

On Best Constants in Two Refinements of the Garfunkel–Bankoff Inequality

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Abstract

In this paper, we establish a sharper refinement of the Garfunkel–Bankoff inequality and determine the optimal constants for both this refinement and the one previously obtained in [12].

1 Introduction

In triangle ABC , we adopt the standard notations: A , B , and C denote the interior angles, while a , b , and c represent the side lengths opposite these angles, respectively. Furthermore, R , r , and s denote the circumradius, inradius, and semiperimeter of ABC . The area of ABC is denoted by S .

In [3], J. Garfunkel established the following inequality:

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \quad (1)$$

The first proof of inequality (1) was provided by L. Bankoff in [1], and thus inequality (1) is often referred to as the Garfunkel–Bankoff inequality [1, 2, 3, 5, 13, 14]. Notably, inequality (1) refines the classical Finsler–Hadwiger and Weitzenböck inequalities (see [13, 9, 10]), and is equivalent to Kooi’s inequality (see [8]).

In [12], the first author proposed the following refinement of the Garfunkel–Bankoff inequality:

Theorem 1. For any triangle ABC , the following inequality holds whenever $0 \leq k \leq 4$:

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + k \cdot \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}. \quad (2)$$

The case $k = 4$ was established in detail in [12]. In this paper, we prove that $k = 4$ is indeed the best constant in inequality (2). The methods we use to determine the best possible constants are also inspired by the articles [15, 16] published in Mathematical Reflections.

Furthermore, we propose and establish another refinement of the Garfunkel–Bankoff inequality:

Theorem 2. For any triangle ABC , the following inequality holds whenever $0 \leq k \leq \frac{1}{16}$:

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + k \cdot \frac{(a-b)^2(b-c)^2(c-a)^2}{R^4r^2}. \quad (3)$$

2 Proofs

Proof of Theorem 1. Assume, for contradiction, that there exists $k > 4$ such that the inequality

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + k \cdot \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}$$

holds for all triangles ABC . Now, let

$$\begin{cases} x = \frac{b+c-a}{2} > 0, \\ y = \frac{c+a-b}{2} > 0, \\ z = \frac{a+b-c}{2} > 0. \end{cases}$$

We then have

$$\begin{aligned} S &= \sqrt{xyz(x+y+z)}, \quad s = x+y+z, \\ R &= \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}, \quad r = \frac{\sqrt{xyz(x+y+z)}}{x+y+z}, \\ \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2} &= \frac{(x-y)^2(y-z)^2(z-x)^2}{(x+y)^2(y+z)^2(z+x)^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} &= \frac{r}{4R} = \frac{xyz}{(x+y)(y+z)(z+x)}, \\ \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} &= \frac{(4R+r)^2}{s^2} - 2 = \frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz(x+y+z)}. \end{aligned}$$

Hence the inequality is equivalent to

$$\frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz(x+y+z)} \geq 2 - \frac{8xyz}{(x+y)(y+z)(z+x)} + k \cdot \frac{(x-y)^2(y-z)^2(z-x)^2}{(x+y)^2(y+z)^2(z+x)^2}.$$

Fix $y = \frac{k-2}{2} > 1$, $z = 1$, and let $x \rightarrow +\infty$. Observing the limits, we derive a contradiction, which shows that no $k > 4$ can satisfy the inequality. Thus $k = 4$ is the best constant in (2). $\square$

We now proceed to the proof of Theorem 2.

Proof of Theorem 2. We begin by recalling an extended form of Gerretsen's inequality established in [7]:

$$16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - \frac{r^2(R-2r)}{R-r}. \quad (4)$$

We also have the following well-known identities:

$$\begin{aligned} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} &= \frac{r}{4R}, \\ \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} &= \frac{(4R+r)^2}{s^2} - 2, \\ \frac{(a-b)^2(b-c)^2(c-a)^2}{16R^4r^2} &= \frac{-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r^4 - 12Rr^3 - 48R^2r^2 - 64R^3r}{4R^4}. \end{aligned}$$

Thus, inequality (3) is equivalent to

$$\frac{1}{4R^4s^2} \left(s^6 + (-4R^2 - 20Rr + 2r^2)s^4 + (-16R^4 + 72R^3r + 48R^2r^2 + 12Rr^3 + r^4)s^2 + 64R^6 + 32R^5r + 4R^4r^2 \right) \geq 0. \quad (5)$$

On the other hand, observe (by inequality (4)) that

$$\begin{aligned} & s^6 + (-4R^2 - 20Rr + 2r^2)s^4 + (-16R^4 + 72R^3r + 48R^2r^2 + 12Rr^3 + r^4)s^2 \\ & \quad + 64R^6 + 32R^5r + 4R^4r^2 \\ & = (4(R-r)(R-2r) + s^2)(4R^2 + 4Rr + 3r^2 - s^2)^2 \\ & \quad + 4r(6R^3 + 5r^2R - 10r^3) \left(4R^2 + 4Rr + 3r^2 - \frac{r^2(R-2r)}{R-r} - s^2 \right) \\ & \quad + \frac{4r^2(R-2r)(R^4 + r(R-r)(R^2 + 4r^2))}{R-r} \geq 0. \end{aligned}$$

Hence, inequality (3) is proved.

Next, assume for contradiction that there exists $k > \frac{1}{16}$ such that inequality (3) holds for all triangles ABC :

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + k \cdot \frac{(a-b)^2(b-c)^2(c-a)^2}{R^4r^2}.$$

As in the proof of Theorem 1, let

$$\begin{cases} x = \frac{b+c-a}{2} > 0, \\ y = \frac{c+a-b}{2} > 0, \\ z = \frac{a+b-c}{2} > 0. \end{cases}$$

We obtain the relations

$$\begin{aligned} S &= \sqrt{xyz(x+y+z)}, \quad s = x+y+z, \\ R &= \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}, \quad r = \frac{\sqrt{xyz(x+y+z)}}{x+y+z}, \\ \frac{(a-b)^2(b-c)^2(c-a)^2}{R^4r^2} &= \frac{256xyz(x+y+z)^3(x-y)^2(y-z)^2(z-x)^2}{(x+y)^4(y+z)^4(z+x)^4}. \end{aligned}$$

Moreover, we recall the standard identities

$$\begin{aligned} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} &= \frac{r}{4R} = \frac{xyz}{(x+y)(y+z)(z+x)}, \\ \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} &= \frac{(4R+r)^2}{s^2} - 2 = \frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz(x+y+z)}. \end{aligned}$$

Thus, inequality (3) is equivalent to

$$\frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz(x+y+z)} \geq 2 - \frac{8xyz}{(x+y)(y+z)(z+x)} + k \cdot \frac{256xyz(x+y+z)^3(x-y)^2(y-z)^2(z-x)^2}{(x+y)^4(y+z)^4(z+x)^4}.$$

Fix $y = 8\sqrt{k} - 1 > 1$, $z = 1$, and let $x \rightarrow +\infty$. We then compute the following limits:

$$\lim_{x \rightarrow +\infty} \frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz(x+y+z)} = \frac{y^2 + z^2}{yz},$$

$$\begin{aligned}
& \lim_{x \rightarrow +\infty} \frac{xyz}{(x+y)(y+z)(z+x)} = 0, \\
& \lim_{x \rightarrow +\infty} \frac{xyz(x+y+z)^3(x-y)^2(y-z)^2(z-x)^2}{(x+y)^4(y+z)^4(z+x)^4} \\
&= \lim_{x \rightarrow +\infty} \frac{yz(1+\frac{y+z}{x})^3(1-\frac{y}{x})^2(y-z)^2(\frac{z}{x}-1)^2}{(1+\frac{y}{z})^4(y+z)^4(\frac{z}{x}+1)^4} \\
&= \frac{yz(y-z)^2}{(y+z)^4}.
\end{aligned}$$

Therefore,

$$\frac{y^2+z^2}{yz} \geq 2+k \cdot \frac{256yz(y-z)^2}{(y+z)^4},$$

which simplifies to

$$\frac{(y-z)^2}{yz} \geq \frac{256kyz(y-z)^2}{(y+z)^4},$$

and hence

$$(y+z)^4 \geq 256ky^2z^2.$$

Thus,

$$(y+z)^2 \geq 16\sqrt{k} \cdot yz.$$

But with $y = 8\sqrt{k} - 1$ and $z = 1$, we obtain the contradiction

$$64k = (y+z)^2 \geq 16\sqrt{k} \cdot yz = 16\sqrt{k}(8\sqrt{k}-1) > 64k.$$

This contradiction shows that no $k > \frac{1}{16}$ can satisfy inequality (3). Therefore, $k = \frac{1}{16}$ is the best possible constant. $\square$

3 Conclusion

In this paper, we revisited the classical Garfunkel–Bankoff inequality and presented two sharp refinements. First, we confirmed that the refinement obtained in [12] admits the best possible constant $k = 4$. Second, we introduced a new refinement involving the ratio $\frac{(a-b)^2(b-c)^2(c-a)^2}{R^4r^2}$ and proved that its optimal constant is $k = \frac{1}{16}$. These results highlight the continuing richness of inequalities in elementary geometry and demonstrate that even classical relations such as the Garfunkel–Bankoff inequality can still be sharpened in meaningful ways.

It would be of further interest to explore whether analogous refinements can be obtained for other fundamental triangle inequalities, such as the Finsler–Hadwiger and Weitzenböck inequalities, or whether similar techniques can be extended to higher-dimensional settings.

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