 0$ and $x = \frac{a+1}{a}$, $y = \frac{b+1}{b}$, $z = \frac{c+1}{c}$ follow. We have, by hypothesis, that $\sigma_1 = a + b + c = 1$. The inequality to prove (17) reads

$$\frac{a+1}{a} \cdot \frac{b+1}{b} \cdot \frac{c+1}{c} + 44 \geq 9 \left(\frac{a+1}{a} + \frac{b+1}{b} + \frac{c+1}{c} \right)$$

or, after computations,

$$\sigma_3 \geq \frac{4\sigma_2 - 1}{9}. \quad (18)$$

If $\sigma_2 < \frac{1}{4}$, then $4\sigma_2 - 1 < 0$, so (18) is clearly true, therefore we can assume further that $\sigma_2 \geq \frac{1}{4}$.

Note again that $\sigma_1 = 1$ implies $\sigma_2 \leq \frac{1}{3}$. Using Theorem 1, we have $\sigma_3 \geq s_1$, hence it will be sufficient to prove

$$s_1 \geq \frac{4\sigma_2 - 1}{9} \Leftrightarrow \frac{1}{27} \left(-2 + 9\sigma_2 - 2\sqrt{(1-3\sigma_2)^3} \right) \geq \frac{4\sigma_2 - 1}{9},$$

which is equivalent to

$$1 - 3\sigma_2 - 2\sqrt{(1-3\sigma_2)^3} \geq 0,$$

or to

$$(1 - 3\sigma_2)(1 - 2\sqrt{1 - 3\sigma_2}) \geq 0. \quad (19)$$

But this follows from $1 - 3\sigma_2 \geq 0$ and $1 - 2\sqrt{1 - 3\sigma_2} \geq 0 \Leftrightarrow 2\sqrt{1 - 3\sigma_2} \leq 1 \Leftrightarrow 1 - 4\sigma_2 \leq 0$, which is true since $\sigma_2 \geq \frac{1}{4}$.

For equality we must have $\sigma_3 = s_1$, and either $\sigma_2 = \frac{1}{3}$, or $\sigma_2 = \frac{1}{4}$. If $\sigma_2 = \frac{1}{3}$ we have $a = b = c = \frac{1}{3}$, thus in (17) we get equality for $x = y = z = 4$. If $\sigma_3 = s_1$ and $\sigma_2 = \frac{1}{4}$, by theorem 1, we must have (for instance) $b = c \geq a$. We thus get $b^2 + 2ab = \frac{1}{4}$ and $2b + a = 1$, yielding $a = \frac{2}{3}$ and $b = c = \frac{1}{6}$, or $a = 0$ and $b = c = \frac{1}{2}$. Since $a > 0$ we only remain with $a = \frac{2}{3}$, $b = c = \frac{1}{6}$, but this triple does not verify $a \leq b$. Thus, finally, the only equality case in (17) is $x = y = z = 4$. $\square$

Application 5. Let a, b, c be nonnegative real numbers such that

$$(a+b)(b+c)(c+a) = 2.$$

Prove that

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) + 8a^2b^2c^2 \leq 1. \quad (20)$$

(Marius Stănean, MR 2(2020), see [2])

Solution. We prove the general inequality

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) + 8a^2b^2c^2 \leq \frac{1}{4}(a+b)^2(b+c)^2(c+a)^2$$

from which the desired inequality immediately follows. By expressing in terms of $\sigma_1, \sigma_2, \sigma_3$, we need to show that

$$\frac{63}{4}\sigma_3^2 + \sigma_1^3\sigma_3 - \frac{1}{4}\sigma_1^2\sigma_2^2 - \frac{11}{2}\sigma_1\sigma_2\sigma_3 + \sigma_2^3 \leq 0. \quad (21)$$

Without loss of generality we may consider $\sigma_1 = 1$, so we now have to prove the inequality

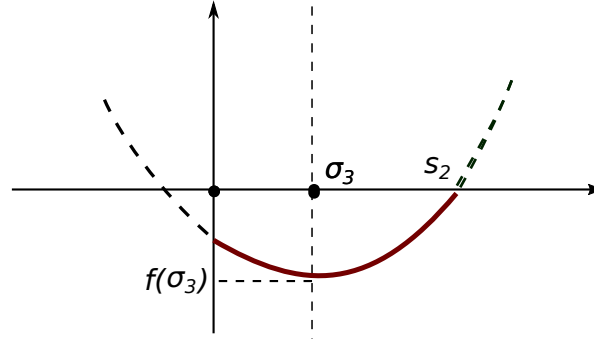
$$f(\sigma_3) = 63\sigma_3^2 + (4 - 22\sigma_2)\sigma_3 + 4\sigma_2^3 - \sigma_2^2 \leq 0.$$

(where $f(t) = 63t^2 + (4 - 22\sigma_2)t + 4\sigma_2^3 - \sigma_2^2$). We distinguish between two cases.

Case 1. When $\sigma_2 < \frac{1}{4}$ we have $f(0) = 4\sigma_2^3 - \sigma_2^2 < 0$ and

$$f(s_2) = \frac{8}{81}(1 - 3\sigma_2)(18\sigma_2^2 - 21\sigma_2 + 4 + (15\sigma_2 - 4)\sqrt{1 - 3\sigma_2}) < 0,$$

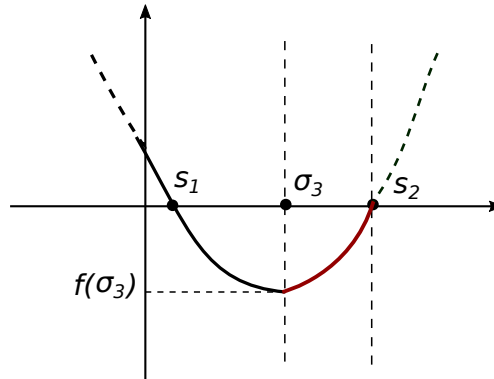
(without equality). Because $\sigma_3 \in [0, s_2]$ (by Theorem 1), and f is a quadratic trinomial with a positive leading coefficient, we get $f(\sigma_3) < 0$, too.



Case 2. Consider now that $\sigma_2 \geq \frac{1}{4}$ (and, of course, we have anyway $\sigma_2 \leq \frac{1}{3}$). In this case $\sigma_3 \in [s_1, s_2]$ (by Theorem 1), and we have

$$f(s_1) = \frac{8}{81}(1 - 3\sigma_2)(18\sigma_2^2 - 21\sigma_2 + 4 - (15\sigma_2 - 4)\sqrt{1 - 3\sigma_2}) \leq 0$$

and $f(s_2) \leq 0$. It follows that $f(\sigma_3) \leq 0$, too, since $\sigma_3 \in [s_1, s_2]$. The equality case holds for $f(s_1) = 0$ or $f(s_2) = 0$. The equation $f(s_1) = 0$ leads to $\sigma_2 = \frac{1}{3}$, or $\sigma_2 = \frac{1}{4}$, while $f(s_2) = 0$ gives $\sigma_2 = \frac{1}{3}$.



From $\sigma_2 = \frac{1}{3}$ we clearly get $a = b = c = \frac{1}{3}$. If $\sigma_2 = \frac{1}{4}$, in the same way as in Application 4, we obtain $a = 0$, $b = c = \frac{1}{2}$, or $a = \frac{2}{3}$, $b = c = \frac{1}{6}$ with $a < b = c$. It remains that $a = b = c = \frac{1}{3}$, or

$a = 0, b = c = \frac{1}{2}$. So the equality case in (21) holds either for $a = b = c$, or for $a = 0$ and $b = c$. By replacing in the condition $(a+b)(b+c)(c+a) = 2$, we see that equality holds in (20) if either $a = b = c = \frac{1}{\sqrt[3]{4}}$, or $a = 0, b = 1, c = 1$ (and all the permutations of the last triple).

Application 6. Find the greatest constant λ such that the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq \lambda \left(\frac{a^2 + b^2 + c^2}{(a+b+c)^2} - \frac{1}{3} \right) \quad (22)$$

holds for all positive real numbers a, b , and c .

(*Nguyen Viet Hung*, see [1])

Solution. The inequality is clearly an equality for $a = b = c$ (and any λ), so we can assume that a, b, c are not all equal. It is equivalent to

$$\frac{\sigma_1^3 - 2\sigma_1\sigma_2 + 3\sigma_3}{\sigma_1\sigma_2 - \sigma_3} - \frac{3}{2} \geq \lambda \left(\frac{\sigma_1^2 - 2\sigma_2}{\sigma_1^2} - \frac{1}{3} \right).$$

Without loss of generality we suppose $\sigma_1 = 1$. Then the inequality to prove rewrites successively

$$\begin{aligned} \frac{1 - 2\sigma_2 + 3\sigma_3}{\sigma_2 - \sigma_3} - \frac{3}{2} &\geq \lambda \left(1 - 2\sigma_2 - \frac{1}{3} \right) \Leftrightarrow \\ \Leftrightarrow \frac{1 - 2\sigma_2 + 3\sigma_3}{\sigma_2 - \sigma_3} - \frac{3}{2} &\geq \lambda \left(\frac{2}{3} - 2\sigma_2 \right) \Leftrightarrow \\ \Leftrightarrow \frac{2 - 4\sigma_2 + 6\sigma_3 - 3\sigma_2 + 3\sigma_3}{2(\sigma_2 - \sigma_3)} &\geq \frac{2\lambda(1 - 3\sigma_2)}{3} \Leftrightarrow \\ \Leftrightarrow \lambda &\leq \frac{3}{4} \cdot \frac{2 - 7\sigma_2 + 9\sigma_3}{(1 - 3\sigma_2)(\sigma_2 - \sigma_3)} = f(\sigma_3), \end{aligned}$$

where

$$f(t) = \frac{3}{4} \cdot \frac{2 - 7\sigma_2 + 9t}{(1 - 3\sigma_2)(\sigma_2 - t)}.$$

Note that

$$1 - 3\sigma_2 = \sigma_1^2 - 3\sigma_2 = \frac{1}{2} ((a-b)^2 + (b-c)^2 + (c-a)^2) > 0$$

since a, b, c are positive and not all equal. Moreover, $\sigma_2 - \sigma_3 = \sigma_1\sigma_2 - \sigma_3 = (a+b)(b+c)(c+a) > 0$, so $\sigma_3 < \sigma_2$. We further have

$$f'(t) = \frac{3(\sigma_2 + 1)}{2(1 - 3\sigma_2)(\sigma_2 - t)^2}$$

for all $t \neq \sigma_2$, so f is strictly increasing for $t < \sigma_2$ (and for $t > \sigma_2$, too, but this is not important here). Also, we have $\sigma_3 \in [s_1, s_2]$ by theorem 1 (thus, actually, $\sigma_3 \in [s_1, \sigma_2]$) and $\sigma_3 = s_1$ can be achieved when two of the numbers are equal, therefore we must also have

$$\lambda \leq f(s_1) = \frac{27}{12\sigma_2 + 4 + 4\sqrt{1 - 3\sigma_2}}$$

for all $\sigma_2 \in \left(0, \frac{1}{3}\right)$. So the best constant would be

$$\lambda_0 = \inf_{0 < t < \frac{1}{3}} g(t)$$

if we denote

$$g(t) = \frac{27}{12t + 4 + 4\sqrt{1-3t}} \text{ for } t < \frac{1}{3}.$$

The above infimum (actually the minimum) is easily seen to be 3, achieved for $t = \frac{1}{4}$, and it indeed represents the best constant for which the inequality holds, because we have

$$3 \leq g(\sigma_2) = f(s_1) \leq f(\sigma_3)$$

since f is increasing for $t < \sigma_2$, and $s_1 \leq \sigma_3 < \sigma_2$.

For $\lambda_0 = 3$ we obtain the inequality

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} &\geq 3 \left(\frac{a^2 + b^2 + c^2}{(a+b+c)^2} - \frac{1}{3} \right) \Leftrightarrow \\ \Leftrightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\geq \frac{3(a^2 + b^2 + c^2)}{(a+b+c)^2} + \frac{1}{2}. \end{aligned}$$

We invite the reader to see if (for $\lambda = 3$) any other case of equality (except for $a = b = c$) exists.

Application 7. Find the greatest real number λ such that the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq \lambda \left(\frac{8}{27} - \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \right) \quad (23)$$

holds for all positive real numbers a, b, c .

Solution. The inequality becomes an equality for $a = b = c$ (and any λ), so we can assume that not all of a, b, c are equal. Now, (23) is equivalent to

$$\frac{\sigma_1^3 - 2\sigma_1\sigma_2 + 3\sigma_3}{\sigma_1\sigma_2 - \sigma_3} - \frac{3}{2} \geq \lambda \left(\frac{8}{27} - \frac{\sigma_1\sigma_2 - \sigma_3}{\sigma_1^3} \right).$$

Without loss of generality we can consider $\sigma_1 = 1$, so we have to show that

$$\frac{1 - 2\sigma_2 + 3\sigma_3}{\sigma_2 - \sigma_3} - \frac{3}{2} \geq \lambda \left(\frac{8}{27} - \sigma_2 + \sigma_3 \right),$$

or

$$\lambda \leq \frac{27}{2} \cdot \frac{2 - 7\sigma_2 + 9\sigma_3}{(\sigma_2 - \sigma_3)(8 - 27\sigma_2 + 27\sigma_3)} = f(\sigma_3)$$

with

$$f(t) = \frac{27}{2} \cdot \frac{2 - 7\sigma_2 + 9t}{(\sigma_2 - t)(8 - 27\sigma_2 + 27t)}.$$

(Note that $8 - 27\sigma_2 + 27\sigma_3 > 0$ for positive a, b, c not all equal. We also have $\sigma_2 > \sigma_3$.) Now, since the derivative of f is

$$f'(t) = \frac{27}{8} \left(\frac{\sigma_2 + 1}{(\sigma_2 - t)^2} + \frac{243(1 - 3\sigma_2)}{(8 - 27\sigma_2 + 27t)^2} \right),$$

hence it is positive, we have that f is increasing for $t < \sigma_2$. By Theorem 1, we also have $s_1 \leq \sigma_3$ (thus $s_1 \leq \sigma_3 < \sigma_2$), therefore any λ must satisfy

$$\lambda \leq f(s_1) = \frac{243}{8(\sqrt{1-3\sigma_2}+1) + 12(\sqrt{1-3\sigma_2}-4)\sigma_2} = g(\sigma_2),$$

where

$$g(t) = \frac{243}{8(\sqrt{1-3t}+1) + 12(\sqrt{1-3t}-4)t}.$$

The required best constant would be

$$\lambda_0 = \inf_{0 < t \leq \frac{1}{3}} g(t) = \frac{27}{32} (13\sqrt{13} - 35) \simeq 10.017$$

(the minimum being achieved at $t = \frac{8\sqrt{13}-20}{27}$). And, indeed, this is the best constant, as it verifies $\lambda_0 \leq g(\sigma_2) = f(s_1) \leq f(\sigma_3)$.

Remark 2. For $\lambda = \frac{81}{16}$ we obtain

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{81}{16} \cdot \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq 3. \quad (24)$$

(*Nguyen Viet Hung*, see [1])

Since we have $\frac{81}{16} < \frac{27}{32} (13\sqrt{13} - 35)$, the inequality from (23) for $\lambda = \frac{27}{32} (13\sqrt{13} - 35)$ is a refinement of the inequality from (24).

We invite the interested reader to find more applications of Theorem 1 in the study of inequalities depending on three variables. For example, just to begin, one could ask if this result is useful in proving the inequalities between the means of three positive real numbers.

References

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