

Junior problems

J733. Solve in real numbers the system of equations

$$\begin{aligned}x(x^2 + 1) + y &= 5 \\y^3 + 3xy(x + y) + z &= 60 \\z^3 + 3z(x + y + z)(x + y) &= -63.\end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Adding the three equations, we get

$$(x + y + z)^3 + (x + y + z) - 2 = 0.$$

Setting $t = x + y + z$, we find

$$t^3 + t - 2 = 0 \iff (t - 1)(t^2 + t + 2) = 0,$$

which gives $t = 1$, i.e. $x + y + z = 1$.

Adding the first two equations, we get $(x + y)^3 + (x + y + z) = 65$, i.e. $(1 - z)^3 = 64$, which gives $1 - z = 4$, i.e. $z = -3$.

So, $x + y = 4$ and from the first equation of the system we get $x^3 + 4 = 5$, i.e. $x^3 = 1$, which gives $x = 1$ and $y = 3$. An easy check shows that $(x, y, z) = (1, 3, -3)$ is a solution to the given system of equations.

Solution 2 by Polyhedra, Polk State College, FL, USA

Let $x + y + z = r$. Adding the first and second equations we get

$$(x + y)^3 + r = 65.$$

The third equation can be written as

$$r^3 - (x + y)^3 = -63,$$

thus $0 = r^3 + r - 2 = (r - 1)(r^2 + r + 2)$. Therefore, $r = 1$ and $x + y = 4$, so $z = -3$, $x = 1$, and $y = 3$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Brenna Meagher, SUNY Brockport, NY, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; Debarun Das, CMC, Hyderabad, India; Dow Lee, St. George's School, Vancouver, Canada; Dragoljub Milošević, Gornji Milanovac, Serbia; G. C. Greubel, Newport News, VA, USA; José Luis Díaz-Barrero, Barcelona, Spain; Luke Paluso, Blinn College, Brenham, TX, USA; Mikhail Balepin, California, USA; Nicole Indelicato, SUNY Brockport, NY, USA; Ritwik Kumar, Noida, India; Rousen Pirguliyeu; Sahel Koley, Tarakeswar, West Bengal, India; Sundares H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Daniel Pascuas, Barcelona, Spain; Ganghun Kim, Fort Lee High School, NJ, USA; Kristine Lee, Mount Juliet High School, TN, USA; Nguyen Tuan Minh; Rachayita Bose, Kolkata, West Bengal, India; Saarang Agarwal, AMMOC; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania;

J734. Let x and y be real numbers such that

$$4^{x+y} - 16^x + 4^{x-y} = 1.$$

Prove that $|x| = |y|$.

Proposed by Adrian Andreescu, Dallas, USA

Solution 1 by the author

Adding or subtracting 2^{2x+1} to both sides of the equation

$$4^{x+y} + 4^{x-y} = 16^x + 1$$

yields

$$(2^{x+y} \pm 2^{x-y})^2 = (2^{2x} \pm 1)^2,$$

where the signs $+$ and $-$ correspond, implying

$$2^{x+y} + 2^{x-y} = 2^{2x} + 1 \quad \text{and} \quad 2^{x+y} - 2^{x-y} = \pm(2^{2x} - 1).$$

It follows that $x + y = 2x$ and $x - y = 0$ or $x + y = 0$ and $x - y = 2x$, yielding $x = y$ or $x = -y$. Hence the conclusion.

Solution 2 by Polyhedra, Polk State College, FL, USA

Indeed, $0 = 16^x - 4^{x+y} - 4^{x-y} + 1 = (4^x - 4^y)(4^x - 4^{-y})$, so $x = y$ or $x = -y$.

Solution 3 by Daniel Pascuas, Barcelona, Spain

Since $4^{x+y} - 16^x + 4^{x-y} = 4^{x+y} - 4^{2x} + 4^{x-y} = 4^x(4^y - 4^x + 4^{-y})$, the equation $4^{x+y} - 16^x + 4^{x-y} = 1$ can be written as $4^y - 4^x + 4^{-y} = 4^{-x}$, that is, $4^x + 4^{-x} = 4^y + 4^{-y}$. In other words, we have that $f(x) = f(y)$, where $f(t) = 4^t + 4^{-t}$, $t \in \mathbb{R}$. Note that f is an even function. Moreover, f is increasing on the interval $[0, \infty)$, because it is the composition of two increasing functions: $t \in [0, \infty) \mapsto 4^t \in [1, \infty)$ and $s \in [1, \infty) \mapsto s + \frac{1}{s} \in \mathbb{R}$. The latter function is increasing due to the inequality

$$\left(s_2 + \frac{1}{s_2}\right) - \left(s_1 + \frac{1}{s_1}\right) = s_2 - s_1 + \frac{s_1 - s_2}{s_1 s_2} = (s_2 - s_1) \left(1 - \frac{1}{s_1 s_2}\right) > 0 \quad (1 \leq s_1 < s_2).$$

Therefore the hypothesis $f(x) = f(y)$ implies that $f(|x|) = f(|y|)$ (since f is even) and so $|x| = |y|$ (since f is injective on $[0, \infty)$, because it is increasing on that interval). And we are done.

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J735. Let $a \geq b \geq 1 \geq c > 0$ such that $a + b + c = 3$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

Proposed by Mircea Lascu, Zalău, Romania

Solution 1 by the author

We can write the inequality as

$$ab + bc + ca \geq abc(a^2 + b^2 + c^2),$$

or

$$ab + (a + b)(3 - a - b) \geq ab(3 - a - b) [(a + b)^2 - 2ab + (3 - a - b)^2].$$

Denote $x = a + b$, $y = ab$, then we rewrite the given inequality as a quadratic in y :

$$2(3 - x)y^2 + (2x^3 - 12x^2 + 27x - 26)y + 3x - x^2 \geq 0.$$

Since $(a - 1)(b - 1) \geq 0$ it follows that $y \geq x - 1$. Also we have $2 \leq x < 3$. Denote the left-hand side by $f(y)$, so that we want to prove $f(y) \geq 0$. Since f quadratic function of y and

$$-\frac{2x^3 - 12x^2 + 27x - 26}{4(3 - x)} - (x - 1) = -\frac{(x - 2)(2x^2 - 12x + 19)}{3 - x} \leq 0,$$

we see that $f(y)$ is an increasing function of y for $y \geq x - 1$. Hence it suffices to prove $f(x - 1) \geq 0$, that is

$$2(3 - x)(x - 1)^2 + (2x^3 - 12x^2 + 27x - 26)(x - 1) + 3x - x^2 \geq 0.$$

After expanding this becomes

$$2(x - 2)^4 \geq 0,$$

which is clearly true.

The equality holds when $x = 2$, $y = 1$ which means $a = b = c = 1$.

Solution 2 by Polyhedra, Polk State College, FL, USA

Let $q = ab + bc + ca$ and $r = abc$. We have $0 \geq (a - 1)(b - 1)(c - 1) = r - q + 2$, so $q \geq r + 2$. Therefore,

$$ab + bc + ca - abc(a^2 + b^2 + c^2) = q - r(9 - 2q) = (2r + 1)q - 9r \geq (2r + 1)(r + 2) - 9r = 2(r - 1)^2 \geq 0,$$

which is equivalent to the claimed inequality.

Solution 3 by Anita Tufisi, Romania

Since $a + b + c = 3$, we have

$$1 - c = (a - 1) + (b - 1).$$

Moreover, $a + c = 3 - b \leq 2$, so by AM-GM,

$$ac \leq \left(\frac{a + c}{2}\right)^2 \leq 1.$$

Similarly, $bc \leq 1$.

Hence

$$c + \frac{1}{c} - a - \frac{1}{a} = (a - c) \left(\frac{1}{ac} - 1\right) \geq 0,$$

which gives

$$c + 1 + \frac{1}{c} \geq a + 1 + \frac{1}{a}.$$

Likewise,

$$c + 1 + \frac{1}{c} \geq b + 1 + \frac{1}{b}.$$

Therefore,

$$\begin{aligned} \frac{1}{c} - c^2 &= (1 - c) \left(c + 1 + \frac{1}{c} \right) \\ &= ((a - 1) + (b - 1)) \left(c + 1 + \frac{1}{c} \right) \\ &\geq (a - 1) \left(a + 1 + \frac{1}{a} \right) + (b - 1) \left(b + 1 + \frac{1}{b} \right) \\ &= \left(a^2 - \frac{1}{a} \right) + \left(b^2 - \frac{1}{b} \right). \end{aligned}$$

Rearranging the last inequality yields

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

Solution 4 by Daniel Pascuas, Barcelona, Spain

We write our inequality as $f(a) + f(b) + f(c) \geq 0$, where

$$f(x) = \frac{1}{x} - x^2 = \frac{1 - x^3}{x} = \frac{(1 - x)^3 - 3x^2 + 3x}{x} = -3(x - 1) - \frac{(x - 1)^3}{x} \quad (x > 0).$$

Since $a + b + c = 3$, we have that $f(a) + f(b) + f(c) = -\frac{(a-1)^3}{a} - \frac{(b-1)^3}{b} - \frac{(c-1)^3}{c}$, so the inequality we want to prove is

$$\frac{(a - 1)^3}{a} + \frac{(b - 1)^3}{b} + \frac{(c - 1)^3}{c} \leq 0. \quad (1)$$

Let $x = a - 1$ and $y = b - 1$. Then $x \geq y \geq 0$ and $c = 3 - (a + b) = 3 - (x + y + 2) = 1 - (x + y)$, so $x + y < 1$, and (1) is rewritten as

$$\frac{x^3}{1 + x} + \frac{y^3}{1 + y} \leq \frac{(x + y)^3}{1 - (x + y)}. \quad (2)$$

Since $x \geq y \geq 0$ and $0 \leq x + y < 1$, we have that $1 + x \geq 1 + y \geq 1 \geq 1 - (x + y) > 0$, so

$$\frac{x^3}{1 + x} + \frac{y^3}{1 + y} \leq \frac{x^3 + y^3}{1 - (x + y)} \leq \frac{x^3 + 3x^2 + 3y^2 + y^3}{1 - (x + y)} = \frac{(x + y)^3}{1 - (x + y)}.$$

Moreover, any of the preceding two inequalities becomes an equality if and only if $x = y = 0$. Therefore the inequality (2) holds and it becomes an equality if and only if $x = y = 0$. Hence we have just proved that the inequality in the statement holds and it becomes an equality only when $a = b = c = 1$.

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J736. Let a, b, c be real numbers such that $b + ca + 3c^2 = 0$. Prove that there are two roots u and v of the polynomial $p(x) = x^3 + ax^2 + bx + c^3$ such that $(c - u)v = c^2$.

Proposed by Li Zhou, Polk State College, USA

Solution 1 by the author

Let x_1, x_2, x_3 be the roots of $p(x)$ and let $k = x_1x_2 - cx_2 + c^2$. Since $x_1x_2x_3 = -c^3$, $kx_3 = -c^3 - cx_2x_3 + c^2x_3$ and $kx_3x_1 = -c^3x_1 + c^4 + c^2x_3x_1$. Therefore,

$$\begin{aligned} k(x_3x_1 - cx_3 + c^2) &= (-c^3x_1 + c^4 + c^2x_3x_1) - c(-c^3 - cx_2x_3 + c^2x_3) + c^2(x_1x_2 - cx_2 + c^2) \\ &= c^2(x_3x_1 + x_2x_3 + x_1x_2) - c^3(x_1 + x_3 + x_2) + 3c^4 \\ &= c^2(b + ca + 3c^2) \\ &= 0, \end{aligned}$$

so either $k = 0$ or $x_1x_3 - cx_3 + c^2 = 0$, completing the proof.

Solution 2 by Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA

First, compute

$$p(c) = c^3 + ac^2 + bc + c^3 = c(b + ca + 3c^2) - c^3 = -c^3.$$

If $c = 0$, then the condition gives $b = 0$, and

$$p(x) = x^3 + ax^2 = x^2(x + a).$$

Thus 0 is a root of p . Taking $u = v = 0$, we get

$$(c - u)v = 0 = c^2.$$

Assume now that $c \neq 0$. Then $p(c) = -c^3 \neq 0$, so c is not a root of p .

We claim that, for every $x \neq c$,

$$(c - x)^3 p\left(\frac{c^2}{c - x}\right) = -c^3 p(x). \quad (1)$$

Indeed,

$$\begin{aligned} (c - x)^3 p\left(\frac{c^2}{c - x}\right) &= (c - x)^3 \left[\left(\frac{c^2}{c - x}\right)^3 + a \left(\frac{c^2}{c - x}\right)^2 + b \left(\frac{c^2}{c - x}\right) + c^3 \right] \\ &= c^6 + ac^4(c - x) + bc^2(c - x)^2 + c^3(c - x)^3 \\ &= c^2[-cx^3 + (b + 3c^2)x^2 - (2b + ca + 3c^2)cx + c^2(b + ca + 2c^2)]. \end{aligned}$$

Using $b + ca + 3c^2 = 0$, we obtain

$$b + 3c^2 = -ca, \quad 2b + ca + 3c^2 = b, \quad b + ca + 2c^2 = -c^2.$$

Hence

$$\begin{aligned} (c - x)^3 p\left(\frac{c^2}{c - x}\right) &= c^2[-cx^3 - cax^2 - cbx - c^4] \\ &= -c^3(x^3 + ax^2 + bx + c^3) \\ &= -c^3 p(x), \end{aligned}$$

as claimed.

Now let u be a root of p . Since c is not a root, we have $u \neq c$. By (1),

$$(c - u)^3 p\left(\frac{c^2}{c - u}\right) = -c^3 p(u) = 0.$$

Therefore $v = \frac{c^2}{c-u}$ is also a root of p . Finally,

$$(c-u)v = (c-u) \cdot \frac{c^2}{c-u} = c^2.$$

Thus there exist two roots u and v of p such that $(c-u)v = c^2$.

Solution 3 by Theo Koupelis, Clark College, Washington, USA

If $c = 0$, then from the given condition $b + ca + 3c^2 = 0$ we get $b = 0$, and thus $p(x) = x^2(x+a)$. Therefore, two of the three roots of $p(x)$ are zero and thus the desired equation $(c-u)v = c^2$ holds for any pair of roots.

Let $c \neq 0$, and let u be a root of $p(x)$; clearly $u \neq c$ because otherwise we would get $c^3 + ac^2 - (ac + 3c^2)c + c^3 = 0 \iff c = 0$. We now define the numbers v, w such that $(c-u)v = c^2$ and $uvw = -c^3$. Multiplying both sides of the former equation by c and using the latter equation, we get $(c-w)u = c^2$; multiplying both sides of the latter equation by c and using $uvw = -c^3$, we get $(c-v)w = c^2$. Then, using $v = c^2/(c-u)$ and $w = (cu - c^2)/u$ we get

$$u + v + w = -a \iff u + \frac{c^2}{c-u} + \frac{cu - c^2}{u} = -a \iff u^3 + au^2 - (ac + 3c^2)u + c^3 = 0,$$

which is obvious. Adding the three equations $(c-u)v = (c-w)u = (c-v)w = c^2$ we get

$$c(u + v + w) - (uv + vw + wu) = 3c^2 \implies uv + vw + wu = -ca - 3c^2 = b.$$

Thus, by Vieta's formulas, the numbers u, v, w are the roots of the polynomial and the desired equation holds.

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J737. Let ABC be an acute triangle with the circumcircle $\Gamma(O, R)$. The points X, Y, Z represent the intersections of the lines AO, BO, CO with the circumscribed circles of the triangles BOC, AOC , and AOB , respectively. If $\max\{AX, BY, CZ\} \leq 3R$, show that $\triangle ABC$ is an equilateral triangle.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Rachayita Bose, Kolkata, West Bengal, India

In cyclic quadrilateral $BXCO$, by Ptolemy's Theorem,

$$OX \cdot BC = OB \cdot CX + OC \cdot BX = R \cdot CX + R \cdot BX = R(CX + BX)$$

$$\implies OX = \frac{R(CX + BX)}{BC}$$

Let R_A be the radius of the circumscribed circle of $\triangle BOC$. Since $\angle BOC = 2\angle A$, we have $\angle BOX = 180^\circ - 2\angle A$.

$$BX = 2R_A \sin(180^\circ - 2C) = 2R_A \sin 2C$$

Similarly,

$$CX = 2R_A \sin 2B, \quad BC = 2R_A \sin 2A.$$

Now,

$$AX = AO + OX = R + R \cdot \frac{CX + BX}{BC} = R \left(1 + \frac{\sin 2B + \sin 2C}{\sin 2A} \right) = R \left[\frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A} \right]$$

Similarly,

$$BY = R \left[\frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2B} \right], \quad CZ = R \left[\frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2C} \right]$$

Therefore,

$$\max\{AX, BY, CZ\} = \frac{R(\sin 2A + \sin 2B + \sin 2C)}{\min\{\sin 2A, \sin 2B, \sin 2C\}} \leq 3R$$

$$\implies \min\{\sin 2A, \sin 2B, \sin 2C\} \geq \frac{\sin 2A + \sin 2B + \sin 2C}{3}$$

But also, by definition of the minimum,

$$\min\{\sin 2A, \sin 2B, \sin 2C\} \leq \frac{\sin 2A + \sin 2B + \sin 2C}{3}$$

Hence equality holds throughout, which forces

$$\sin 2A = \sin 2B = \sin 2C.$$

Since ABC is acute, this implies $A = B = C$, so $\triangle ABC$ is equilateral.

Solution 2 by Polyhedra, Polk State College, FL, USA

First, $\angle OXB = \angle OCB = 90^\circ - A$ and $\angle BOX = 180^\circ - 2C$. Therefore, $\angle XBO = 2C + A - 90^\circ = C - B + 90^\circ$, so by the law of sines and trigonometric identities,

$$\frac{OX}{R} = \frac{OX}{OB} = \frac{\sin \angle XBO}{\sin \angle OXB} = \frac{\cos(C - B)}{\cos A} = \frac{2 \sin A \cos(C - B)}{2 \sin A \cos A} = \frac{\sin 2B + \sin 2C}{\sin 2A}.$$

Therefore,

$$6R \geq OX + OY + OZ = R \left(\frac{\sin 2B}{\sin 2A} + \frac{\sin 2A}{\sin 2B} + \frac{\sin 2C}{\sin 2B} + \frac{\sin 2B}{\sin 2C} + \frac{\sin 2A}{\sin 2C} + \frac{\sin 2C}{\sin 2A} \right) \geq 6R,$$

with equality if and only if $A = B = C = 60^\circ$.

Solution 3 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Let X', Y', Z' be the second intersections of AO, BO, CO with the circumcircles of BOC, AOC, AOB , respectively. These are exactly the points X, Y, Z .

Because ABC is acute, each of X, Y, Z lies on the ray opposite to the corresponding vertex, so

$$AX = AO + OX = R + OX,$$

and similarly

$$BY = R + OY, \quad CZ = R + OZ.$$

Hence, from the hypothesis $\max\{AX, BY, CZ\} \leq 3R$, we get

$$OX, OY, OZ \leq 2R.$$

Now we use the classical lemma from the standard solution of IMO Shortlist 1996/16: for the second intersections X, Y, Z defined above, one has

$$OX \cdot OY \cdot OZ \geq 8R^3,$$

with equality if and only if ABC is equilateral.

Combining this lemma with $OX, OY, OZ \leq 2R$, we obtain

$$8R^3 \leq OX \cdot OY \cdot OZ \leq (2R)^3 = 8R^3.$$

Therefore equality holds throughout, so

$$OX = OY = OZ = 2R.$$

The equality case of the lemma forces ABC to be equilateral. Thus, $\triangle ABC$ is equilateral.

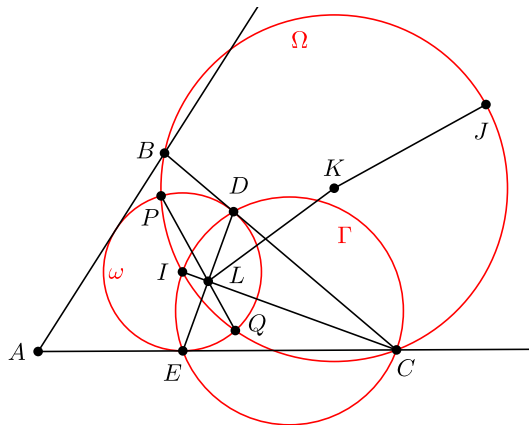
Also solved by Theo Koupelis, Clark College, Washington, USA; Kristine Lee, Mount Juliet High School, TN, USA; Shilong Zhuang, California, USA.

J738. Let J be the A -excenter of $\triangle ABC$ and K be the center of the circumcircle Ω of $\triangle JBC$. Suppose that the incircle ω of $\triangle ABC$ intersects Ω at P, Q and is tangent to BC, CA at D, E , respectively. Let L be the point where DE intersects PQ . Prove that DL, KJ , and KL form the sides of a right triangle.

Proposed by Li Zhou, Polk State College, USA

Solution 1 by the author

Let I be the center of ω , then I is on the circumcircle Γ of $\triangle CDE$. It is also well known (and easy to prove) that I is on Ω . Since PQ is the radical axis of ω with Ω and DE is the radical axis of ω with Γ , L is on the radical axis IC of Γ with Ω , thus L is the midpoint of DE . By the power of a point, $DL^2 = IL \cdot LC = PL \cdot LQ = KJ^2 - KL^2$, from which the claim follows by the Pythagorean theorem.



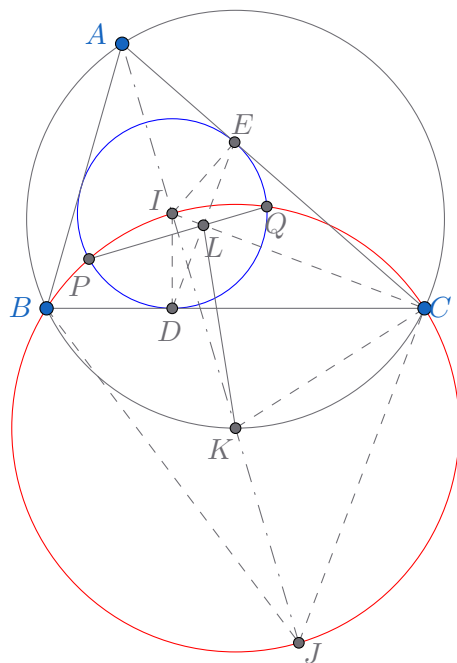
Solution 2 by Theo Koupelis, Clark College, Washington, USA

Using the incenter-excenter lemma, we get that A, I, K, J are collinear, the point K is on the circumcircle of $\triangle ABC$, and the points B, P, I, Q, C, J are on the circle Ω with center K . Let L' be the intersection point of CI and DE . The quadrilateral $IDCE$ is cyclic because $\angle IDC = \angle IEC = 90^\circ$, and thus $DL' \perp CI$ and $DL' = EL'$. The power of the point L' with respect to the incircle is $IL'^2 - ID^2 = IL'^2 - IL' \cdot IC = -IL' \cdot CL'$, which is also its power with respect to the circle Ω . Thus, L' is on the radical axis of the circles (I) and Ω , and therefore L' is on PQ . Thus, $L = L'$.

Let $a = BC$, $b = AC$, $c = AB$, and s the semiperimeter of the triangle. We have $\angle LCK = \angle ICK = (A + C)/2$. Also, $CE = s - c$ and $a = 2 \cdot KC \cos \frac{A}{2}$. Using the law of cosines in $\triangle KCL$ and the half-angle formulae for $\cos(A/2)$, $\cos(C/2)$, and $\sin(B/2)$, we get

$$\begin{aligned} KL^2 - KC^2 &= CL^2 - 2KC \cdot CL \cdot \cos \frac{A+C}{2} \\ &= CL^2 - \frac{a}{\cos \frac{A}{2}} \cdot (s-c) \cos \frac{C}{2} \cdot \sin \frac{B}{2} \\ &= CL^2 - a \sqrt{\frac{bc}{s(s-a)}} \cdot (s-c) \sqrt{\frac{s(s-c)}{ab}} \cdot \sqrt{\frac{(s-a)(s-c)}{ac}} \\ &= CL^2 - (s-c)^2 = CL^2 - CE^2 = -DL^2. \end{aligned}$$

Thus, because $KC = KJ$, we get $KJ^2 = KL^2 + DL^2$; therefore, DL, KJ , and KL form the sides of a right triangle.



Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Andrea Fanchini, Cantù, Italy; Polyhedra, Polk State College, FL, USA; Shilong Zhuang, California, USA.

Senior problems

S733. Solve in integers the equation

$$\frac{x^2 + y^2}{2} + \left| \frac{x + y}{2} + 1 \right| = 2026.$$

Proposed by Titu Andreescu, Dallas, USA and Alessandro Ventullo, Milan, Italy

Solution by the authors

Let $s = x + y$ and $d = x - y$. Then

$$x = \frac{s + d}{2}, \quad y = \frac{s - d}{2},$$

and

$$x^2 + y^2 = \frac{(s + d)^2 + (s - d)^2}{4} = \frac{s^2 + d^2}{2}.$$

So, the given equation can be rewritten as

$$\frac{s^2 + d^2}{4} + \frac{|s + 2|}{2} = 2026.$$

Multiplying by 4, we get

$$s^2 + d^2 + 2|s + 2| = 8104. \tag{1}$$

We have two cases.

(i) If $s \geq -2$, then $|s + 2| = s + 2$ and (1) becomes

$$s^2 + d^2 + 2(s + 2) = 8104 \implies (s + 1)^2 + d^2 = 8101.$$

Since $8101 = 90^2 + 1^2$ and this is the only representation as the sum of two perfect squares, then up to signs and swapping we have

$$(s + 1, d) \in \{(\pm 1, \pm 90), (\pm 90, \pm 1)\}.$$

With $s \geq -2$, we obtain $s \in \{-2, 0, 89\}$, and therefore

$$(44, -46), (-46, 44), (45, -45), (-45, 45), (45, 44), (44, 45).$$

(ii) If $s < -2$, then $|s + 2| = -(s + 2)$ and (1) becomes

$$s^2 + d^2 - 2(s + 2) = 8104 \implies (s - 1)^2 + d^2 = 8109.$$

Since $8109 = 90^2 + 3^2 = 78^2 + 45^2$, up to signs and swapping we get

$$(s - 1, d) \in \{(\pm 90, \pm 3), (\pm 3, \pm 90), (\pm 78, \pm 45), (\pm 45, \pm 78)\}.$$

As $s < -2$, we get $s - 1 \in \{-90, -78, -45\}$, so

$$(17, -61), (-61, 17), (-16, -61), (-61, -16), (-43, -46), (-46, -43).$$

In conclusion, all integer solutions (x, y) are

$$(45, -45), (-45, 45), (45, 44), (44, 45), (44, -46), (-46, 44), \\ (17, -61), (-61, 17), (-16, -61), (-61, -16), (-43, -46), (-46, -43).$$

Editor's Comment.

Daniel Pascuas provided a very similar solution to this one, but justified the above representations as sums of two squares.

First, he observe that $8101 = 90^2 + 1^2$ and 8101 is a prime congruent to 1 modulo 4. Hence, by the classical two-square theorem, this representation is unique up to signs and order.

For $8109 = 90^2 + 3^2 = 78^2 + 45^2$, he invoked Jacobi's two-square theorem. Since $8109 = 3^2 \cdot 17 \cdot 53$, the number of representations of 8109 as a sum of two squares, counting signs and order, is

$$r_2(8109) = 4(d_1(8109) - d_3(8109)),$$

where $d_1(n)$ is the number of divisors of n congruent to 1 modulo 4 and $d_3(n)$ is the number of divisors congruent to 3 modulo 4. Here $d_1(8109) = 8$ and $d_3(8109) = 4$, so

$$r_2(8109) = 4(8 - 4) = 16.$$

The two displayed representations $8109 = 90^2 + 3^2 = 78^2 + 45^2$ already produce exactly 16 representations when signs and order are counted. Therefore there are no further representations.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; G. C. Greubel, Newport News, VA, USA; José Luis Díaz-Barrero, Barcelona, Spain; Mohammed Aassila, Strasbourg, France; Ritwik Kumar, Noida, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Nguyen Tuan Minh; Shilong Zhuang, California, USA.

S734. Prove that for any positive integer n ,

$$\sum_{k=1}^n \gcd(k, n) = n \cdot \sum_{d|n} \frac{\varphi(d)}{d},$$

where φ is Euler's totient function.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Ritwik Kumar, Noida, India

For each divisor $d \mid n$, the condition $\gcd(k, n) = d$ is equivalent to:

$$k = d\ell, \quad 1 \leq \ell \leq \frac{n}{d}, \quad \gcd\left(\ell, \frac{n}{d}\right) = 1.$$

Thus the number of $k \in \{1, \dots, n\}$ with $\gcd(k, n) = d$ is $\varphi(n/d)$. Therefore:

$$\sum_{k=1}^n \gcd(k, n) = \sum_{d|n} d \varphi\left(\frac{n}{d}\right).$$

Substitute $e = n/d$. Then $d = n/e$, and:

$$\sum_{k=1}^n \gcd(k, n) = n \sum_{e|n} \frac{\varphi(e)}{e}.$$

This is the desired identity.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; Henry Ricardo, Westchester Area Math Circle, New York, USA; José Luis Díaz-Barrero, Barcelona, Spain; The Ring Lords, Baden-Württemberg, Germany; Ganghun Kim, Fort Lee High School, NJ, USA; Nguyen Tuan Minh; Shilong Zhuang, California, USA.

S735. Let $n \geq 3$ be an integer and $A_0A_1 \dots A_{n-1}$ a regular polygon of center O . On the ray OA_i , a point B is taken such that $OB = A_iA_{i+1}$. It is known that the triangle $A_{i-1}BA_{i+1}$ is equilateral. Find n .

Proposed by Mircea Becheanu, Montreal, Canada

Solution 1 by the author

From geometric considerations it is clear that cases $n = 3, 4, 5, 6$ are not possible, so $OB < OA_i$. We can assume that the polygon is inscribed in the unit circle of center O and the point B is on the ray OA_0 . Then the equilateral triangle is A_1BA_{n-1} . We will use complex numbers such that A_0 is the affix of the complex number $z = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, B is the affix of the complex number $b = |z - 1| + 0i$ and A_{n-1} is the affix of $\bar{z} = \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n}$.

Let $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ be the complex cubic root of the unit such that $-\omega^2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$. The condition that the triangle is A_1BA_{n-1} is equilateral can be written as follows:

$$-\omega^2(\bar{z} - b) = z - b$$

After algebraic transformations it is equivalent to

$$b + \omega^2 z + \overline{\omega^2 z} = 0.$$

Since $\omega^2 z = \cos\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right) + i \sin\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right)$ it is equivalent to

$$\left|1 - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n}\right| + 2 \cos\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right) = 0.$$

It follows that

$$\cos\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right) < 0$$

which requires

$$\frac{4\pi}{3} + \frac{2\pi}{n} < \frac{3\pi}{n} \Leftrightarrow n > 12.$$

By squaring the equality of positive real numbers

$$\left|1 - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n}\right| = -2 \cos\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right)$$

we obtain

$$\begin{aligned} 4 \cos^2\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right) &= 2 \left(1 - \cos \frac{2\pi}{n}\right) \Leftrightarrow 2 \cos^2\left(\frac{4\pi}{3} + \frac{2\pi}{n}\right) - 1 = -\cos \frac{2\pi}{n} \Leftrightarrow \\ \cos\left(\frac{8\pi}{3} + \frac{4\pi}{n}\right) + \cos \frac{2\pi}{n} &= 0 \Leftrightarrow \cos\left(\frac{4\pi}{3} + \frac{3\pi}{n}\right) \cdot \cos\left(\frac{4\pi}{3} + \frac{\pi}{n}\right) = 0. \end{aligned}$$

Solving this trigonometric equation we obtain

$$\cos\left(\frac{4\pi}{3} + \frac{3\pi}{n}\right) = 0 \quad \text{or} \quad \cos\left(\frac{4\pi}{3} + \frac{\pi}{n}\right) = 0.$$

In the first case we have

$$\frac{4\pi}{3} + \frac{3\pi}{n} = (2k+1)\frac{\pi}{2} \Leftrightarrow \frac{4}{3} + \frac{3}{n} = k + \frac{1}{2} \Leftrightarrow \frac{5}{6} + \frac{3}{n} = k < 2.$$

Hence $\frac{5}{6} + \frac{3}{n} = 1$, giving that $n = 18$.

In the second case

$$\frac{4}{3} + \frac{1}{n} = k + \frac{1}{2} \Leftrightarrow \frac{5}{6} + \frac{1}{n} = k.$$

Since $n > 12$ we have

$$\frac{5}{6} + \frac{1}{n} < \frac{5}{6} + \frac{1}{12} = \frac{11}{12} < 1.$$

Therefore, $n = 18$ is the unique solution.

Solution 2 by Ritwik Kumar, Noida, India

Let R be the circumradius and set $t = \frac{\pi}{n}$. Choose coordinates so that:

$$O = (0, 0), \quad A_i = (R, 0), \quad B = (2R \sin t, 0), \quad A_{i\pm 1} = (R \cos 2t, \pm R \sin 2t).$$

Since $A_{i-1}BA_{i+1}$ is equilateral we have $A_{i-1}B = A_{i-1}A_{i+1} = 2R \sin 2t$. Squaring, we get

$$(R \cos 2t - 2R \sin t)^2 + (R \sin 2t)^2 = 4R^2 \sin^2 2t \iff 8 \sin^3 t - 6 \sin t + 1 = 0.$$

Using $\sin 3t = 3 \sin t - 4 \sin^3 t$, we get $\sin 3t = \frac{1}{2}$.

Since $0 < t < \pi/3$, we have $0 < 3t < \pi$, so $3t = \frac{\pi}{6}$ or $3t = \frac{5\pi}{6}$. Thus $t = \frac{\pi}{18}$ or $t = \frac{5\pi}{18}$.

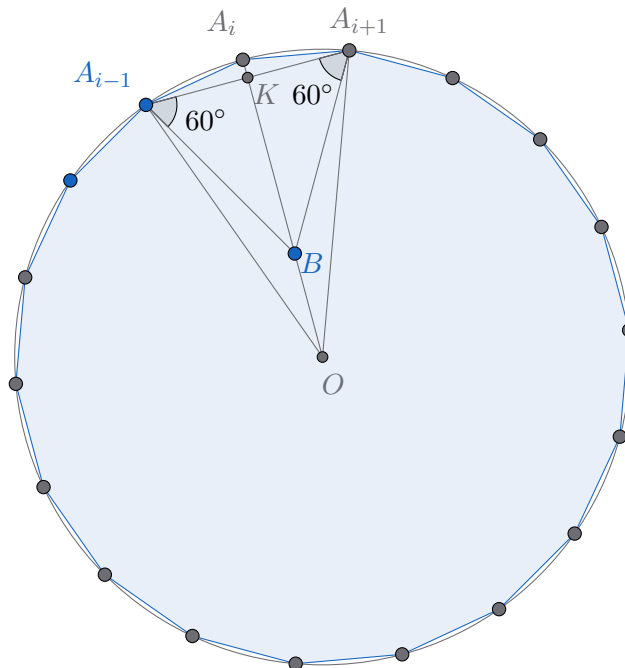
Since $t = \pi/n$ and n is a positive integer we conclude that $n = 18$.

Solution 3 by Theo Koupelis, Clark College, Washington, USA

Let R be the radius of circle (O) , and let $\angle A_0OA_1 = 2\theta$. Then $A_iA_{i+1} = 2R \sin \theta$. It is given that $n \geq 3$, and thus $2\theta = 360^\circ/n \leq 120^\circ \implies \theta \leq 60^\circ$. Let K be the midpoint of $A_{i-1}A_{i+1}$. Because the polygon is regular, the points O, K, B, A_i are collinear, and the line OK is the perpendicular bisector of $A_{i-1}A_{i+1}$. Thus, $OK = R \cos(2\theta)$, $A_{i+1}K = R \sin(2\theta)$, and $BK = |OB - OK| = |2R \sin \theta - R \cos(2\theta)|$. It is known that $A_{i-1}A_{i+1} = A_{i-1}B = A_{i+1}B$, and thus

$$\begin{aligned} A_{i-1}A_{i+1} &= 2A_{i+1}K = 2R \sin(2\theta) \\ &= \sqrt{A_{i+1}K^2 + BK^2} = \sqrt{R^2 \sin^2(2\theta) + R^2(2 \sin \theta - \cos(2\theta))^2}. \end{aligned}$$

Squaring and simplifying we get $(2 \sin \theta + 1)(2 \sin(3\theta) - 1) = 0$. Therefore, $\sin(3\theta) = 1/2$, and thus $\theta = 10^\circ$ or $\theta = 50^\circ$. The latter option does not lead to an integer n , and thus the only solution is $n = 18$.



Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; José Luis Díaz-Barrero, Barcelona, Spain; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Ganghun Kim, Fort Lee High School, NJ, USA; Nguyen Tuan Minh; Shilong Zhuang, California, USA.

S736. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 3 \geq \frac{27}{100} (a^3 + b^3 + c^3 - 3).$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

Since $ab + bc + ca \leq 3$ there exists $t \in [0, 1)$, such that $ab + bc + ca = 3(1 - t^2)$.

Applying the Cauchy-Schwarz Inequality, we have

$$3 + 6t^2 = a^2 + b^2 + c^2 \geq a^2 + \frac{(b+c)^2}{2} = a^2 + \frac{(3-a)^2}{2}$$

which implies

$$(a-1)^2 \leq 4t^2 \iff 1-2t \leq a \leq 1+2t,$$

and analogously for b, c . It follows that

$$(a-1-2t)(b-1-2t)(c-1-2t) \leq 0 \iff abc \leq (1+2t)(1-t)^2.$$

We have

$$\begin{aligned} a^3 + b^3 + c^3 &= 3abc + (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) \\ &= 3abc + 27 - 27(1-t^2) = 3abc + 27t^2 \\ &\leq 3(1+2t)(1-t)^2 + 27t^2 = 3(1+6t^2+2t^3). \end{aligned}$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \sum_{cyc} \frac{1}{a} &= \frac{3}{1+2t} + \sum_{cyc} \left(\frac{1}{a} - \frac{1}{1+2t} \right) \\ &= \frac{3}{1+2t} + \sum_{cyc} \frac{1+2t-a}{a(1+2t)} \\ &= \frac{3}{1+2t} + \sum_{cyc} \frac{(1+2t-a)^2}{a(1+2t)(1+2t-a)} \\ &\geq \frac{3}{1+2t} + \frac{(\sum_{cyc} (1+2t-a))^2}{(1+2t) \sum_{cyc} a(1+2t-a)} \\ &= \frac{3}{1+2t} + \frac{6t}{(1+2t)(1-t)} = \frac{3(1+t)}{(1-t)(1+2t)}. \end{aligned}$$

Therefore, it suffices to prove that

$$\frac{3(1+t)}{(1-t)(1+2t)} - 3 \geq \frac{27 \cdot 6t^2(3+t)}{100},$$

or

$$\frac{6t^2}{(1-t)(1+2t)} \geq \frac{27 \cdot 6t^2(3+t)}{100},$$

or

$$\frac{3t^2(3t-1)^2(6t+19)}{50(1-t)(1+2t)} \geq 0$$

which is clearly true.

The equality holds when $t = 0$ which means $a = b = c = 1$ or $t = \frac{1}{3}$ which means $a = \frac{5}{3}$, $b = c = \frac{2}{3}$ and its cyclic permutations.

Solution 2 by Shilong Zhuang, California, USA

Denote

$$f(a, b, c) = \sum_{\text{cyc}} \left(\frac{1}{a} - 1 \right) - \lambda \sum_{\text{cyc}} (a^3 - 1).$$

Without loss of generality, we may assume $a = \max\{a, b, c\}$. Let $\lambda = \frac{27}{100}$ and $x = \frac{b+c}{2}$. Then $a + 2x = 3$, we have $a \geq 1 \geq x$, $b + c \leq 2$ and $bc \leq \frac{(b+c)^2}{4} \leq 1$.

First, we show that $f(a, b, c) \geq f(a, x, x)$. Notice that

$$\begin{aligned} f(a, b, c) - f(a, x, x) &= \left(\frac{1}{b} + \frac{1}{c} - \frac{2}{x} \right) - \lambda(b^3 + c^3 - 2x^3) \\ &= \frac{(b-c)^2}{bc(b+c)} - \lambda \left((b+c)(b^2 - bc + c^2) - \frac{(b+c)^3}{4} \right) \\ &= \frac{(b-c)^2}{bc(b+c)} - \frac{3\lambda(b+c)(b-c)^2}{4} \\ &= \frac{(b-c)^2(4 - 3\lambda bc(b+c)^2)}{4bc(b+c)} \end{aligned}$$

It is clear that $f(a, b, c) - f(a, x, x) \geq 0$ because $4 - 3\lambda bc(b+c)^2 \geq 4 - 12\lambda > 0$.

Next, we show that $f(a, x, x) \geq 0$. Since $a - 1 = 2(1 - x)$, we obtain

$$\begin{aligned} f(a, x, x) &= \frac{1}{a} - 1 + \frac{2}{x} - 2 - \lambda(a^3 - 1 + 2x^3 - 2) \\ &= \frac{1-a}{a} + \frac{2(1-x)}{x} - \lambda((a-1)(a^2 + a + 1) + 2(x-1)(x^2 + x + 1)) \\ &= \frac{1-a}{a} + \frac{a-1}{x} - \lambda(a-1)(a^2 + a + 1 - x^2 - x - 1) \\ &= \frac{(a-1)(a-x)}{ax} - \lambda(a-1)(a-x)(a+x+1) \\ &= \frac{(a-1)(a-x)(1 - \lambda ax(a+x+1))}{ax}. \end{aligned}$$

By the AM-GM inequality, we have

$$\lambda ax(a+x+1) = \frac{1}{10} \lambda(2a)(5x)(a+x+1) \leq \frac{\lambda}{10} \left(\frac{2a+5x+a+x+1}{3} \right)^3 = \frac{\lambda}{10} \left(\frac{10}{3} \right)^3 = \frac{100\lambda}{27} = 1.$$

It is clear that $a-1 \geq 0$ and $a-x \geq 0$ and it follows that $f(a, x, x) \geq 0$. Equality holds when $a = b = c = 1$ or $a = \frac{5}{3}, b = c = \frac{2}{3}$.

Solution 3 by Theo Koupelis, Clark College, Washington, USA

For the function $f(x) = \frac{1}{x} - \frac{27x^3}{100}$ we get

$$f''(x) = \frac{100 - 81x^4}{50x^3}.$$

Thus, the function has one inflection point for $x > 0$. According to the $n-1$ equal value principle, it suffices to consider the case when two of the three variables are equal.

Let $b = c = (3 - a)/2$, where $0 < a < 3$. Therefore, it suffices to show that

$$\frac{1}{a} + \frac{4}{3-a} - 3 - \frac{27}{100} \left[a^3 + 2 \cdot \frac{(3-a)^3}{8} - 3 \right] \geq 0 \iff \frac{3(a-1)^2(3a-5)^2(3a+16)}{400a(3-a)} \geq 0,$$

which is obvious. Equality occurs when $a = 1$ or $a = 5/3$; that is, equality occurs when $(a, b, c) = (1, 1, 1)$ or $\left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and cyclic permutations.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Arkady Alt, San Jose, California, USA; Corneliu Mănescu-Avram, Ploiești, Romania; José Luis Díaz-Barrero, Barcelona, Spain; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Ritwik Kumar, Noida, India; The Ring Lords, Baden-Württemberg, Germany; Teodor Cristian Petrescu, Romania.

S737. What is the smallest positive exponent k such that

$$a^k + b^k + c^k \geq 3$$

for any nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq d$ and $ab + bc + cd + da = 4$?

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution by the author

For $a = b = 2$ and $c = d = 0$, the inequality becomes

$$2^{k+1} \geq 3,$$

therefore

$$k \geq m = \frac{\ln 3}{\ln 2} - 1 \approx 0.58496, \quad 2^{m+1} = 3.$$

To show that m is the smallest positive k , we need to prove the inequality for $k = m$. In addition, since

$$4 = ab + bc + cd + da \leq ab + bc + c^2 + ca = ab + (a + b)c + c^2,$$

it suffices to show that $ab + (a + b)c + c^2 \geq 4$ involves

$$a^m + b^m + c^m \geq 3.$$

Since the left side of the required inequality increases if a increases, it suffices to consider the case

$$ab + (a + b)c + c^2 = 4.$$

For fixed c , consider b as function of a , and write the required inequality as $f(a) \geq 3$, where

$$f(a) = a^m + b^m + c^m.$$

Differentiating the equality constraint with respect to a yields

$$(a + c)b' + b + c = 0.$$

Therefore,

$$\begin{aligned} \frac{f'(a)}{m} &= a^{m-1} + b^{m-1}b' = a^{m-1} - b^{m-1} \cdot \frac{b + c}{a + c} = \frac{a^m - b^m + c(a^{m-1} - b^{m-1})}{a + c} \\ &\geq \frac{a^m - b^m + b(a^{m-1} - b^{m-1})}{a + c} = \frac{a^{m-1}(a + b) - 2b^m}{a + c} \geq \frac{2a^{m-1}(ab)^{1/2} - 2b^m}{a + c} \\ &= \frac{2b^{1/2}(a^{m-1/2} - b^{m-1/2})}{a + c} \geq 0. \end{aligned}$$

Thus $f(a)$ is increasing and has the minimum value for minimal a , hence for $a = b$. So, it suffices to consider this case, that is to show that

$$2b^m + c^m \geq 3$$

for $b + c = 2$. Write this inequality in the homogeneous form

$$\frac{2b^m + c^m}{(b + c)^m} \geq \frac{3}{2^m}.$$

Due to homogeneity, we may set $b = 1$, hence $c \leq b = 1$. So, we need to prove that $g(c) \geq 0$, where

$$g(c) = \frac{c^m + 2}{(c + 1)^m} - \frac{3}{2^m}.$$

We have

$$\frac{1}{m}g'(c) = \frac{1 - 2c^{1-m}}{c^{1-m}(c+1)^{1+m}}.$$

Since $g'(c) \geq 0$ for $c \in [0, c_0]$, and $g'(c) \leq 0$ for $c \in [c_0, 1]$, where $c_0 = 2^{-1/(1-m)}$, $g(c)$ is increasing on $[0, c_0]$ and decreasing on $[c_0, 1]$. Since $g(0) = g(1) = 0$, we have $g(c) \geq 0$ for $c \in [0, 1]$.

The proof is completed. For $k = m$, the equality occurs for $a = b = c = d = 1$, as well as for $a = b = 2$ and $c = d = 0$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Corneliu Mănescu-Avram, Ploiești, Romania; Theo Koupelis, Clark College, Washington, USA.

S738. Let ABC be a triangle with an arbitrary point $D \in BC$. Triangles ABD , ADC , and ABC have circumradii R_1 , R_2 , and R . If $\max\{R_1, R_2\} = R \cos \frac{A}{2}$, prove that

$$(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{BC} = 0.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Shilong Zhuang, California, USA

Denote $|\overrightarrow{BC}| = BC$, $|\overrightarrow{AC}| = AC$ and $|\overrightarrow{AB}| = AB$. By the Law of cosines, we have

$$\cos B = \frac{BC^2 + AB^2 - AC^2}{2BC \cdot AB} \quad \text{and} \quad \cos C = \frac{BC^2 + AC^2 - AB^2}{2BC \cdot AC}.$$

It follows that

$$(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{BC} = \overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{AC} \cdot \overrightarrow{BC} = -|\overrightarrow{AB}| \cdot |\overrightarrow{BC}| \cos B + |\overrightarrow{AC}| \cdot |\overrightarrow{BC}| \cos C = AC^2 - AB^2.$$

Clearly, to show that $(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{BC} = 0$, it suffices to show that $AC = AB$.

Denote $\angle ADB = \alpha$, then $\angle ADC = 180^\circ - \alpha$. By the Law of sines, we have

$$2R_1 = \frac{AB}{\sin \alpha} \quad \text{and} \quad 2R_2 = \frac{AC}{\sin(180^\circ - \alpha)} = \frac{AC}{\sin \alpha}.$$

If $R_1 = R_2$, then $AC = AB$. Next, we consider $R_1 \neq R_2$. WLOG, we may assume $R_1 > R_2$. From $\max\{R_1, R_2\} = R \cos \frac{A}{2}$, we have $R_2 < R_1 = R \cos \frac{A}{2}$.

Applying the Law of sines again, we have

$$BD = 2R_1 \sin \angle BAD = 2R_1 \sin(\alpha + B) \quad \text{and} \quad CD = 2R_2 \sin \angle CAD = 2R_2 \sin(\alpha - C).$$

Notice that

$$\begin{aligned} BC &= BD + CD = 2R_1 \sin(\alpha + B) + 2R_2 \sin(\alpha - C) \\ &< 2R \cos \frac{A}{2} (\sin(\alpha + B) + \sin(\alpha - C)) \\ &= 4R \cos \frac{A}{2} \sin\left(\alpha + \frac{B - C}{2}\right) \cos \frac{B + C}{2} \\ &= 2R \sin A \sin\left(\alpha + \frac{B - C}{2}\right) \\ &= BC \sin\left(\alpha + \frac{B - C}{2}\right) \leq BC, \end{aligned}$$

which leads to a contradiction. Hence, $R_1 = R_2$. Then $AC = AB$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $a = BC$, $b = AC$, $c = AB$, $\angle BAD = \theta$, and $\angle DAC = A - \theta$. Then $b = 2R_2 \sin \angle ADC$, $c = 2R_1 \sin \angle ADB$, and thus $R_2/R_1 = b/c$. Without loss of generality, let $b \geq c$. Therefore, $R_2 \geq R_1$ and thus $R_2 = R \cos \frac{A}{2}$, and $R_1 = R \cdot \frac{c}{b} \cdot \cos \frac{A}{2}$. We now have

$$\begin{aligned} a &= DC + DB = 2R_2 \sin(A - \theta) + 2R_1 \sin \theta \\ &= 2R \cos \frac{A}{2} \cdot \sin(A - \theta) + 2R \cdot \frac{c}{b} \cdot \cos \frac{A}{2} \cdot \sin \theta. \end{aligned}$$

But $a = 2R \sin A$, and thus

$$\sin A = \cos \frac{A}{2} \cdot \sin(A - \theta) + \frac{c}{b} \cdot \cos \frac{A}{2} \cdot \sin \theta,$$

or, because $\cos(A/2) > 0$,

$$2 \sin \frac{A}{2} = \sin(A - \theta) + \frac{c}{b} \sin \theta \leq \sin(A - \theta) + \sin \theta = 2 \sin \frac{A}{2} \cdot \cos \frac{A - 2\theta}{2}.$$

The above is impossible unless $\cos \frac{A - 2\theta}{2} = 1$ and $b = c$; that is, unless the triangle is isosceles ($b = c$) and $\theta = A/2$. In this case $D = M$, the midpoint of BC , and AD is the perpendicular bisector of BC . Thus,

$$(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{BC} = 2\overrightarrow{AM} \cdot \overrightarrow{BC} = 0.$$

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Nguyen Tuan Minh.

Undergraduate problems

U733. Evaluate

$$\int \frac{x-2}{x^4+16x-12} dx.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Mohammed Aassila, Strasbourg, France

We have

$$\begin{aligned} \frac{x-2}{x^4+16x-12} &= \frac{x-2}{(x^2+2x-2)(x^2-2x+6)} \\ &= \frac{1}{4} \left(\frac{1}{x^2-2x+6} - \frac{1}{x^2+2x-2} \right) \\ &= \frac{1}{4} \left(\frac{1}{(x-1)^2+5} - \frac{1}{(x+1)^2-3} \right). \end{aligned}$$

Since

$$\int \frac{dx}{(x-a)^2+b^2} = \frac{1}{b} \arctan\left(\frac{x-a}{b}\right) \quad \text{and} \quad \int \frac{dx}{(x-a)^2-b^2} = \frac{1}{2b} \ln \left| \frac{x-a-b}{x-a+b} \right|$$

we deduce that

$$\int \frac{x-2}{x^4+16x-12} dx = \frac{1}{4\sqrt{5}} \arctan\left(\frac{x-1}{\sqrt{5}}\right) - \frac{1}{8\sqrt{3}} \ln \left| \frac{x+1-\sqrt{3}}{x+1+\sqrt{3}} \right| + C.$$

Also solved by Anita Tufisi, Romania; Arkady Alt, San Jose, California, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Jiang Lianjun, Guilin, China; José Luis Díaz-Barrero, Barcelona, Spain; Nicole Indelicato, SUNY Brockport, NY, USA; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Nikos Hliopoulos, Aristotle University of Thessaloniki, Greece; Ritwik Kumar, Noida, India; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Sundaresh H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania.

U734. For $x \in \left(0, \frac{\pi}{2}\right)$ evaluate

$$\int \frac{1 - \sin x}{3 \sin x + 5(1 + \cos x)e^x} dx.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Teodor Cristian Petrescu, Romania

We have

$$\int \frac{1 - \sin x}{3 \sin x + 5(1 + \cos x)e^x} dx = \int \left(\frac{\frac{1 - \sin x}{1 + \cos x}}{\frac{3 \sin x}{1 + \cos x} + 5e^x} \right) dx.$$

Considering the function $f : \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$, $f(x) = \frac{3 \sin x}{1 + \cos x} + 5e^x$, we have that

$$f'(x) = \frac{3}{1 + \cos x} + 5e^x,$$

so our integral is equivalent to

$$\frac{1}{3} \int \frac{f'(x) - f(x)}{f(x)} dx = \frac{1}{3} \int \left(\frac{f'(x)}{f(x)} - 1 \right) dx = \frac{1}{3} (\ln |f(x)| - x) + C = \frac{1}{3} \ln \left(\frac{3 \sin x}{1 + \cos x} + 5e^x \right) - \frac{x}{3} + C.$$

Solution 2 by Anita Tufisi, Romania

Since

$$\frac{1 + \cos x}{\sin x} = \cot \frac{x}{2} \implies I = \int \frac{1 - \sin x}{\sin x (3 + 5e^x \cot \frac{x}{2})} dx.$$

Set

$$u = e^x \cot \frac{x}{2}.$$

Then

$$\frac{u'}{u} = 1 + \frac{d}{dx} \ln \cot \frac{x}{2} = 1 - \frac{1}{\sin x} = -\frac{1 - \sin x}{\sin x},$$

hence

$$\frac{1 - \sin x}{\sin x} dx = -\frac{du}{u}.$$

Therefore,

$$I = - \int \frac{du}{u(3 + 5u)}.$$

Now

$$\frac{1}{u(3 + 5u)} = \frac{1}{3u} - \frac{5}{3(3 + 5u)},$$

thus

$$I = -\frac{1}{3} \int \frac{du}{u} + \frac{5}{3} \int \frac{du}{3 + 5u} = -\frac{1}{3} \ln u + \frac{1}{3} \ln(3 + 5u) + C = \frac{1}{3} \ln \left(\frac{3 + 5u}{u} \right) + C.$$

Since

$$\frac{1}{u} = e^{-x} \tan \frac{x}{2}, \implies I = \frac{1}{3} \ln \left(5 + 3e^{-x} \tan \frac{x}{2} \right) + C.$$

Solution 3 by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

We are required to find

$$I = \int \frac{1 - \sin x}{3 \sin x + 5(1 + \cos x)e^x} dx, \quad x \in (0, \frac{\pi}{2}).$$

Rewrite the integral as:

$$I = \int \frac{\frac{1-\sin x}{(1+\cos x)e^x}}{\frac{3\sin x}{(1+\cos x)e^x} + 5} dx.$$

Since $\tan \frac{x}{2} = \frac{\sin x}{1+\cos x}$, then

$$I = \int \frac{\frac{1-\sin x}{(1+\cos x)e^x}}{3e^{-x} \tan \frac{x}{2} + 5} dx.$$

Setting $u = 3e^{-x} \tan \frac{x}{2} + 5$ for $x \in (0, \frac{\pi}{2})$, then

$$\begin{aligned} du &= \left(-3e^{-x} \tan \frac{x}{2} + \frac{3}{2}e^{-x} \sec^2 \frac{x}{2}\right) dx \\ &= 3e^{-x} \left(\frac{1}{2\cos^2 \frac{x}{2}} - \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}\right) dx \\ &= 3e^{-x} \left(\frac{1 - 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}}\right) dx. \end{aligned}$$

Since

$$2\sin \frac{x}{2} \cos \frac{x}{2} = \sin x \quad \text{and} \quad 2\cos^2 \frac{x}{2} = 1 + \cos x,$$

then

$$du = 3e^{-x} \frac{1 - \sin x}{1 + \cos x} dx,$$

or

$$\frac{1}{3}du = \frac{1 - \sin x}{(1 + \cos x)e^x} dx.$$

The indefinite integral with this substitution therefore becomes

$$I = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \log |u| + C,$$

where C is an arbitrary constant of integration. Thus

$$\int \frac{1 - \sin x}{3\sin x + 5(1 + \cos x)e^x} dx = \frac{1}{3} \log \left(3e^{-x} \tan \left(\frac{x}{2}\right) + 5\right) + C, \quad x \in (0, \frac{\pi}{2}).$$

Also solved by Arkady Alt, San Jose, California, USA; Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras; G. C. Greubel, Newport News, VA, USA; Ritwik Kumar, Noida, India; Sundaresh H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA.

U735. Prove that $\frac{7}{3}$ is the smallest positive exponent k which does not depend on n such that the inequality

$$\frac{a_1^4 + a_2^4 + \cdots + a_n^4}{n} \leq \left(\frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n} \right)^k$$

holds for any $n \geq 2$ and nonnegative real numbers a_i with at most one $a_i < 1$ and $a_1 + a_2 + \cdots + a_n = n$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution by the author

For $a_1 = 2$, $a_2 = \cdots = a_{n-1} = 1$ and $a_n = 0$, the constraints are satisfied and the inequality becomes

$$\left(\frac{n+6}{n} \right)^k \geq \frac{n+14}{n}, \quad k \geq \frac{\ln(n+14) - \ln n}{\ln(n+6) - \ln n}.$$

As $n \rightarrow \infty$, we get the necessary condition

$$k \geq \lim_{n \rightarrow \infty} \frac{\ln(n+14) - \ln n}{\ln(n+6) - \ln n} = \lim_{n \rightarrow \infty} \frac{7(n+6)}{3(n+14)} = \frac{7}{3}.$$

To prove that $\frac{7}{3}$ is the smallest value of k , we need to show that $F \geq 0$, where

$$F = \left(\frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n} \right)^{7/3} - \frac{a_1^4 + a_2^4 + \cdots + a_n^4}{n}.$$

For $n = 2$, we need to show that $a_1 + a_2 = 2$ implies

$$\left(\frac{a_1^3 + a_2^3}{2} \right)^{7/3} \geq \frac{a_1^4 + a_2^4}{2}.$$

Since $4(a_1^3 + a_2^3) \geq (a_1 + a_2)^3 = 8$, it suffices to show that

$$\left(\frac{a_1^3 + a_2^3}{2} \right)^2 \geq \frac{a_1^4 + a_2^4}{2}.$$

Denoting $a_1 = 1 + \sqrt{t}$ and $a_2 = 1 - \sqrt{t}$, where $t \in [0, 1]$, the inequality is equivalent to

$$(3t+1)^2 \geq t^2 + 6t + 1, \quad 8t^2 \geq 0.$$

For $n \geq 3$, assume that

$$a_1 = \max\{a_1, a_2, \dots, a_n\} \geq 1, \quad a_n = \min\{a_1, a_2, \dots, a_n\} \leq 1, \quad a_2, \dots, a_{n-1} \geq 1.$$

Let

$$z = \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}.$$

By the known inequality $n^2(a_1^3 + a_2^3 + \cdots + a_n^3) \geq (a_1 + a_2 + \cdots + a_n)^3$, we obtain $z \geq 1$. For fixed $a_1, \dots, a_{n-2}$, assume that a_n and F are functions of a_2 . We have $a'_n(a_2) = -1$ and

$$\begin{aligned} nF'(a_2) &= 7a_2^2 z^{4/3} - 4a_2^3 + (7a_n^2 z^{4/3} - 4a_n^3) a'_n = 7(a_2^2 - a_n^2) z^{4/3} - 4(a_2^3 - a_n^3) \\ &= (a_2 - a_n)[7(a_2 + a_n) z^{4/3} - 4(a_2^2 + a_2 a_n + a_n^2)] \geq 2(a_2 - a_n)[3(a_2 + a_n) - 2(a_2^2 + a_2 a_n + a_n^2)]. \end{aligned}$$

We will show that $F'(a_2) \geq 0$. This is true if

$$3(a_2 + a_n) \geq 2(a_2^2 + a_2 a_n + a_n^2).$$

From

$$n = a_1 + a_2 + (a_3 + \cdots + a_{n-1}) + a_n \geq 2a_2 + (n-3) + a_n,$$

we get $3 \geq 2a_2 + a_n$. Therefore,

$$3(a_2 + a_n) - 2(a_2^2 + a_2a_n + a_n^2) \geq (2a_2 + a_n)(a_2 + a_n) - 2(a_2^2 + a_2a_n + a_n^2) = a_n(a_2 - a_n) \geq 0.$$

From $F'(a_2) \geq 0$, it follows that $F(a_2)$ is increasing and has the minimum value for minimal a_2 , hence for $a_2 = 1$. So, it suffices to prove that $F \geq 0$ for $a_2 = 1$. By extending this result to all $a_2, \dots, a_{n-1}$, it suffices to prove that $F \geq 0$ for $a_2 = \cdots = a_{n-1} = 1$. So, we need to show that

$$\left(\frac{a_1^3 + a_n^3 + n - 2}{n} \right)^{7/3} - \frac{a_1^4 + a_n^4 + n - 2}{n} \geq 0$$

for $a_1 + a_n = 2$. Denoting $a_1 = 1 + \sqrt{t}$ and $a_n = 1 - \sqrt{t}$, where $t \in [0, 1]$, the inequality becomes

$$\left(\frac{6t + n}{n} \right)^{7/3} - \frac{2t^2 + 12t + n}{n} \geq 0.$$

By Bernoulli's inequality, we have

$$\left(1 + \frac{6t}{n} \right)^{7/3} - \frac{2t^2 + 12t + n}{n} \geq 1 + \frac{14t}{n} - \frac{2t^2 + 12t + n}{n} = \frac{2t(1-t)}{n} \geq 0.$$

So, the proof is completed. If $k = \frac{7}{3}$, then the equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Ritwik Kumar, Noida, India; Theo Koupelis, Clark College, Washington, USA; Shilong Zhuang, California, USA.

U736. Given $a, b, c \in \mathbb{R}_{\geq 0}$, prove that the following statements are true.

- (i) $(a - b)^2 + (b - c)^2 + (c - a)^2 \neq 0$ is a necessary and sufficient condition for the existence of an infinite, non-constant sequence x_n such that

$$\frac{3}{2} \leq x_n \leq \sum_{\text{cyc}} \frac{a}{b+c}, \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n = \frac{3}{2}.$$

- (ii) For all $t \geq 0$, the following inequality holds:

$$\sum_{\text{cyc}} \frac{ab}{(b+c)(c+a)} \geq \sum_{\text{cyc}} \frac{(a+t)(b+t)}{(b+c+2t)(c+a+2t)}.$$

Determine the conditions under which equality occurs.

Proposed by Shilong Zhuang, California, USA

Solution 1 by Ritwik Kumar, Noida, India

We assume that the expressions are defined, i.e. no two of a, b, c are zero.

- (i) By Nesbitt's inequality,

$$\sum_{\text{cyc}} \frac{a}{b+c} \geq \frac{3}{2},$$

with equality if and only if $a = b = c$.

Let

$$U = \sum_{\text{cyc}} \frac{a}{b+c}.$$

If $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$, then $a = b = c$, and hence $U = \frac{3}{2}$. Thus any sequence satisfying

$$\frac{3}{2} \leq x_n \leq U$$

must be identically equal to $\frac{3}{2}$, so no non-constant sequence exists.

Conversely, suppose $(a - b)^2 + (b - c)^2 + (c - a)^2 \neq 0$. Then a, b, c are not all equal, so by the equality case of Nesbitt's inequality, $U > \frac{3}{2}$.

Define

$$x_n = \frac{3}{2} + \frac{U - \frac{3}{2}}{n+1}, \quad n \in \mathbb{N}.$$

Then

$$\frac{3}{2} < x_n < U \leq \sum_{\text{cyc}} \frac{a}{b+c}$$

for every n , the sequence is non-constant, and

$$\lim_{n \rightarrow \infty} x_n = \frac{3}{2}.$$

Therefore, $(a - b)^2 + (b - c)^2 + (c - a)^2 \neq 0$ is necessary and sufficient.

- (ii) We use the identity

$$\sum_{\text{cyc}} \frac{ab}{(b+c)(c+a)} = 1 - \frac{2abc}{(a+b)(b+c)(c+a)}.$$

Indeed,

$$\begin{aligned}\sum_{\text{cyc}} \frac{ab}{(b+c)(c+a)} &= \frac{\sum_{\text{cyc}} ab(a+b)}{(a+b)(b+c)(c+a)} \\ &= \frac{(a+b+c)(ab+bc+ca) - 3abc}{(a+b)(b+c)(c+a)} \\ &= 1 - \frac{2abc}{(a+b)(b+c)(c+a)}.\end{aligned}$$

Applying the same identity to $a+t$, $b+t$, $c+t$, we get

$$\sum_{\text{cyc}} \frac{(a+t)(b+t)}{(b+c+2t)(c+a+2t)} = 1 - \frac{2(a+t)(b+t)(c+t)}{(a+b+2t)(b+c+2t)(c+a+2t)}.$$

Thus it suffices to prove that

$$\frac{abc}{(a+b)(b+c)(c+a)} \leq \frac{(a+t)(b+t)(c+t)}{(a+b+2t)(b+c+2t)(c+a+2t)}.$$

If one of a , b , c is zero, then the left-hand side is 0, and the inequality is immediate. Hence assume first that $a, b, c > 0$. Define

$$G(t) = \frac{(a+t)(b+t)(c+t)}{(a+b+2t)(b+c+2t)(c+a+2t)}.$$

We prove that $G(t)$ is nondecreasing for $t \geq 0$. Taking logarithmic derivatives gives

$$\frac{G'(t)}{G(t)} = \sum_{\text{cyc}} \frac{1}{a+t} - 2 \sum_{\text{cyc}} \frac{1}{a+b+2t}.$$

Now, by AM-HM,

$$\frac{1}{a+t} + \frac{1}{b+t} \geq \frac{4}{a+b+2t}.$$

Adding the three cyclic inequalities, we obtain

$$2 \sum_{\text{cyc}} \frac{1}{a+t} \geq 4 \sum_{\text{cyc}} \frac{1}{a+b+2t},$$

or equivalently,

$$\sum_{\text{cyc}} \frac{1}{a+t} - 2 \sum_{\text{cyc}} \frac{1}{a+b+2t} \geq 0.$$

Therefore, $G'(t) \geq 0$. Hence

$$G(t) \geq G(0),$$

which is exactly the desired inequality.

It remains to discuss equality. If $t = 0$, then the two sides of the original inequality are identical, so equality holds for all admissible a, b, c .

Now suppose $t > 0$. If one of a, b, c is zero, then $G(0) = 0$ while $G(t) > 0$, so the inequality is strict. Thus equality can occur only when $a, b, c > 0$.

For $a, b, c > 0$, equality in the AM-HM step occurs if and only if $a+t = b+t = c+t$, that is, $a = b = c$.

Therefore, for $t > 0$, equality holds if and only if $a = b = c > 0$.

Hence equality in part (ii) occurs precisely when $t = 0$ or $a = b = c > 0$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

(i) Clearing denominators we get

$$\sum_c \frac{a}{b+c} - \frac{3}{2} = \frac{2(a^3+b^3+c^3) - \sum_c a^2(b+c)}{2 \prod_c (a+b)}. \quad (*)$$

Given that $a, b, c \in \mathbb{R}_{\geq 0}$, it is necessary that $(a - b)^2 + (b - c)^2 + (c - a)^2 \neq 0$ because otherwise we could have one or more of the quantities $(a + b)$, $(b + c)$, $(c + a)$ be equal to zero. Also, by AM-GM we get $a^3 + b^3 + c^3 \geq 3abc$, and thus by Schur's inequality we get $2 \sum_c a^3 - \sum_c a^2(b + c) \geq 0$, with equality when $a = b = c$. Thus, a necessary and sufficient condition for the existence of x_n is that $a \neq b \neq c \neq a$. Under this condition, the right-hand-side of (*) is positive, and thus we can define x_n as

$$x_n = \frac{3}{2} + \frac{1}{n} \cdot \frac{2(a^3 + b^3 + c^3) - \sum_c a^2(b + c)}{2 \prod_c (a + b)},$$

which satisfies the required conditions.

(ii) Clearing denominators we get

$$\frac{\sum_c ab(a + b)}{\prod_c (a + b)} - \frac{\sum_c (a + t)(b + t)(a + b + 2t)}{\prod_c (a + b + 2t)} \geq 0 \iff \frac{2t \cdot S(t)}{\prod_c (a + b) \cdot \prod_c (a + b + 2t)} \geq 0,$$

where

$$\begin{aligned} S(t) &= At^2 + Bt + C, \\ A &= \sum_c a^2(b + c) - 6abc, \\ B &= 2 \sum_c a^2b^2 + \sum_c a^3(b + c) - 4abc(a + b + c), \\ C &= \sum_c a^3(b^2 + c^2) - 2abc \sum_c ab. \end{aligned}$$

Using AM-GM we get $\sum_c a^2(b + c) \geq 6abc$, and thus $A \geq 0$. Also, $ac^3 + 2a^2b^2 + a^3c \geq 4a^2bc$, and cyclic permutations; thus, $B \geq 0$. Finally, $a^3b^2 + a^2b^3 + a^3c^2 + b^3c^2 \geq 4a^2b^2c$, and cyclic permutations; thus $C \geq 0$. Therefore, $S(t) \geq 0$ and the desired inequality holds. Equality occurs when $t = 0$ or when $a = b = c$.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo.

U737. Calculate

$$\lim_{n \rightarrow \infty} n \sqrt[n]{\int_0^1 \left| \arctan x - x + \frac{x^3}{3} - \dots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right|^n dx}.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

Solution 1 by the authors

The limit equals $\frac{1}{4}$. Let

$$I_n = n \sqrt[n]{\int_0^1 \left| \arctan x - x + \frac{x^3}{3} - \dots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right|^n dx}.$$

We have, based on Taylor series expansion of $\arctan$, that

$$\begin{aligned} \arctan x - x + \frac{x^3}{3} - \dots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} &= \sum_{k=0}^{\infty} (-1)^{n+k} \frac{x^{2n+2k+1}}{2n+2k+1} \\ &= (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \frac{x^{2n+3}}{2n+3} + \dots \\ &= \int_0^x (-1)^n t^{2n} dt + \int_0^x (-1)^{n+1} t^{2n+2} dt + \dots \\ &= (-1)^n \int_0^x t^{2n} (1 - t^2 + t^4 - \dots) dt \\ &= (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt, \quad x \in [0, 1]. \end{aligned}$$

Another proof of the above equality which is based on the Squeeze Theorem is as follows:

$$\begin{aligned} \sum_{k=0}^m (-1)^{n+k} \frac{x^{2n+2k+1}}{2n+2k+1} &= \sum_{k=0}^m (-1)^{n+k} \int_0^x t^{2n+2k} dt \\ &= \int_0^x \sum_{k=0}^m (-t^2)^{n+k} dt \\ &= \int_0^x (-t^2)^n \cdot \frac{1 - (-t^2)^{m+1}}{1+t^2} dt \end{aligned}$$

and it follows that

$$\begin{aligned} \left| \sum_{k=0}^m (-1)^{n+k} \frac{x^{2n+2k+1}}{2n+2k+1} - (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt \right| &= \int_0^x \frac{t^{2n+2m+2}}{1+t^2} dt \\ &\leq \int_0^x t^{2n+2m+2} dt = \frac{x^{2n+2m+3}}{2n+2m+3} < \frac{1}{2n+2m+3}. \end{aligned}$$

Letting $m \rightarrow \infty$ in the above inequality we get that

$$\sum_{k=0}^{\infty} (-1)^{n+k} \frac{x^{2n+2k+1}}{2n+2k+1} = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt.$$

It follows that

$$(-1)^n \left(\arctan x - x + \frac{x^3}{3} - \dots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right) = \int_0^x \frac{t^{2n}}{1+t^2} dt,$$

which in turn implies that

$$\left| \arctan x - x + \frac{x^3}{3} - \cdots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right| = \int_0^x \frac{t^{2n}}{1+t^2} dt.$$

We have, since $\frac{1}{2} \leq \frac{1}{1+t^2}$ and $\frac{t^2}{1+t^2} \leq \frac{x^2}{1+x^2}$ that

$$\frac{x^{2n+1}}{2(2n+1)} \leq \int_0^x \frac{t^{2n}}{1+t^2} dt = \int_0^x \frac{t^2}{1+t^2} t^{2n-2} dt \leq \frac{x^2}{1+x^2} \cdot \frac{x^{2n-1}}{2n-1} = \frac{x^{2n+1}}{1+x^2} \cdot \frac{1}{2n-1} \leq \frac{x^2}{1+x^2} \cdot \frac{1}{2n-1}$$

which in turn implies that

$$\frac{x^{2n^2+n}}{[2(2n+1)]^n} \leq \left| \arctan x - x + \frac{x^3}{3} - \cdots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right|^n \leq \left(\frac{x^2}{1+x^2} \right)^n \cdot \frac{1}{(2n-1)^n}. \quad (1)$$

It follows that

$$\frac{n}{2(2n+1)} \cdot \frac{1}{\sqrt[n]{2n^2+n+1}} \leq I_n \leq \frac{n}{2n-1} \sqrt[n]{\int_0^1 \left(\frac{x^2}{1+x^2} \right)^n dx}.$$

Passing to the limit, as $n \rightarrow \infty$, in the preceding inequalities, and using the fact that $\lim_{n \rightarrow \infty} \sqrt[n]{\int_0^1 \left(\frac{x^2}{1+x^2} \right)^n dx} = \sup_{x \in [0,1]} \frac{x^2}{1+x^2} = \frac{1}{2}$, we get that the desired limit equals $\frac{1}{4}$ and the problem is solved.

Remark Observe, it follows from (1), that if $k > 0$ is a real number, then the following equality holds true

$$\lim_{n \rightarrow \infty} n \sqrt[n]{\int_0^1 x^k \left| \arctan x - x + \frac{x^3}{3} - \cdots - (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right|^n dx} = \frac{1}{4}.$$

Solution 2 by Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia

Denote the limit to be found by L . We claim $L = \frac{1}{4}$.

Since

$$\arctan(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{2k-1}, \quad |x| < 1,$$

the limit L can be rewritten as

$$L = \lim_{n \rightarrow \infty} n \left[\int_0^1 \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{2k-1} \right|^n dx \right]^{\frac{1}{n}}.$$

Reindexing $k \mapsto k+n+1$ yields

$$L = \lim_{n \rightarrow \infty} n \left[\int_0^1 \left| \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2n+1}}{2k+2n+1} \right|^n dx \right]^{\frac{1}{n}}.$$

Let us rewrite the limit as

$$L = \lim_{n \rightarrow \infty} n(I_n)^{1/n}.$$

Here

$$I_n = \int_0^1 |S_n(x)|^n dx,$$

such that

$$S_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2n+1}}{2k+2n+1}.$$

Since

$$\frac{x^{2k+2n+1}}{2k+2n+1} = \int_0^x t^{2k+2n} dt,$$

then

$$S_n(x) = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k+2n} dt = \int_0^x t^{2n} \left(\sum_{k=0}^{\infty} (-1)^k t^{2k} \right) dt = \int_0^x \frac{t^{2n}}{1+t^2} dt.$$

Observe that $S_n(x) > 0$ for all $n \geq 0$ and $x \in (0, 1)$. Enforcing a substitution of $t = ux$ produces

$$S_n(x) = x^{2n+1} \int_0^1 \frac{u^{2n}}{1+u^2x^2} du.$$

We now find an asymptotic expansion for $S_n(x)$ for large n . For large n , the term u^{2n} in the integrand concentrates heavily near the point $u = 1$. Integrating by parts, we have

$$S_n(x) = x^{2n+1} \left(\frac{1}{(2n+1)(1+x^2)} + \frac{2x^2}{2n+1} \int_0^1 \frac{u^{2n+2}}{(1+x^2u^2)^2} du \right),$$

or

$$S_n(x) \sim \frac{x^{2n+1}}{(2n+1)(1+x^2)} \sim \frac{x^{2n}}{2n(1+x^2)},$$

as $n \rightarrow \infty$. Thus

$$I_n = \int_0^1 |S_n(x)|^n dx = \int_0^1 [S_n(x)]^n dx \sim \frac{1}{(2n)^n} \int_0^1 \frac{x^{2n^2}}{(1+x^2)^n} dx.$$

When n is large, the integrand of this integral concentrates almost entirely near the point $x = 1$. Here then $1+x^2 \sim 2$. Letting $x = 1 - \frac{y}{2n^2}$, we see that when $x = 0$, $y = 2n^2 \rightarrow \infty$ as $n \rightarrow \infty$. Thus

$$I_n \sim \frac{1}{(2n)^n} \int_0^\infty \left(1 - \frac{y}{2n^2}\right)^{2n^2} \frac{1}{2n} \frac{dy}{2n^2} = \frac{1}{(4n)^n 2n^2} \int_0^\infty \left(1 - \frac{y}{2n^2}\right)^{2n^2} dy.$$

Noting that as $n \rightarrow \infty$, one has

$$\left(1 - \frac{y}{2n^2}\right)^{2n^2} \sim e^{-y}.$$

Then

$$I_n \sim \frac{1}{(4n)^n 2n^2} \int_0^\infty e^{-y} dy = \frac{1}{(4n)^n 2n^2}.$$

Thus

$$(I_n)^{1/n} \sim \left(\frac{1}{(4n)^n 2n^2} \right)^{1/n} = \frac{1}{4n(2n^2)^{1/n}}.$$

Returning to the limit L , we have

$$L = \lim_{n \rightarrow \infty} (I_n)^{1/n} = \lim_{n \rightarrow \infty} n \cdot \frac{1}{4n(2n^2)^{1/n}} = \frac{1}{4} \lim_{n \rightarrow \infty} \frac{1}{(2n^2)^{1/n}} = \frac{1}{4},$$

as announced.

Also solved by Ritwik Kumar, Noida, India; The Ring Lords, Baden-Württemberg, Germany.

U738. Let $n \geq 3$ and let

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

be the Dihedral group of order $2n$. Fix a divisor $d \mid n$ and let $H = \langle r^d \rangle \leq \langle r \rangle$. Determine the largest possible size of a set

$$\mathcal{R} \subseteq \{sr^0, sr^1, \dots, sr^{n-1}\}$$

of reflections with the property that for every distinct $x, y \in \mathcal{R}$, then $xy \in H$. What are the families $\mathcal{R}$ achieving this maximum?

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

First, we prove that for reflections sr^i and sr^j one has

$$(sr^i)(sr^j) = r^{j-i} \quad \text{and} \quad (sr^j)(sr^i) = r^{i-j}.$$

Using $srs = r^{-1}$ we have $sr = r^{-1}s$ and hence $sr^i = r^{-i}s$ for all i . Thus

$$(sr^i)(sr^j) = s(r^i s)r^j = s(sr^{-i})r^j = r^{j-i},$$

and swapping i and j gives $(sr^j)(sr^i) = r^{i-j}$.

Now, let $\mathcal{R}$ be as stated and write

$$I(\mathcal{R}) = \{k \in \{0, 1, \dots, n-1\} : sr^k \in \mathcal{R}\}.$$

By what we have seen, for distinct $i, j \in I(\mathcal{R})$ the condition $(sr^i)(sr^j) \in H = \langle r^d \rangle$ is equivalent to $r^{i-j} \in \langle r^d \rangle$, i.e. $d \mid (i-j)$. Therefore all indices in $I(\mathcal{R})$ lie in a single residue class modulo d . Conversely, if all indices lie in one residue class modulo d , then for any distinct i and j we have $d \mid (i-j)$, hence $(sr^i)(sr^j) = r^{i-j} \in \langle r^d \rangle = H$.

For a fixed residue class a modulo d , the set

$$I_a = \{a, a+d, a+2d, \dots, a + (\frac{n}{d} - 1)d\} \quad (\text{taken modulo } n)$$

has cardinality n/d , and it is maximal with the divisibility property above. Thus

$$\max |\mathcal{R}| = \frac{n}{d}$$

and the only extremal families are

$$\mathcal{R}_a = \{sr^k : k \equiv a \pmod{d}\} \quad (a = 0, 1, \dots, d-1).$$

Also solved by Ritwik Kumar, Noida, India; The Ring Lords, Baden-Württemberg, Germany.

Olympiad problems

O733. At a party with $n \geq 3$ people, it is known that among any three people, there are at least two pairs of them who know each other. Prove that:

- (a) you can always line them up in a row so that each person is next to only acquaintances;
- (b) if $n \geq 4$ you can seat them around a round table so that everyone has acquaintances on both sides.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

We reformulate the given conditions in terms of graph theory. Let $n \geq 3$ and let G be a simple graph on n vertices with the property that every 3-vertex set spans at least 2 edges in G . Consider the complement $\overline{G}$. The given condition means that no 3-vertex set can span 2 or 3 edges in $\overline{G}$. Hence, in $\overline{G}$ no vertex can have degree ≥ 2 : if a vertex v were nonadjacent in G to two distinct vertices x, y , then the triple $\{v, x, y\}$ would span at most one edge in G , contradicting the hypothesis.

Thus every vertex of $\overline{G}$ has degree at most 1, so $\overline{G}$ is a matching (possibly empty). Equivalently,

$$G = K_n \setminus M,$$

where M is a matching in K_n .

- (a) We will prove that G contains a Hamiltonian path. A complete graph minus a matching always has a Hamiltonian path. Indeed, let $M = \{a_1a'_1, \dots, a_ka'_k\}$ be the matching of non-edges and let $U = \{u_1, u_2, \dots, u_m\}$ be the set of vertices not incident to M in $\overline{G}$ (possibly $m = 0$). We order the vertices as

$$a_1, a_2, \dots, a_k, u_1, u_2, \dots, u_m, a'_1, a'_2, \dots, a'_k$$

Consecutive pairs are never matched partners, so all consecutive pairs are adjacent in G , proving that this is a Hamiltonian path in G .

- (b) If $n \geq 4$, we will prove that G is Hamiltonian. One way to see this is via Ore's Theorem. For any two nonadjacent vertices x, y in G , necessarily $xy \in M$, so both x and y miss at most one neighbor each (their unique partners in M). Therefore

$$\deg_G(x) + \deg_G(y) \geq (n - 1 - 1) + (n - 1 - 1) = 2n - 4 \geq n$$

for $n \geq 4$. By Ore's Theorem, G contains a Hamiltonian cycle.

Remark

Observe that (b) could have been solved easily also by Dirac's Theorem. Indeed, since $\overline{G}$ is a matching, then every vertex in G has degree at least $n - 2$, that is $\delta(G) \geq n - 2$. Since $n \geq 4$ we have

$$\delta(G) \geq n - 2 \geq \frac{n}{2}$$

and the conclusion follows by Dirac's Theorem.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Ritwik Kumar, Noida, India; Theo Koupelis, Clark College, Washington, USA; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; Ganghun Kim, Fort Lee High School, NJ, USA; Shilong Zhuang, California, USA; William Han, Horace Mann School, Bronx, NY.

O734. Let $p > 3$ be a prime, and let

$$k = \left\lfloor \frac{3p^2 - 6p - 1}{4p} \right\rfloor$$

(where $\lfloor u \rfloor$ denotes the integer part of the real number u). Show that among the numbers $1 + p, 1 + 2p, \dots, 1 + kp$ there exists at least one that can be written as the sum of three nonzero squares.

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution 1 by the authors

Let $p > 3$ be a prime, and let

$$k = \left\lfloor \frac{3p^2 - 6p - 1}{4p} \right\rfloor$$

(where $\lfloor u \rfloor$ denotes the integer part of the real number u). Show that among the numbers $1 + p, 1 + 2p, \dots, 1 + kp$ there exists at least one that can be written as the sum of three nonzero squares.

Solution. It is well-known that, for an integer a , the congruence $x^2 + y^2 \equiv a \pmod{p}$ has $p + (-1)^{\frac{p-1}{2}}(p-1)$ solutions for $a \equiv 0 \pmod{p}$, and it has $p - (-1)^{\frac{p-1}{2}}$ solutions whenever $a \not\equiv 0 \pmod{p}$. Using these facts one can easily show that the congruence $x^2 + y^2 + z^2 \equiv 0 \pmod{p}$ has

$$p^2 + \left(\frac{-a}{p}\right)p = p^2 + (-1)^{\frac{p-1}{2}}\left(\frac{a}{p}\right)p$$

solutions for any integer a , where $\left(\frac{t}{p}\right)$ is the Legendre symbol of t modulo p , defined to be 0 if $t \equiv 0 \pmod{p}$, 1 if t is a nonzero quadratic residue modulo p , and -1 if t is a nonzero quadratic non-residue modulo p . Note that two solutions (x_1, y_1) and (x_2, y_2) are considered distinct if $x_1 \not\equiv x_2 \pmod{p}$, or $y_1 \not\equiv y_2 \pmod{p}$ (and similarly for triples).

Thus the congruence $x^2 + y^2 + z^2 \equiv 1 \pmod{p}$ has $p^2 + (-1)^{\frac{p-1}{2}}p$ solutions (as 1 is, evidently, a quadratic residue modulo p). Among these we have the six solutions $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ with two 0 components, and yet a number of $3\left(p - (-1)^{\frac{p-1}{2}} - 4\right)$ solutions, each with precisely one component equal to 0 modulo p (for instance we have $p - (-1)^{\frac{p-1}{2}} - 4$ solutions $(x, y, 0)$ with (x, y) different from any of the pairs $(\pm 1, 0)$ and $(0, \pm 1)$). Notice that the inequality

$$p^2 + (-1)^{\frac{p-1}{2}}p > 3\left(p - (-1)^{\frac{p-1}{2}} - 4\right) + 6$$

holds for p being a prime greater than 3, for it is equivalent to $p^2 - 2p + 9 > 0$ when $p \equiv 1 \pmod{4}$, and to $p^2 - 4p + 3 > 0$ when $p \equiv 3 \pmod{4}$. This means that the number of all the solutions to $x^2 + y^2 + z^2 \equiv 1 \pmod{p}$ is greater than the number of the solutions of this congruence having at least one component equal to 0 modulo p , so there exists a solution with all components nonzero modulo p . Moreover, since x from a solution (x, y, z) can be replaced by $-x$, or by $p - x$, and we still get solutions (and the same is true for y and z), there must exist a solution (x, y, z) satisfying

$$x^2 + y^2 + z^2 \equiv 1 \pmod{p}, \text{ and } x, y, z \in \left\{1, 2, \dots, \frac{p-1}{2}\right\}.$$

For this solution we clearly have

$$3 \leq x^2 + y^2 + z^2 \leq \frac{3(p-1)^2}{4},$$

so, in the first place, $x^2 + y^2 + z^2 = 1 + sp$ for some positive integer s . On the other hand, the number k defined in the statement of the problem satisfies

$$k \leq \frac{3p^2 - 6p - 1}{4p} < k + 1,$$

whence

$$1 + kp \leq \frac{3(p-1)^2}{4} < 1 + (k+1)p.$$

Since

$$x^2 + y^2 + z^2 = 1 + sp \leq \frac{3(p-1)^2}{4}$$

it follows that s is at most equal to k , therefore $x^2 + y^2 + z^2 = 1 + sp$ is one of the numbers $1+p, 1+2p, \dots, 1+kp$, as we intended to prove.

Solution 2 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

We use Legendre's three-square theorem: a positive integer is a sum of three squares if and only if it is not of the form

$$4^a(8b+7), \quad a, b \geq 0.$$

We split into cases.

Case 1: $p = 5$. Then

$$k = \left\lfloor \frac{3 \cdot 25 - 6 \cdot 5 - 1}{4 \cdot 5} \right\rfloor = \left\lfloor \frac{44}{20} \right\rfloor = 2.$$

So $1 + 2p = 11$ is among the listed numbers, and

$$11 = 1^2 + 1^2 + 3^2.$$

Case 2: $p = 7$. Then

$$k = \left\lfloor \frac{3 \cdot 49 - 6 \cdot 7 - 1}{4 \cdot 7} \right\rfloor = \left\lfloor \frac{104}{28} \right\rfloor = 3.$$

So $1 + 3p = 22$ is among the listed numbers, and

$$22 = 1^2 + 3^2 + 3^2.$$

Case 3: $p \geq 11$. Then

$$\frac{3p^2 - 6p - 1}{4p} \geq 6,$$

so

$$k \geq 6.$$

Hence the numbers

$$1 + p, \quad 1 + 2p, \quad 1 + 3p, \quad 1 + 6p$$

all lie among

$$1 + p, 1 + 2p, \dots, 1 + kp.$$

Now consider $p \pmod{8}$.

- If $p \equiv 1 \pmod{8}$, take $1 + 2p$. Then

$$1 + 2p \equiv 3 \pmod{8}.$$

- If $p \equiv 3 \pmod{8}$, take $1 + 6p$. Then

$$1 + 6p \equiv 3 \pmod{8}.$$

- If $p \equiv 5 \pmod{8}$, take $1 + p$. Then

$$1 + p \equiv 6 \pmod{8}.$$

- If $p \equiv 7 \pmod{8}$, take $1 + 3p$. Then

$$1 + 3p \equiv 6 \pmod{8}.$$

So in every case we obtain a number N from the list such that

$$N \equiv 3 \pmod{8} \quad \text{or} \quad N \equiv 6 \pmod{8}.$$

In either case, N is not of the form $4^a(8b+7)$. Therefore, by Legendre's theorem, N is a sum of three squares.

It remains to see that these three squares are nonzero.

If $N \equiv 3 \pmod{8}$, then in any three-square representation

$$N = x^2 + y^2 + z^2,$$

each of x^2, y^2, z^2 must be odd modulo 8. Hence x, y, z are all odd, and in particular all are nonzero.

If $N \equiv 6 \pmod{8}$, then N cannot be the sum of two squares, since a sum of two squares is congruent modulo 8 only to 0, 1, 2, 4, 5. Therefore no square can be zero in a three-square representation of N .

Thus N is a sum of three nonzero squares. Hence among $1+p, 1+2p, \dots, 1+kp$ there is at least one number that can be written as the sum of three nonzero squares.

Also solved by Juan José Granier, Cencosud, Chile; Ritwik Kumar, Noida, India; Theo Koupelis, Clark College, Washington, USA; Benjamin Cha, Choate Rosemary Hall, Wallingford, CT, USA; William Han, Horace Mann School, Bronx, NY.

O735. Prove that in all acute triangles ABC and for any $\lambda > 0$,

$$\sum_{cyc} \frac{\cos A}{\sqrt{\lambda + \cos^2 A}} \leq \frac{3(R+r)}{\sqrt{(9\lambda+1)R^2 + 2Rr + r^2}}.$$

Proposed by Mihály Bencze, Braşov, Romania

Solution 1 by the author

Construct a matrix of order 3×2 , set $A = \begin{bmatrix} 1 & \sqrt{\lambda} \sec A \\ 1 & \sqrt{\lambda} \sec B \\ 1 & \sqrt{\lambda} \sec C \end{bmatrix}$ and $p = 2 > q = -1$ based on Lin-Phoong inequality, we have:

$$\begin{aligned} \frac{1}{\sum \frac{1}{\sqrt{1+\lambda \sec^2 A}}} &\geq \sqrt{\left(\frac{1}{1+1+1}\right)^2 + \left(\frac{\sqrt{\lambda}}{\sum \cos A}\right)^2} = \sqrt{\frac{1}{9} + \frac{\lambda R^2}{(R+r)^2}} = \frac{\sqrt{(9\lambda+1)R^2 + 2Rr + r^2}}{3(R+r)} \\ \frac{1}{\sum \frac{\cos A}{\sqrt{\lambda + \cos^2 A}}} &\geq \frac{\sqrt{(9\lambda+1)R^2 + 2Rr + r^2}}{3(R+r)} \end{aligned}$$

Finally

$$\sum \frac{\cos A}{\sqrt{\lambda + \cos^2 A}} \leq \frac{3(R+r)}{\sqrt{(9\lambda+1)R^2 + 2Rr + r^2}}$$

Solution 2 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Let

$$f(x) = \frac{x}{\sqrt{\lambda + x^2}}, \quad x > 0.$$

Then

$$f'(x) = \frac{\lambda}{(\lambda + x^2)^{3/2}} > 0$$

and

$$f''(x) = -\frac{3\lambda x}{(\lambda + x^2)^{5/2}} < 0.$$

Hence f is concave on $(0, \infty)$.

Since ABC is acute, we have $\cos A, \cos B, \cos C > 0$. Therefore, by Jensen's inequality,

$$\sum_{cyc} f(\cos A) \leq 3f\left(\frac{\cos A + \cos B + \cos C}{3}\right).$$

Thus

$$\sum_{cyc} \frac{\cos A}{\sqrt{\lambda + \cos^2 A}} \leq 3 \cdot \frac{\frac{\cos A + \cos B + \cos C}{3}}{\sqrt{\lambda + \left(\frac{\cos A + \cos B + \cos C}{3}\right)^2}}.$$

Now use the well-known identity

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Then

$$\sum_{cyc} \frac{\cos A}{\sqrt{\lambda + \cos^2 A}} \leq 3 \cdot \frac{1 + \frac{r}{R}}{\sqrt{\lambda + \left(\frac{1 + \frac{r}{R}}{3}\right)^2}}.$$

Simplifying the right-hand side gives

$$3 \cdot \frac{\frac{1 + \frac{r}{R}}{3}}{\sqrt{\lambda + \left(\frac{1 + \frac{r}{R}}{3}\right)^2}} = \frac{1 + \frac{r}{R}}{\sqrt{\lambda + \frac{(R+r)^2}{9R^2}}}.$$

Since

$$1 + \frac{r}{R} = \frac{R+r}{R},$$

we get

$$\frac{1 + \frac{r}{R}}{\sqrt{\lambda + \frac{(R+r)^2}{9R^2}}} = \frac{\frac{R+r}{R}}{\sqrt{\frac{9\lambda R^2 + (R+r)^2}{9R^2}}} = \frac{3(R+r)}{\sqrt{9\lambda R^2 + (R+r)^2}}.$$

Finally,

$$9\lambda R^2 + (R+r)^2 = (9\lambda + 1)R^2 + 2Rr + r^2.$$

Hence

$$\sum_{\text{cyc}} \frac{\cos A}{\sqrt{\lambda + \cos^2 A}} \leq \frac{3(R+r)}{\sqrt{(9\lambda + 1)R^2 + 2Rr + r^2}}.$$

Equality in Jensen's inequality occurs if and only if $\cos A = \cos B = \cos C$, that is, $A = B = C = 60^\circ$. Therefore equality holds if and only if ABC is equilateral.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Ritwik Kumar, Noida, India; Sahel Koley, Tarakeswar, West Bengal, India; Theo Koupelis, Clark College, Washington, USA; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania.

O736. For every positive integer x , let $\alpha(x)$ denote the largest positive divisor z of 2025 for which there is an integer y such that $x = y^z$. Let $n \geq 2$ be a positive integer and let $x_1, x_2, \dots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \dots + x_n = 2$. Determine the largest possible value of

$$\sum_{1 \leq i < j \leq n} (\alpha(j-i) - 1)x_i x_j.$$

Proposed by Andrei-Giovani Chirita, University of Cambridge, UK

Solution by the author

For $x_1 = x_2 = 1$ and $x_3 = \dots = x_n = 0$ the expression equals 2024.

We will prove that

$$\sum_{1 \leq i < j \leq n} (\alpha(j-i) - 1)x_i x_j \leq 2024.$$

First recall the well-known identity $\sum_{d|k} \varphi(d) = k$, where φ is the Euler totient function (the identity follows, for instance, by considering the fractions $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$ in simplified form and double counting).

In light of this we can write

$$\begin{aligned} \sum_{1 \leq i < j \leq n} (\alpha(j-i) - 1)x_i x_j &= \sum_{1 \leq i < j \leq n} \left(\sum_{d|\alpha(j-i), d>1} \varphi(d) \right) x_i x_j \\ &= \sum_{d|2025, d>1} \varphi(d) \left(\sum_{d|\alpha(j-i)} x_i x_j \right). \end{aligned} \quad (1)$$

Given a divisor $d > 1$ of 2025, define the undirected graph G_d whose vertices are $1, 2, \dots, n$ and two vertices are connected if their difference has the form x^d for $x \in \mathbb{Z}_{>0}$. We have the following two claims.

Claim 1. G_d is triangle-free.

Proof. Assume otherwise. Let $i < j < k$ be three pairwise adjacent vertices. If $j - i = a^d, k - j = b^d$ and $k - i = c^d$, then $a^d + b^d = c^d$ which is impossible by Fermat's Last Theorem since $d \geq 3$.

Claim 2. We have

$$\sum_{(i,j) \in G_d} x_i x_j \leq 1$$

Proof. As long as there are at least three nonzero numbers among $x_1, \dots, x_n$, run the following procedure: choose $x_i, x_j > 0$ such that i and j are not adjacent in G_d (which is possible by Claim 1) and vary x_i, x_j while keeping $x_i + x_j$ constant. Since the left hand side is a linear function in x_i , it follows that the sum increases in one of the instances where $x_i = 0$ or $x_j = 0$, and we can go with that case. This procedure will decrease the number of nonzero values until there are at most two nonzero numbers left. But then for any $a, b \geq 0$ with $a + b = 2$ we have $ab \leq 1$, so our inequality is proved.

Finally, from Claim 2 it follows that for all $d | 2025, d > 1$, we have

$$\sum_{d|\alpha(j-i)} x_i x_j \leq 1. \quad (2)$$

Therefore, by combining (1) and (2), it follows that

$$\sum_{1 \leq i < j \leq n} (\alpha(j-i) - 1)x_i x_j \leq \sum_{d|2025, d>1} \varphi(d) = 2024.$$

Comment. The solution works if 2025 is replaced by any odd number.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo.

O737. Let $x, y, z \in \left[0, \frac{7}{4}\right]$ be real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$9(x + y + z) \geq 23 + 4xyz.$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

Without loss of generality, we may assume that $x \geq y \geq z$, then $x \geq 1$. Choose a number $t \geq 0$ for which

$$x^2 + t^2 + t^2 + x \cdot t \cdot t = x^2 + y^2 + z^2 + xyz = 4$$

so

$$y^2 + z^2 - 2t^2 = x(t^2 - yz), \quad x = 2 - t^2.$$

From $1 \leq x \leq \frac{7}{4}$ it follows that $t^2 = 2 - x \geq 2 - \frac{7}{4} = \frac{1}{4}$ so

$$\frac{1}{2} \leq t \leq 1.$$

If we assume that $t^2 < yz$, then it follows that $y^2 + z^2 < 2t^2$. On the other hand

$$t^2 < yz \leq \frac{y^2 + z^2}{2} \implies 2t^2 < y^2 + z^2$$

which is a contradiction. It follows that $t^2 \geq yz$ and $y^2 + z^2 \geq 2t^2$. Denote

$$f(x, y, z) = 9(x + y + z) - 23 - 4xyz.$$

We notice that

$$f(x, y, z) - f(x, t, t) = -9(2t - y - z) + 4x(t^2 - yz).$$

But, from

$$y^2 + z^2 - 2t^2 = x(t^2 - yz) \implies 4t^2 - (y + z)^2 = (2 - x)(t^2 - yz),$$

so

$$\begin{aligned} f(x, y, z) - f(x, t, t) &= 4x(t^2 - yz) - \frac{9(2 - x)(t^2 - yz)}{2t + y + z} \\ &= \frac{(t^2 - yz)(4(2 - t^2)(2t + y + z) - 9t^2)}{2t + y + z} \end{aligned}$$

We will prove that $f(x, y, z) \geq f(x, t, t)$ which means

$$y + z \geq \frac{9t^2}{4(2 - t^2)} - 2t = \frac{8t^3 + 9t^2 - 16t}{4(2 - t^2)},$$

which is true for $8t^2 + 9t - 16 \leq 0 \implies t \leq \frac{\sqrt{593} - 9}{16} \approx 0.9594$.

On the other hand, we have

$$\begin{aligned} (y + z)^2 + (y - z)^2 &= 2(4 - x^2 - xyz) = 8 - 2x^2 - 2xyz \\ &\geq 8 - 2x^2 - \frac{x(y + z)^2}{2}. \end{aligned}$$

It follows that

$$(y + z)^2 \geq 4(2 - x) - \frac{2(y - z)^2}{2 + x},$$

or

$$(y+z)^2 \geq 4t^2 - \frac{2(y-z)^2}{4-t^2}.$$

Since

$$(y-z)^2 \leq \left(\frac{7}{4} - 0\right)^2 = \frac{49}{16}$$

we deduce that

$$(y+z)^2 \geq 4t^2 - \frac{49}{8(4-t^2)} = \frac{-32t^4 + 128t^2 - 49}{8(4-t^2)}.$$

Since $-32t^4 + 128t^2 - 49 \geq 0$ for

$$0.6548 \approx \sqrt{\frac{16 - \sqrt{158}}{8}} \leq t \leq \sqrt{\frac{16 + \sqrt{158}}{8}} \approx 1.8897.$$

Hence, for $1 \geq t > \frac{\sqrt{593}-9}{16}$ it remains to prove that

$$\frac{-32t^4 + 128t^2 - 49}{8(4-t^2)} \geq \frac{(8t^3 + 9t^2 - 16t)^2}{16(2-t^2)^2},$$

or after performing the calculations this is written as follows

$$\frac{144t^7 + 81t^6 - 864t^5 - 422t^4 + 1152t^3 + 392t^2 - 392}{16(4-t^2)(2-t^2)^2} \geq 0$$

which is true using the computer.

Finally, to prove the inequality, it is necessary to show that

$$\begin{aligned} f(x, t, t) \geq 0 &\iff 9(x+2t) - 23 - 4xt^2 \geq 0 \\ &\iff 9(2-t^2+2t) - 23 - 4(2-t^2)t^2 \geq 0 \\ &\iff (t-1)^2(2t-1)(2t+5) \geq 0 \end{aligned}$$

which is clearly true.

The equality holds when $t = 1$ which means $x = y = z = 1$ or $t = \frac{1}{2}$ which means $x = \frac{7}{4}$, $y = z = \frac{1}{2}$ and its cyclic permutations.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $f(x, y, z) = 9(x+y+z) - 23 - 4xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 + xyz - 4 = 0$. Both f and g have continuous derivatives and the gradient of g is never zero. Thus, by the theorem of Lagrange multipliers, any extrema occur on the boundary or where $\nabla f = \lambda \nabla g$, that is

$$\left. \begin{aligned} 9 - 4yz &= \lambda(2x + yz), \\ 9 - 4xz &= \lambda(2y + xz), \\ 9 - 4xy &= \lambda(2z + xy), \end{aligned} \right\} \iff \left\{ \begin{aligned} (x-y)[4z - \lambda(2-z)] &= 0, \\ (y-z)[4x - \lambda(2-x)] &= 0, \\ (z-x)[4y - \lambda(2-y)] &= 0. \end{aligned} \right.$$

(i) Let $x \neq y \neq z \neq x$. From the above we get $x, y, z \neq 2$ and

$$\frac{4x}{2-x} = \frac{4y}{2-y} = \frac{4z}{2-z},$$

and thus $x = y = z$, which contradicts the assumption.

(ii) Let $x = y = z$. In this case, from the condition $g(x, y, z) = 0$ we get $x^3 + 3x^2 = 4 \iff (x-1)(x+2)^2 = 0$; thus $x = y = z = 1$, and the desired inequality holds as a strict equality.

(iii) Without loss of generality, let $y = z \neq x$. In this case, from the condition $g(x, y, z) = 0$ we get $x^2 + 2y^2 + xy^2 = 4 \iff y^2 = 2 - x$, where $0 \leq x \leq 2$. Substituting into the desired inequality we get the equivalent expression

$$9(x + 2y) \geq 23 + 4xy^2 \iff 18\sqrt{2-x} \geq 23 - 4x^2 - x = (x+2)(7-4x) + 9.$$

The right-hand-side of the above inequality is positive in $[0, \frac{7}{4}]$, and thus, after squaring and simplifying we get

$$(x-1)^2(7-4x)(4x+17) \geq 0,$$

which is obvious. For $x = 1$ we get $y = z = 1$, which contradicts the assumption. Equality occurs when $x = 7/4$, and thus $y = z = 1/2$.

Examining the values of f on the boundary, let $z = 0$; without loss of generality let $x \geq y$. From the condition $g = 0$ we get $x^2 + y^2 = 4$ where $\sqrt{2} \leq x \leq 7/4$. The desired inequality now becomes $9\sqrt{4-x^2} \geq 23-9x > 0$, or, after squaring and simplifying we get $162x^2 - 414x + 205 \leq 0$, which is obvious in $[\sqrt{2}, 7/4]$ as a strict inequality. Also, if $x = 7/4$, from the condition $g = 0$ we get $4(y+z) = \sqrt{4yz+15}$; using AM-GM we get $y+z \geq 2\sqrt{yz}$, and thus $yz \leq 1/4$. The desired inequality in this case becomes equivalent to $9\sqrt{4yz+15} \geq 28yz+29$; squaring and simplifying we get $2(1-4yz)(98yz+187) \geq 0$, which is obvious. Equality occurs as in case (iii) above.

Thus, the desired inequality holds, with equality when $(x, y, z) = (1, 1, 1)$, or $(x, y, z) = (7/4, 1/2, 1/2)$ and cyclic permutations.

Also solved by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo; Arkady Alt, San Jose, California, USA; José Luis Díaz-Barrero, Barcelona, Spain; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Shilong Zhuang, California, USA.

O738. Let $a_1, a_2, \dots, a_n$ be positive real numbers such that at most one of them is less than 1 and $a_1^3 + a_2^3 + \dots + a_n^3 = n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1^4 + a_2^4 + \dots + a_n^4.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution 1 by the author

Without loss of generality, assume that

$$a_1 \leq 1 \leq a_2 \leq \dots \leq a_n.$$

Write the inequality as

$$\frac{1 - a_1^5}{a_1} + \frac{1 - a_2^5}{a_2} + \dots + \frac{1 - a_n^5}{a_n} \geq 0.$$

Since

$$\frac{1 - x^5}{x} = \frac{(1 - x)(1 + x + x^2 + x^3 + x^4)}{x} = (1 - x^3)f(x),$$

where

$$f(x) = \frac{1 + x + x^2 + x^3 + x^4}{x(1 + x + x^2)} = 1 + \frac{1 + x^4}{x + x^2 + x^3},$$

the inequality is equivalent to $E \geq 0$, where

$$\begin{aligned} E &= (1 - a_1^3)f(a_1) + (1 - a_2^3)f(a_2) + \dots + (1 - a_n^3)f(a_n) \\ &= [(a_2^3 - 1) + \dots + (a_n^3 - 1)]f(a_1) + (1 - a_2^3)f(a_2) + \dots + (1 - a_n^3)f(a_n) \\ &= (a_2^3 - 1)[f(a_1) - f(a_2)] + \dots + (a_n^3 - 1)[f(a_1) - f(a_n)]. \end{aligned}$$

It suffices to show that $f(a_1) - f(a_k) \geq 0$ for $k = 2, \dots, n$. Since

$$\begin{aligned} f(a_1) - f(a_k) &= \frac{1 + a_1^4}{a_1 + a_1^2 + a_1^3} - \frac{1 + a_k^4}{a_k + a_k^2 + a_k^3} \\ &= \frac{(a_k - a_1)(1 - a_1^3 a_k^3) + (a_k^2 - a_1^2)(1 - a_1^2 a_k^2) + (a_k^3 - a_1^3)(1 - a_1 a_k)}{(a_1 + a_1^2 + a_1^3)(a_k + a_k^2 + a_k^3)}, \end{aligned}$$

it suffices to show that $1 \geq a_1 a_k$. Indeed, we have

$$1 - a_1^3 a_n^3 = (1 - a_1^3)(a_n^3 - 1) + 2 - a_1^3 - a_n^3 \geq 2 - a_1^3 - a_n^3 = (a_2^3 + \dots + a_{n-1}^3) - (n - 2) \geq 0,$$

hence $1 \geq a_1 a_n \geq a_1 a_k$.

The equality occurs for $a_1 = a_2 = \dots = a_n = 1$.

Solution 2 by Adela Llapashtica, Specialized Mathematical Gymnasium, Prishtine, Kosovo

Set

$$f(a) = \frac{1}{a} - a^4.$$

We first observe that

$$f(a) - \frac{5}{3}(1 - a^3) = -\frac{(a - 1)^3(3a^2 + 4a + 3)}{3a}.$$

Since

$$a^3 - 1 = (a - 1)(a^2 + a + 1),$$

we may rewrite this as

$$f(a) = \frac{5}{3}(1 - a^3) + \eta(a)(1 - a^3)^3,$$

where

$$\eta(a) = \frac{3a^2 + 4a + 3}{3a(a^2 + a + 1)^3}.$$

Clearly $\eta(a) > 0$ for all $a > 0$. Also,

$$\eta'(a) = -\frac{5a^4 + 10a^3 + 10a^2 + 4a + 1}{a^2(a^2 + a + 1)^4} < 0,$$

so η is strictly decreasing on $(0, \infty)$.

Now put

$$u_i = 1 - a_i^3, \quad i = 1, 2, \dots, n.$$

Since

$$a_1^3 + a_2^3 + \dots + a_n^3 = n,$$

we have

$$u_1 + u_2 + \dots + u_n = 0.$$

If all $a_i \geq 1$, then the condition

$$a_1^3 + a_2^3 + \dots + a_n^3 = n$$

forces

$$a_1 = a_2 = \dots = a_n = 1,$$

and the inequality is immediate.

Otherwise, since at most one of the a_i 's is less than 1, we may relabel the variables so that

$$a_1 < 1, \quad a_2, a_3, \dots, a_n \geq 1.$$

Then

$$u_1 > 0, \quad u_2, u_3, \dots, u_n \leq 0.$$

Using the decomposition of f , we get

$$\sum_{i=1}^n f(a_i) = \frac{5}{3} \sum_{i=1}^n u_i + \sum_{i=1}^n \eta(a_i) u_i^3 = \sum_{i=1}^n \eta(a_i) u_i^3.$$

Since η is decreasing and

$$a_1 < 1 \leq a_i \quad (i = 2, \dots, n),$$

we have

$$\eta(a_1) \geq \eta(a_i) \quad (i = 2, \dots, n).$$

Because $u_i^3 \leq 0$ for $i = 2, \dots, n$, it follows that

$$\eta(a_i) u_i^3 \geq \eta(a_1) u_i^3 \quad (i = 2, \dots, n).$$

Therefore,

$$\sum_{i=1}^n \eta(a_i) u_i^3 \geq \eta(a_1) \sum_{i=1}^n u_i^3.$$

It remains to prove that

$$\sum_{i=1}^n u_i^3 \geq 0.$$

Since

$$u_1 = -(u_2 + \dots + u_n) = \sum_{i=2}^n (-u_i),$$

and since $-u_i \geq 0$ for $i = 2, \dots, n$, we have

$$u_1^3 = \left(\sum_{i=2}^n (-u_i) \right)^3 \geq \sum_{i=2}^n (-u_i)^3 = - \sum_{i=2}^n u_i^3.$$

Hence

$$\sum_{i=1}^n u_i^3 \geq 0.$$

Combining the above estimates, we obtain

$$\sum_{i=1}^n \left(\frac{1}{a_i} - a_i^4 \right) = \sum_{i=1}^n f(a_i) \geq \eta(a_1) \sum_{i=1}^n u_i^3 \geq 0.$$

Therefore,

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1^4 + a_2^4 + \dots + a_n^4.$$

Solution 3 by Daniel Pascuas, Barcelona, Spain

If $a_1, a_2, \dots, a_n \geq 1$, then $a_1^3, a_2^3, \dots, a_n^3 \geq 1$, so $a_1 = a_2 = \dots = a_n = 1$ (by the hypothesis $a_1^3 + a_2^3 + \dots + a_n^3 = n$), and therefore

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n = a_1^4 + a_2^4 + \dots + a_n^4.$$

Otherwise, only one of the a_j 's is less than 1. Thus, without loss of generality, we may assume that $0 < a_1 < 1$ and $a_2, \dots, a_n \geq 1$. Write $a_j^3 = 1 + x_j$, where $x_j \geq 0$, for $j = 2, \dots, n$. It follows that $a_1^3 = n - (a_2^3 + \dots + a_n^3) = 1 - x_1$, where $x_1 = x_2 + \dots + x_n \in (0, 1)$. Therefore it is enough to prove the following assertion:

If $x_2, \dots, x_n \geq 0$ satisfy that $x_1 = x_2 + \dots + x_n \in (0, 1)$, then

$$\frac{1}{(1-x_1)^{\frac{1}{3}}} + \frac{1}{(1+x_2)^{\frac{1}{3}}} + \dots + \frac{1}{(1+x_n)^{\frac{1}{3}}} > (1-x_1)^{\frac{4}{3}} + (1+x_2)^{\frac{4}{3}} + \dots + (1+x_n)^{\frac{4}{3}}. \quad (1)$$

First we show the assertion for $n = 2$, that is,

$$\frac{1}{(1-x)^{\frac{1}{3}}} + \frac{1}{(1+x)^{\frac{1}{3}}} > (1-x)^{\frac{4}{3}} + (1+x)^{\frac{4}{3}} \quad (0 < x < 1), \quad (2)$$

The binomial formula

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad (\alpha \in \mathbb{R}, -1 < x < 1),$$

where

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-(k-1))}{k!}, \quad \text{for } k = 1, 2, \dots,$$

allow us to prove (2) easily:

$$\frac{1}{(1-x)^{\frac{1}{3}}} + \frac{1}{(1+x)^{\frac{1}{3}}} = 2 \sum_{k=0}^{\infty} \binom{-\frac{1}{3}}{2k} x^{2k} > 2 \sum_{k=0}^{\infty} \binom{\frac{4}{3}}{2k} x^{2k} = (1-x)^{\frac{4}{3}} + (1+x)^{\frac{4}{3}} \quad (0 < x < 1),$$

since $\binom{-\frac{1}{3}}{0} = 1 = \binom{\frac{4}{3}}{0}$, $\binom{-\frac{1}{3}}{2} = \frac{2}{9} = \binom{\frac{4}{3}}{2}$, and

$$\binom{-\frac{1}{3}}{2k} = \frac{1}{(2k)!} 4 \prod_{j=2}^{2k-1} \left(\frac{1}{3} + j \right) > \frac{1}{(2k)!} 4 \prod_{j=2}^{2k-1} \left(j - \frac{4}{3} \right) = \binom{\frac{4}{3}}{2k}, \quad \text{for } k = 2, 3, \dots$$

Now we show the assertion for $n > 2$. Write (1) as

$$\frac{1}{(1-x_1)^{\frac{1}{3}}} + f(x_2) + \cdots + f(x_n) > (1-x_1)^{\frac{4}{3}}, \quad (3)$$

where

$$f(x) = \frac{1}{(1+x)^{\frac{1}{3}}} - (1+x)^{\frac{4}{3}} \quad (x \geq 0).$$

Note that $f'(x) = -\frac{1}{3}(1+x)^{-\frac{4}{3}} - \frac{4}{3}(1+x)^{\frac{1}{3}}$ and $f''(x) = \frac{4}{9}((1+x)^{-\frac{7}{3}} - (1+x)^{-\frac{2}{3}}) \leq 0$, for $x \geq 0$, so f is concave on $[0, \infty)$. Since f is concave on $[0, \infty)$ and $f(0) = 0$, f is subadditive, that is, $f(x) + f(y) \geq f(x+y)$, for any $x, y \geq 0$. It follows that

$$\frac{1}{(1-x_1)^{\frac{1}{3}}} + f(x_2) + \cdots + f(x_n) \geq \frac{1}{(1-x_1)^{\frac{1}{3}}} + f(x_2 + \cdots + x_n) = \frac{1}{(1-x_1)^{\frac{1}{3}}} + f(x_1)$$

By applying (2) to $x = x_1$, we get that

$$\frac{1}{(1-x_1)^{\frac{1}{3}}} + f(x_1) > (1-x_1)^{\frac{4}{3}},$$

which completes the proof of (3).

Observe that we have also proved that the inequality in the statement becomes an equality if and only if $a_1 = a_2 = \cdots = a_n = 1$.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Theo Koupelis, Clark College, Washington, USA.